Orbital dynamics

- Test particle in gravitational potential Φ
- Cylindrical polar coordinates (r, ϕ, z)
- Newtonian dynamics

- Assume: $\Phi = \Phi(r, z)$ axisymmetric

  $\Phi(r, -z) = \Phi(r, z)$ symmetric

- Special case: $\Phi = -GM(r^2 + z^2)^{-1/2}$

point-mass potential → Keplerian orbits

Orbital dynamics

- Equation of motion

$$\ddot{\boldsymbol{r}} = -\boldsymbol{\nabla}\Phi \qquad \left\{ \begin{aligned} \ddot{r} - r\dot{\phi}^2 &= -\Phi_{,r} \\ r\ddot{\phi} + 2\dot{r}\dot{\phi} &= 0 \\ \ddot{z} &= -\Phi_{,z} \end{aligned} \right.$$

- Specific energy $\varepsilon = \frac{1}{2}|\dot{\boldsymbol{r}}|^2 + \Phi = \text{const}$
- Specific angular momentum $h = r^2\dot{\phi} = \text{const}$
- Reduces to 2D problem

$$\ddot{r} = -\Phi^{\text{eff}}_{,r}$$

$$\ddot{z} = -\Phi^{\text{eff}}_{,z}$$

- Effective potential

$$\Phi^{\text{eff}} = \Phi + \frac{h^2}{2r^2}$$

Orbital dynamics

- Circular orbit in midplane $(z = 0)$

$$0 = \Phi^{\mathrm{eff}}_{,z}(r,0) \qquad \checkmark \text{ by symmetry}$$

$$\left.\begin{aligned} 0 &= \Phi^{\mathrm{eff}}_{,r}(r,0) = \Phi_{,r}(r,0) - \frac{h^2}{r^3} \\ \varepsilon &= \frac{h^2}{2r^2} + \Phi(r,0) \end{aligned}\right\} \text{ defining } \begin{array}{l} h_\circ(r) \\ \varepsilon_\circ(r) \end{array}$$

- Important relation

$$\frac{\mathrm{d}\varepsilon_\circ}{\mathrm{d}r} = \frac{h_\circ}{r^2}\frac{\mathrm{d}h_\circ}{\mathrm{d}r} - \cancel{\frac{h_\circ^2}{r^3}} + \cancel{\Phi_{,r}(r,0)}$$

$$\frac{\mathrm{d}\varepsilon_\circ}{\mathrm{d}h_\circ} = \frac{h_\circ}{r^2} = \dot{\phi} = \Omega_\circ$$

orbital angular velocity

Orbital dynamics

- Keplerian case

$$\Phi(r,0) = -\frac{GM}{r}$$

$$h_\circ = (GMr)^{1/2}$$

$$\varepsilon_\circ = -\frac{GM}{2r}$$

$$\Omega_\circ = \left(\frac{GM}{r^3}\right)^{1/2}$$

Orbital dynamics

- Reminder of general Keplerian orbits

$$\ddot{\boldsymbol{r}} = -\frac{GM\boldsymbol{r}}{|\boldsymbol{r}|^3}$$

$$\frac{\mathrm{d}\boldsymbol{h}}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{r} \times \dot{\boldsymbol{r}}) = \dot{\boldsymbol{r}} \times \dot{\boldsymbol{r}} + \boldsymbol{r} \times \ddot{\boldsymbol{r}} = \boldsymbol{0}$$

- Orbit is confined to plane $\perp$ $\boldsymbol{h}$, so introduce polar coordinates (r, ϕ):

$$\ddot{r} - r\dot{\phi}^2 = -\frac{GM}{r^2} \qquad\qquad h = r^2\dot{\phi} = \mathrm{const}$$

- Let $r = 1/u$ and note that $\frac{\mathrm{d}}{\mathrm{d}t} = \dot{\phi}\frac{\mathrm{d}}{\mathrm{d}\phi} = hu^2\frac{\mathrm{d}}{\mathrm{d}\phi}$:

$$hu^2\frac{\mathrm{d}}{\mathrm{d}\phi}\left[hu^2\frac{\mathrm{d}}{\mathrm{d}\phi}\left(\frac{1}{u}\right)\right] - h^2u^3 = -GMu^2$$

$$\frac{\mathrm{d}^2u}{\mathrm{d}\phi^2} + u = \frac{GM}{h^2}$$

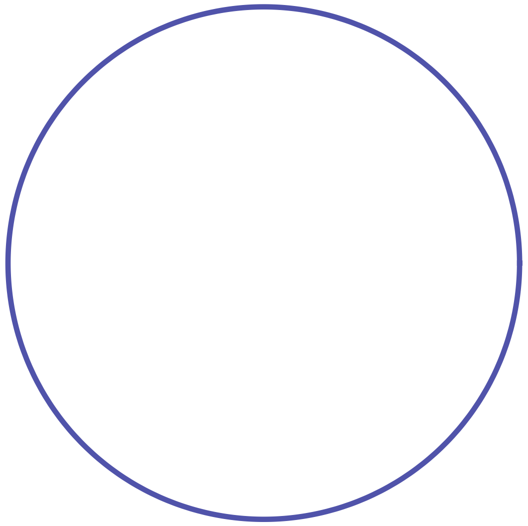
Orbital dynamics

$$\frac{\mathrm{d}^2u}{\mathrm{d}\phi^2} + u = \frac{GM}{h^2}$$

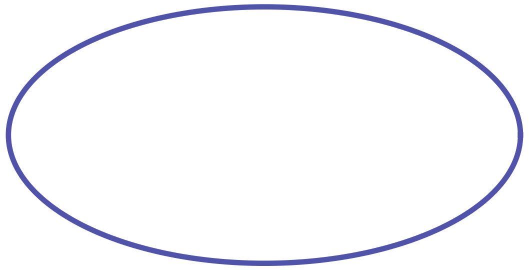
- General solution (with two arbitrary constants)

$$u = \frac{GM}{h^2}[1 + e\cos(\phi - \varpi)] \qquad \Rightarrow \ r = \frac{\lambda}{1 + e\cos(\phi - \varpi)}$$

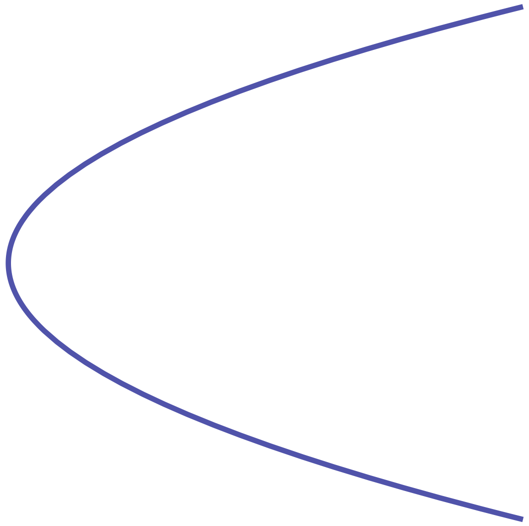
- Polar equation of conic section:



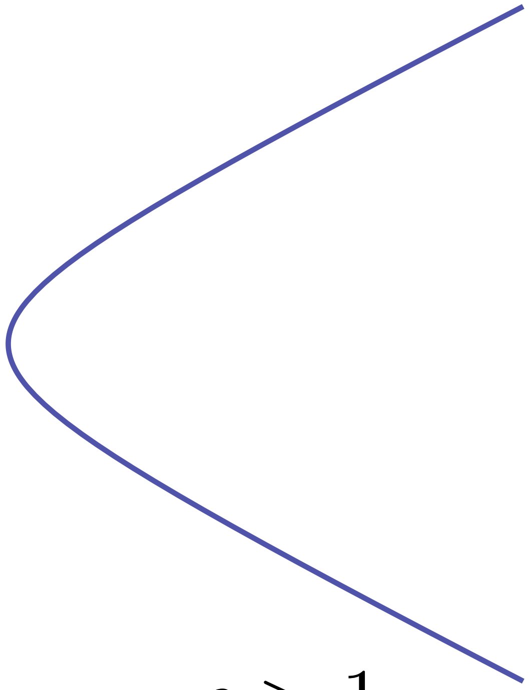
$e = 0$
circle
bound
$(\varepsilon < 0)$



$0 < e < 1$
ellipse
bound



$e = 1$
parabola
marginally
unbound



$e > 1$
hyperbola
unbound
$(\varepsilon > 0)$

Orbital dynamics

- Perturbations $(\delta r, \delta z)$ of circular orbits in midplane (h fixed)

$$\ddot{\delta r} = -\Omega_r^2\, \delta r \qquad \Omega_r^2 = \Phi^{\text{eff}}_{,rr}(r,0)$$

$$\ddot{\delta z} = -\Omega_z^2\, \delta z \qquad \Omega_z^2 = \Phi^{\text{eff}}_{,zz}(r,0)$$

[$\Phi^{\text{eff}}_{,rz}(r,0) = 0$ by symmetry]

Ω_r	usually called	κ	(horizontal) epicyclic frequency
Ω_z	sometimes called	μ	vertical (epicyclic) frequency

- Orbit is stable if $\Omega_r^2 > 0$ (i.e. $\kappa^2 > 0$) and $\Omega_z^2 > 0$
  i.e. if orbit is of minimum energy for given h

Orbital dynamics

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- Now

$$\kappa^2 = \Phi_{,rr}(r,0) + \frac{3h_\circ^2}{r^4}$$

$$= \frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{h_\circ^2}{r^3}\right) + \frac{3h_\circ^2}{r^4}$$

$$= \frac{1}{r^3}\frac{\mathrm{d}h_\circ^2}{\mathrm{d}r}$$

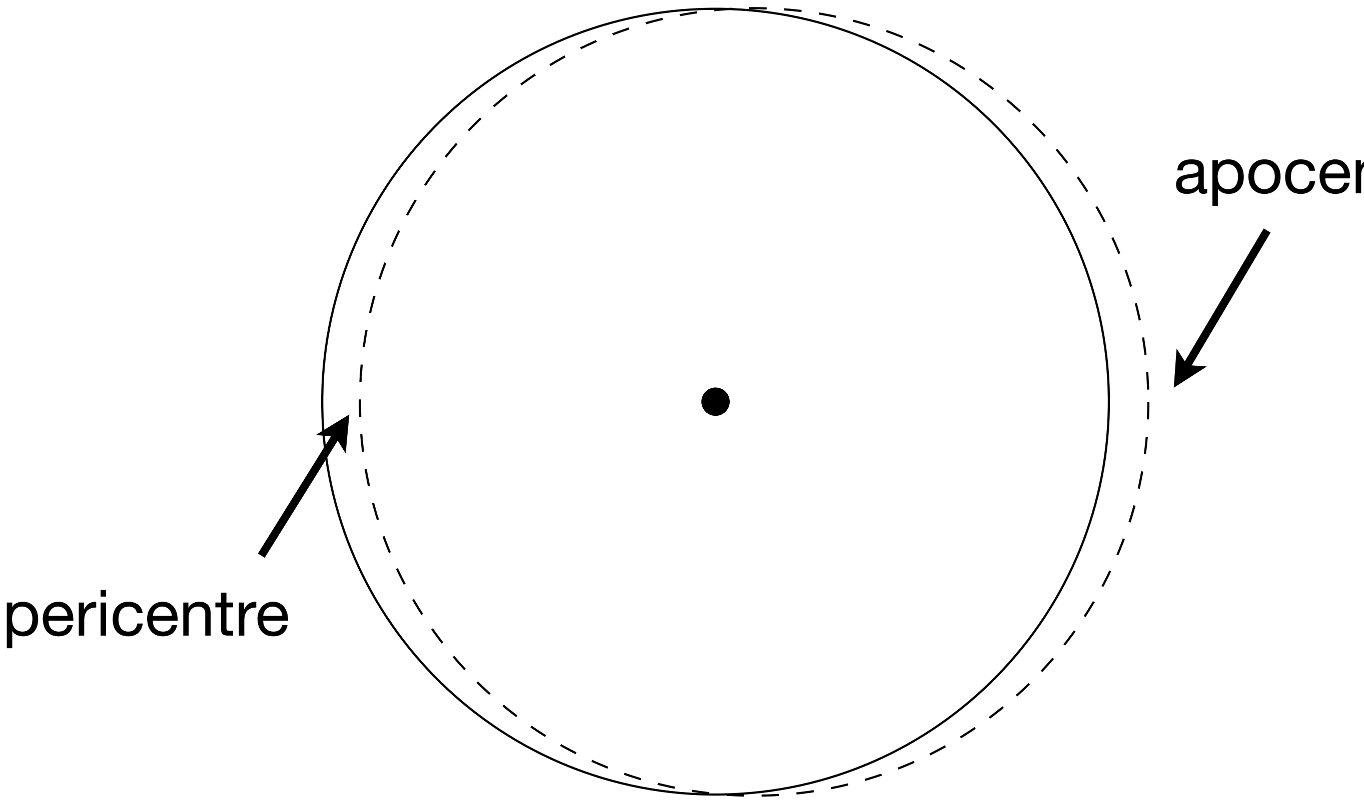
$$= 4\Omega_\circ^2 + 2r\Omega_\circ\frac{\mathrm{d}\Omega_\circ}{\mathrm{d}r}$$

$$\Omega_z^2 = \Phi_{,zz}(r,0)$$

Orbital dynamics

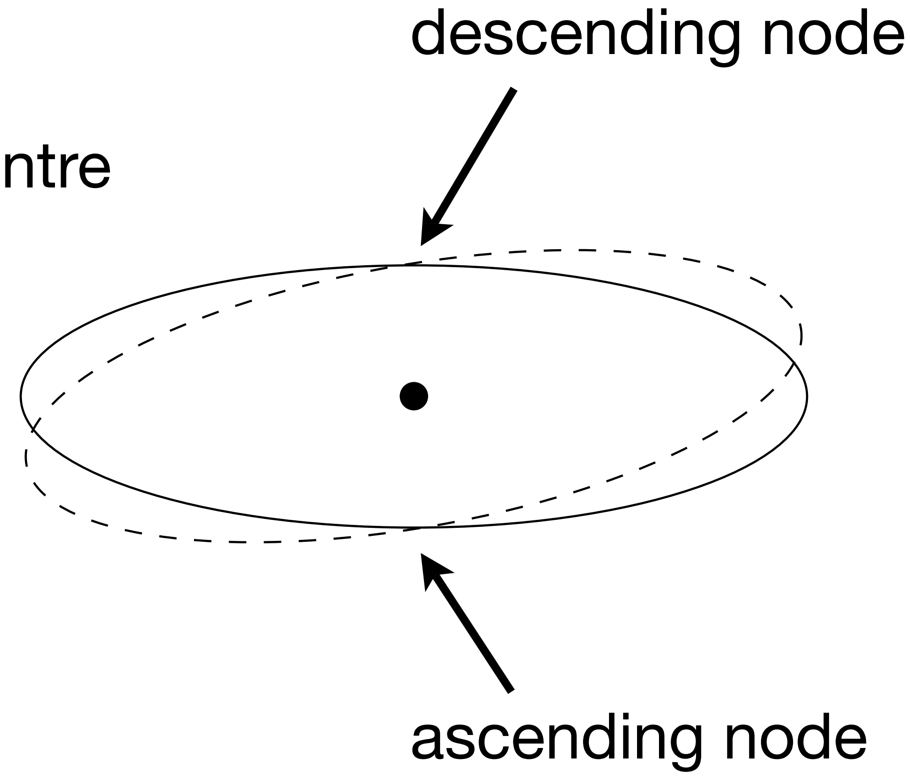
- Keplerian case

$$\kappa = \Omega_z = \Omega$$



$$\kappa = \Omega$$

eccentric Keplerian orbit



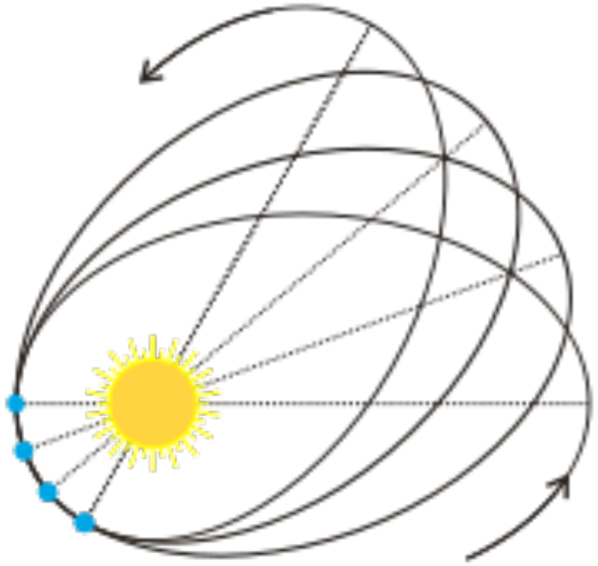
$$\Omega_z = \Omega$$

inclined Keplerian orbit

Orbital dynamics

- Precession
  - If $\kappa \approx \Omega$ and/or $\Omega_z \approx \Omega$, describe as slowly precessing orbit
  - Minimum r (pericentre) occurs at time intervals $\Delta t = \dfrac{2\pi}{\kappa}$

$$\begin{aligned}\Delta\phi &= \frac{2\pi\Omega}{\kappa} \\ &= 2\pi\left(\frac{\Omega}{\kappa} - 1\right) + 2\pi \\ &= 2\pi\left(\frac{\Omega}{\kappa} - 1\right) \mod 2\pi\end{aligned}$$



  - Apsidal precession rate $\dfrac{\Delta\phi}{\Delta t} = \Omega - \kappa$
  - Similarly, nodal precession rate $= \Omega - \Omega_z$

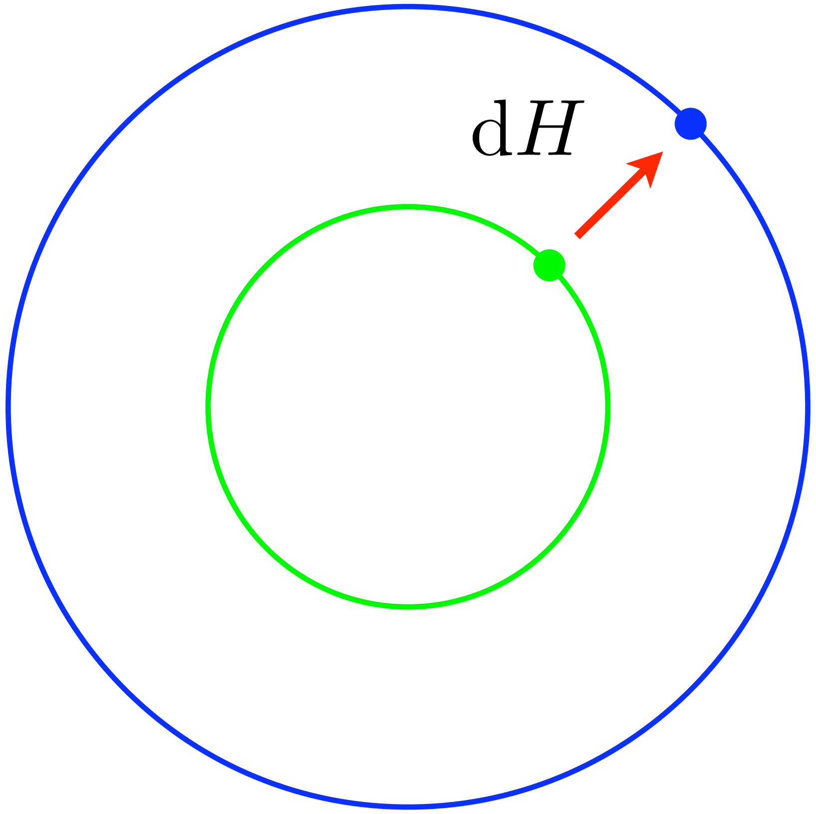
Mechanics of accretion

- Consider two particles in circular orbits in the midplane
- Can energy be released by an exchange of angular momentum?
- Total energy and angular momentum:

$$E = E_1 + E_2 = m_1\varepsilon_1 + m_2\varepsilon_2$$

$$H = H_1 + H_2 = m_1h_1 + m_2h_2$$

- In an infinitesimal exchange:



$$\mathrm{d}E = \mathrm{d}E_1 + \mathrm{d}E_2 = m_1\Omega_1\,\mathrm{d}h_1 + m_2\Omega_2\,\mathrm{d}h_2$$

$$\mathrm{d}H = \mathrm{d}H_1 + \mathrm{d}H_2 = m_1\,\mathrm{d}h_1 + m_2\,\mathrm{d}h_2$$

- If $\mathrm{d}H = 0$ then

$$\mathrm{d}E = (\Omega_1 - \Omega_2)\,\mathrm{d}H_1$$

- In practice $\mathrm{d}\Omega/\mathrm{d}r < 0$
- Energy released by transferring angular momentum outwards

Mechanics of accretion

- Generalize argument to allow for exchange of mass:

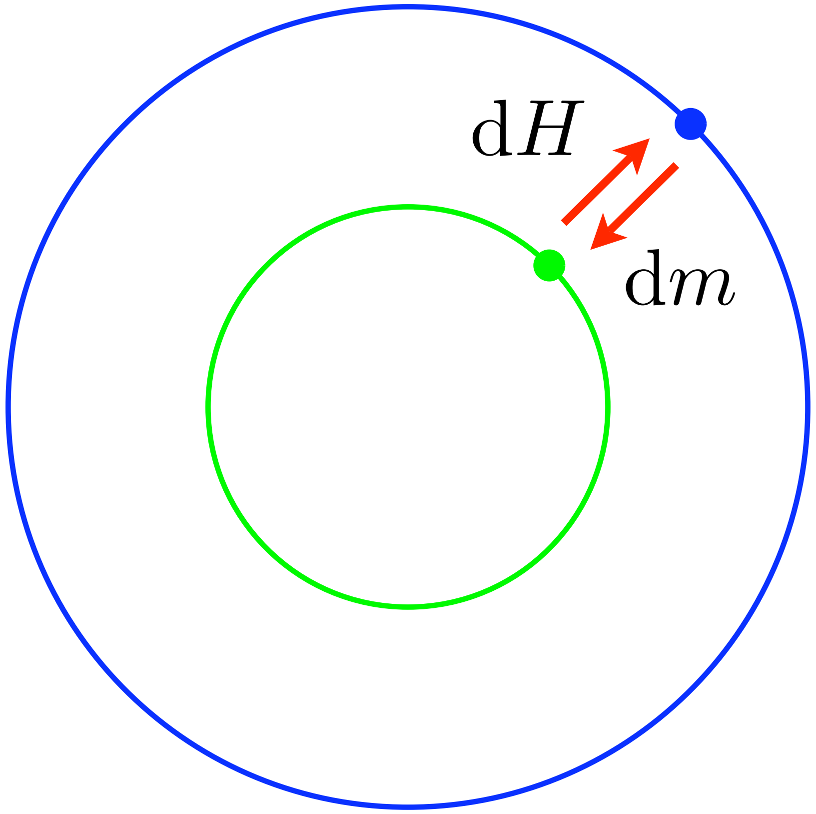
$$\mathrm{d}M = \mathrm{d}m_1 + \mathrm{d}m_2 = 0$$

$$\mathrm{d}H = \mathrm{d}H_1 + \mathrm{d}H_2 = 0 \qquad \mathrm{d}H_i = m_i\,\mathrm{d}h_i + h_i\,\mathrm{d}m_i$$

$$\begin{aligned}\mathrm{d}E_i &= m_i\Omega_i\,\mathrm{d}h_i + \varepsilon_i\,\mathrm{d}m_i \\ &= \Omega_i\,\mathrm{d}H_i + (\varepsilon_i - h_i\Omega_i)\,\mathrm{d}m_i\end{aligned}$$

$$\mathrm{d}E = (\Omega_1 - \Omega_2)\,\mathrm{d}H_1 + [(\varepsilon_1 - h_1\Omega_1) - (\varepsilon_2 - h_2\Omega_2)]\mathrm{d}m_1$$

- In practice $\mathrm{d}(\varepsilon - h\Omega)/\mathrm{d}r = -h\,\mathrm{d}\Omega/\mathrm{d}r > 0$
- Energy released by transferring angular momentum outwards and mass inwards
- This is the physical basis of an accretion disc



Equations of astrophysical fluid dynamics

- Astrophysical fluid dynamics (AFD):
- Basic model: Newtonian gas dynamics:

$$\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u} \cdot \boldsymbol{\nabla} \boldsymbol{u} = -\boldsymbol{\nabla} \Phi - \frac{1}{\rho} \boldsymbol{\nabla} p$$

$$\frac{\partial \rho}{\partial t} + \boldsymbol{u} \cdot \boldsymbol{\nabla} \rho = -\rho \boldsymbol{\nabla} \cdot \boldsymbol{u}$$

$$\frac{\partial p}{\partial t} + \boldsymbol{u} \cdot \boldsymbol{\nabla} p = -\gamma p \boldsymbol{\nabla} \cdot \boldsymbol{u}$$

$\boldsymbol{u}$	velocity
Φ	gravitational potential
ρ	density
p	pressure
γ	adiabatic exponent

  - Compressible
  - Ideal (inviscid, adiabatic)
  - Non-relativistic (Galilean-invariant)
  - Lagrangian (material) derivative:

$$\frac{\mathrm{D}}{\mathrm{D}t} = \frac{\partial}{\partial t} + \boldsymbol{u} \cdot \boldsymbol{\nabla}$$

Equations of astrophysical fluid dynamics

- Gravity:
  - Non-self-gravitating fluid:
    - Φ is prescribed (fixed / external potential)
  - Self-gravitating fluid:
    - Φ is determined (in part) from the density of the fluid:

$$\nabla^2\Phi = 4\pi G\rho$$

Equations of astrophysical fluid dynamics

- Extensions of the basic model:
  - Viscosity:
    - Usually extremely small
    - May be needed to provide small-scale dissipation
    - May be introduced to model turbulent transport

$$\frac{\partial \boldsymbol{u}}{\partial t} = \cdots + \frac{1}{\rho}\boldsymbol{\nabla}\cdot\mathbf{T}$$

$$\mathbf{T} = 2\mu\mathbf{S} + \mu_{\mathrm{b}}(\boldsymbol{\nabla}\cdot\boldsymbol{u})\mathbf{I}$$

$$\mathbf{S} = \frac{1}{2}\left[\boldsymbol{\nabla}\boldsymbol{u} + (\boldsymbol{\nabla}\boldsymbol{u})^{\mathrm{T}}\right] - \frac{1}{3}(\boldsymbol{\nabla}\cdot\boldsymbol{u})\mathbf{I}$$

$\mathbf{T}$	viscous stress tensor
μ	(shear) viscosity
μ_{b}	bulk viscosity
$\mathbf{S}$	shear tensor
$\mathbf{I}$	unit tensor

kinematic viscosity $\nu = \mu/\rho$

Equations of astrophysical fluid dynamics

- Non-adiabatic effects:
  - Thermal energy equation:

$$\rho T \frac{\mathrm{D}s}{\mathrm{D}t} = \mathcal{H} - \mathcal{C}$$

T	temperature
s	specific entropy
$\mathcal{H}$	heating / unit volume
$\mathcal{C}$	cooling / unit volume (non-adiabatic effects)

  - Heating:
    - Viscous:

$$\mathcal{H} = \mathbf{T} : \boldsymbol{\nabla}\boldsymbol{u} = 2\mu\mathbf{S}^2 + \mu_{\mathrm{b}}(\boldsymbol{\nabla}\cdot\boldsymbol{u})^2$$

  - Cooling:
    - Radiative: $\mathcal{C} = \boldsymbol{\nabla}\cdot\boldsymbol{F}$
    - Diffusion approximation: (optically thick regions)

$$\boldsymbol{F} = -\frac{16\sigma T^3}{3\kappa\rho}\boldsymbol{\nabla}T$$

σ	Stefan-Boltzmann constant
κ	opacity (Rosseland mean)

Equations of astrophysical fluid dynamics

- Equation of state:

$$p = p(\rho, T)$$

  - Ideal gas with radiation:

$$p = p_\mathrm{g} + p_\mathrm{r} = \frac{k\rho T}{\mu_\mathrm{m} m_\mathrm{p}} + \frac{4\sigma T^4}{3c}$$

  - p_r important at very high T
  - $\mu_\mathrm{m} = 0.5$ for fully ionized H, $\mu_\mathrm{m} = 2$ for molecular H, etc.

k	Boltzmann constant
μ_m	mean molecular weight
m_p	proton mass
c	speed of light

- Thermal energy equation in dynamical variables:

$$\rho T \, \mathrm{d}s = \left(\frac{1}{\gamma_3 - 1}\right)\left(\mathrm{d}p - \frac{\gamma_1 p}{\rho}\mathrm{d}\rho\right)$$

$$\Rightarrow \left(\frac{1}{\gamma_3 - 1}\right)\left(\frac{\mathrm{D}p}{\mathrm{D}t} - \frac{\gamma_1 p}{\rho}\frac{\mathrm{D}\rho}{\mathrm{D}t}\right) = \mathcal{H} - \mathcal{C}$$

- For ideal gas of constant ratio of specific heats, $\gamma_1 = \gamma_2 = \gamma_3 = \gamma$

Equations of astrophysical fluid dynamics

- Extensions of the basic model:
  - Magnetohydrodynamics (MHD)
  - Radiation hydrodynamics (RHD)
  - Relativistic formulations
  - Kinetic theory / plasma physics
- Simplifications of the basic model:
  - Incompressible fluid: $\boldsymbol{\nabla} \cdot \boldsymbol{u} = 0$
  - Boussinesq / anelastic approximations
  - Barotropic fluid: $p = p(\rho)$

Equations of astrophysical fluid dynamics

- Magnetohydrodynamics (MHD):
- Electrically conducting fluid (plasma, metal, weakly ionized gas)
- Pre-Maxwell equations (without displacement current):

$$\frac{\partial \boldsymbol{B}}{\partial t} = -\boldsymbol{\nabla} \times \boldsymbol{E}$$

$$\boldsymbol{\nabla} \cdot \boldsymbol{B} = 0$$

solenoidal constraint

$$\boldsymbol{\nabla} \times \boldsymbol{B} = \mu_0 \boldsymbol{J}$$

($\boldsymbol{\nabla} \cdot \boldsymbol{E}$ equation not required)

$\boldsymbol{B}$	magnetic field
$\boldsymbol{E}$	electric field
$\boldsymbol{J}$	electric current density
μ_0	permeability of free space

- Galilean invariance:

$$\boldsymbol{x}' = \boldsymbol{x} - \boldsymbol{v}t \qquad \boldsymbol{B}' = \boldsymbol{B}$$

$$t' = t \qquad \boldsymbol{E}' = \boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B}$$

$$\boldsymbol{J}' = \boldsymbol{J}$$

Equations of astrophysical fluid dynamics

- Ohm's law:

$$\boldsymbol{J}' = \sigma \boldsymbol{E}'$$ in rest frame of conductor

σ : electrical conductivity

$$\Rightarrow \quad \boldsymbol{J} = \sigma(\boldsymbol{E} + \boldsymbol{u} \times \boldsymbol{B})$$ for conducting fluid with velocity $\boldsymbol{u}(\boldsymbol{x}, t)$

- Combine with Maxwell:

$$\begin{aligned}\frac{\partial \boldsymbol{B}}{\partial t} &= -\boldsymbol{\nabla} \times \boldsymbol{E} \\ &= \boldsymbol{\nabla} \times (\boldsymbol{u} \times \boldsymbol{B}) - \boldsymbol{\nabla} \times \left(\frac{\boldsymbol{J}}{\sigma}\right) \\ &= \boldsymbol{\nabla} \times (\boldsymbol{u} \times \boldsymbol{B}) - \boldsymbol{\nabla} \times (\eta \boldsymbol{\nabla} \times \boldsymbol{B}) \\ &= \boldsymbol{\nabla} \times (\boldsymbol{u} \times \boldsymbol{B}) + \eta \nabla^2 \boldsymbol{B} \qquad \text{if } \eta \text{ uniform}\end{aligned}$$

magnetic diffusivity

$$\eta = \frac{1}{\mu_0 \sigma}$$

$\propto$ resistivity

- "Induction equation": vector advection-diffusion equation

cf. vorticity equation $\dfrac{\partial \boldsymbol{\omega}}{\partial t} = \boldsymbol{\nabla} \times (\boldsymbol{u} \times \boldsymbol{\omega}) + \nu \nabla^2 \boldsymbol{\omega}$ for $\boldsymbol{\omega} = \boldsymbol{\nabla} \times \boldsymbol{u}$

Equations of astrophysical fluid dynamics

- Ideal MHD (perfect conductor: $\sigma \to \infty$, $\eta \to 0$):

$$\begin{aligned}\frac{\partial \boldsymbol{B}}{\partial t} &= \boldsymbol{\nabla} \times (\boldsymbol{u} \times \boldsymbol{B}) \\ &= \boldsymbol{B} \cdot \boldsymbol{\nabla} \boldsymbol{u} - \boldsymbol{u} \cdot \boldsymbol{\nabla} \boldsymbol{B} - \boldsymbol{B}(\boldsymbol{\nabla} \cdot \boldsymbol{u}) + \cancel{\boldsymbol{u}(\boldsymbol{\nabla} \cdot \boldsymbol{B})}\end{aligned}$$

- Magnetic field is “frozen in” to fluid:
  - Field lines behave as material lines
  - Magnetic flux through an open material surface is conserved
- Valid for large magnetic Reynolds number

$$\mathrm{Rm} = \frac{LU}{\eta} \qquad \text{cf.} \quad \mathrm{Re} = \frac{LU}{\nu} \quad \text{(advection versus diffusion)}$$

- Much easier to achieve on astrophysical scales

Equations of astrophysical fluid dynamics

- Lorentz force per unit volume

$$\begin{aligned}\mathbf{F}_\mathrm{m} &= \boldsymbol{J} \times \boldsymbol{B} \\ &= \frac{1}{\mu_0}(\boldsymbol{\nabla} \times \boldsymbol{B}) \times \boldsymbol{B} \\ &= \frac{1}{\mu_0}\boldsymbol{B} \cdot \boldsymbol{\nabla}\boldsymbol{B} - \boldsymbol{\nabla}\left(\frac{|\boldsymbol{B}|^2}{2\mu_0}\right)\end{aligned}$$

curvature force:
magnetic tension

gradient of
magnetic pressure

$$T_\mathrm{m} = \frac{|\boldsymbol{B}|^2}{\mu_0} \qquad p_\mathrm{m} = \frac{|\boldsymbol{B}|^2}{2\mu_0}$$

(= magnetic energy density)

$$\mathbf{F}_\mathrm{m} = \boldsymbol{\nabla} \cdot \mathrm{\mathbf{M}}$$

$$\mathbf{M} = \frac{\boldsymbol{B}\boldsymbol{B}}{\mu_0} - \frac{|\boldsymbol{B}|^2}{2\mu_0}\mathbf{I}$$

Maxwell stress tensor

If $\boldsymbol{B} = B\,\boldsymbol{e}_z$,

$$\mathbf{M} = \begin{pmatrix} -p_\mathrm{m} & 0 & 0 \\ 0 & -p_\mathrm{m} & 0 \\ 0 & 0 & T_\mathrm{m} - p_\mathrm{m} \end{pmatrix}$$

Equations of astrophysical fluid dynamics

- Lorentz force:
    - Magnetic tension + frozen-in field → Alfvén waves

$$v_{\mathrm{a}} = \left(\frac{T_{\mathrm{m}}}{\rho}\right)^{1/2}$$ cf. elastic string

$$\boldsymbol{v}_{\mathrm{a}} = (\mu_0\rho)^{-1/2}\boldsymbol{B}$$ vector Alfvén velocity

    - Magnetic pressure → magnetoacoustic waves

Equations of astrophysical fluid dynamics

- Ideal MHD equations:

$$\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u} \cdot \boldsymbol{\nabla} \boldsymbol{u} = -\boldsymbol{\nabla}\Phi - \frac{1}{\rho}\boldsymbol{\nabla} p + \frac{1}{\mu_0 \rho}(\boldsymbol{\nabla} \times \boldsymbol{B}) \times \boldsymbol{B}$$

$$\frac{\partial \rho}{\partial t} + \boldsymbol{u} \cdot \boldsymbol{\nabla} \rho = -\rho \boldsymbol{\nabla} \cdot \boldsymbol{u}$$

$$\frac{\partial p}{\partial t} + \boldsymbol{u} \cdot \boldsymbol{\nabla} p = -\gamma p \boldsymbol{\nabla} \cdot \boldsymbol{u}$$

$$\frac{\partial \boldsymbol{B}}{\partial t} = \boldsymbol{\nabla} \times (\boldsymbol{u} \times \boldsymbol{B})$$

$$\boldsymbol{\nabla} \cdot \boldsymbol{B} = 0$$

- Or can expand out $\times\times$
- $\boldsymbol{E}$ and $\boldsymbol{J}$ eliminated
- Nonlinearities in equation of motion and induction equation

Equations of astrophysical fluid dynamics

- Total energy equation in ideal MHD:

$$\frac{\partial}{\partial t}\left[\rho\left(\tfrac{1}{2}|\boldsymbol{u}|^2+\Phi+\underset{\substack{\big\uparrow \\ \text{specific} \\ \text{internal} \\ \text{energy}}}{e}\right)+\frac{|\boldsymbol{B}|^2}{2\mu_0}\right]+\boldsymbol{\nabla}\cdot\left[\rho\boldsymbol{u}\left(\tfrac{1}{2}|\boldsymbol{u}|^2+\Phi+\underset{\substack{\big\uparrow \\ \text{specific} \\ \text{enthalpy}}}{w}\right)+\underset{\substack{\big\uparrow \\ \text{Poynting} \\ \text{flux}}}{\frac{\boldsymbol{E}\times\boldsymbol{B}}{\mu_0}}\right]=0$$

$$\boldsymbol{E}=-\boldsymbol{u}\times\boldsymbol{B}$$

- For ideal gas of constant γ :

$$e=\frac{p}{(\gamma-1)\rho}$$

$$w=e+\frac{p}{\rho}=\frac{\gamma p}{(\gamma-1)\rho}$$

- With self-gravity, $\Phi = \Phi_{\mathrm{int}}+\Phi_{\mathrm{ext}}$ and only $\frac{1}{2}\Phi_{\mathrm{int}}+\Phi_{\mathrm{ext}}$ contributes to the energy density

Equations of astrophysical fluid dynamics

- Forces as the divergences of a stress tensor:
- Equation of motion can be written

$$\rho\frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t}=-\rho\boldsymbol{\nabla}\Phi+\boldsymbol{\nabla}\cdot\mathbf{T}$$

- Related to conservative form for momentum:

$$\frac{\partial}{\partial t}(\rho\boldsymbol{u})+\boldsymbol{\nabla}\cdot(\rho\boldsymbol{u}\boldsymbol{u}-\mathbf{T})=-\rho\boldsymbol{\nabla}\Phi$$

- Contributions to stress tensor $\mathbf{T}$:
  - pressure $-p\,\mathbf{I}$
  - viscous $2\mu\mathbf{S}+\mu_{\mathrm{b}}(\boldsymbol{\nabla}\cdot\boldsymbol{u})\mathbf{I}$
  - self-gravity $-\frac{\boldsymbol{g}\boldsymbol{g}}{4\pi G}+\frac{|\boldsymbol{g}|^2}{8\pi G}\mathbf{I}$ (check using Poisson's equation)

    $\nabla^2\Phi=4\,\pi G\rho \qquad \boldsymbol{g}=-\boldsymbol{\nabla}\Phi$
  - magnetic $\frac{\boldsymbol{B}\boldsymbol{B}}{\mu_0}-\frac{|\boldsymbol{B}|^2}{2\mu_0}\mathbf{I}$
  - also turbulent stresses from correlations of fluctuating fields

Evolution of an accretion disc

- Regulated by conservation of mass and angular momentum
- Mass conservation in 3D:

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot (\rho \boldsymbol{u}) = 0$$

- Momentum conservation in 3D:

$$\frac{\partial}{\partial t}(\rho \boldsymbol{u}) + \boldsymbol{\nabla} \cdot (\rho \boldsymbol{u} \boldsymbol{u} - \mathbf{T}) = -\rho \boldsymbol{\nabla} \Phi$$

- Φ is axisymmetric gravitational potential we considered for orbits (later, allow non-axisymmetric potential and tidal torque)
- $\mathbf{T}$ represents collective effects, including self-gravity

Evolution of an accretion disc

- Write in cylindrical polar coordinates (r, ϕ, z)
- Reduce from 3D to 1D
- Mass conservation:

$$\frac{\partial \rho}{\partial t} + \frac{1}{r}\frac{\partial}{\partial r}(r\rho u_r) + \frac{1}{r}\frac{\partial}{\partial \phi}(\rho u_\phi) + \frac{\partial}{\partial z}(\rho u_z) = 0$$

- Integrate $r \int_{-\infty}^{\infty} \int_0^{2\pi} \cdot \, \mathrm{d}\phi \, \mathrm{d}z$ over cylinder of radius r

$$\Rightarrow \quad \frac{\partial \mathcal{M}}{\partial t} + \frac{\partial \mathcal{F}}{\partial r} = 0$$ assuming no vertical mass loss or gain

- 1D mass density $\mathcal{M}(r,t) = \int_{-\infty}^{\infty} \int_0^{2\pi} \rho \, r \, \mathrm{d}\phi \, \mathrm{d}z$
- Radial mass flux $\mathcal{F}(r,t) = \int_{-\infty}^{\infty} \int_0^{2\pi} \rho u_r \, r \, \mathrm{d}\phi \, \mathrm{d}z$

Evolution of an accretion disc

- Angular momentum conservation: (r,ϕ,z)

$$\frac{\partial}{\partial t}(\rho u_\phi)+\frac{1}{r^2}\frac{\partial}{\partial r}[r^2(\rho u_r u_\phi-T_{r\phi})]+\frac{1}{r}\frac{\partial}{\partial\phi}(\rho u_\phi^2-T_{\phi\phi})+\frac{\partial}{\partial z}(\rho u_\phi u_z-T_{\phi z})=0$$

- Integrate $r^2\int_{-\infty}^{\infty}\int_0^{2\pi}\cdot\,\mathrm{d}\phi\,\mathrm{d}z$ over cylinder of radius r
- Assume that $ru_\phi=h(r)$ from orbital dynamics (examine this assumption later)

$$\Rightarrow\quad\frac{\partial}{\partial t}(\mathcal{M}h)+\frac{\partial}{\partial r}(\mathcal{F}h+\mathcal{G})=0$$

assuming no vertical loss or gain

- Internal torque $\mathcal{G}(r,t)=-\int_{-\infty}^{\infty}\int_0^{2\pi}r^2T_{r\phi}\,\mathrm{d}\phi\,\mathrm{d}z$

Evolution of an accretion disc

- Since h depends only on r :

$$\frac{\partial \mathcal{M}}{\partial t} + \frac{\partial \mathcal{F}}{\partial r} = 0$$

$$\frac{\partial \mathcal{M}}{\partial t} h + \frac{\partial}{\partial r}(\mathcal{F}h + \mathcal{G}) = 0$$

- Eliminate $\mathcal{M}$:

$$\mathcal{F}\frac{\mathrm{d}h}{\mathrm{d}r} + \frac{\partial \mathcal{G}}{\partial r} = 0 \qquad \text{which determines } \mathcal{F}$$

$$\frac{\partial \mathcal{M}}{\partial t} = \frac{\partial}{\partial r}\left[\left(\frac{\mathrm{d}h}{\mathrm{d}r}\right)^{-1} \frac{\partial \mathcal{G}}{\partial r}\right]$$

- Interpretation:
  - angular momentum determines orbital radius
  - angular momentum transport determines mass evolution

Evolution of an accretion disc

- More usual notation:

$$\mathcal{M} = 2\pi r \Sigma$$

$$\mathcal{F} = 2\pi r \Sigma \bar{u}_r$$

$$\mathcal{G} = -2\pi \bar{\nu} \Sigma r^3 \frac{\mathrm{d}\Omega}{\mathrm{d}r}$$

$\Sigma(r,t)$	surface density
$\bar{u}_r(r,t)$	mean radial velocity
$\bar{\nu}(r,t)$	mean effective kinematic viscosity

- Equivalent to:

$$\Sigma = \int_{-\infty}^{\infty} \langle \rho \rangle \, \mathrm{d}z$$

$$\langle \cdot \rangle = \frac{1}{2\pi} \int_0^{2\pi} \cdot \, \mathrm{d}\phi$$

$$\Sigma \bar{u}_r = \int_{-\infty}^{\infty} \langle \rho u_r \rangle \, \mathrm{d}z$$

$$\bar{\nu} \Sigma = \int_{-\infty}^{\infty} \langle \mu \rangle \, \mathrm{d}z$$

$$T_{r\phi} = \mu r \frac{\mathrm{d}\Omega}{\mathrm{d}r}$$

Evolution of an accretion disc

- Then obtain:

$$\frac{\partial \Sigma}{\partial t} = \frac{1}{r}\frac{\partial}{\partial r}\left[\left(\frac{\mathrm{d}h}{\mathrm{d}r}\right)^{-1}\frac{\partial}{\partial r}\left(-\bar{\nu}\Sigma r^3 \frac{\mathrm{d}\Omega}{\mathrm{d}r}\right)\right]$$

- Keplerian disc:

$$\Omega \propto r^{-3/2}$$

$$h = r^2\Omega \propto r^{1/2}$$

$$\frac{\partial \Sigma}{\partial t} = \frac{3}{r}\frac{\partial}{\partial r}\left[r^{1/2}\frac{\partial}{\partial r}(r^{1/2}\bar{\nu}\Sigma)\right]$$

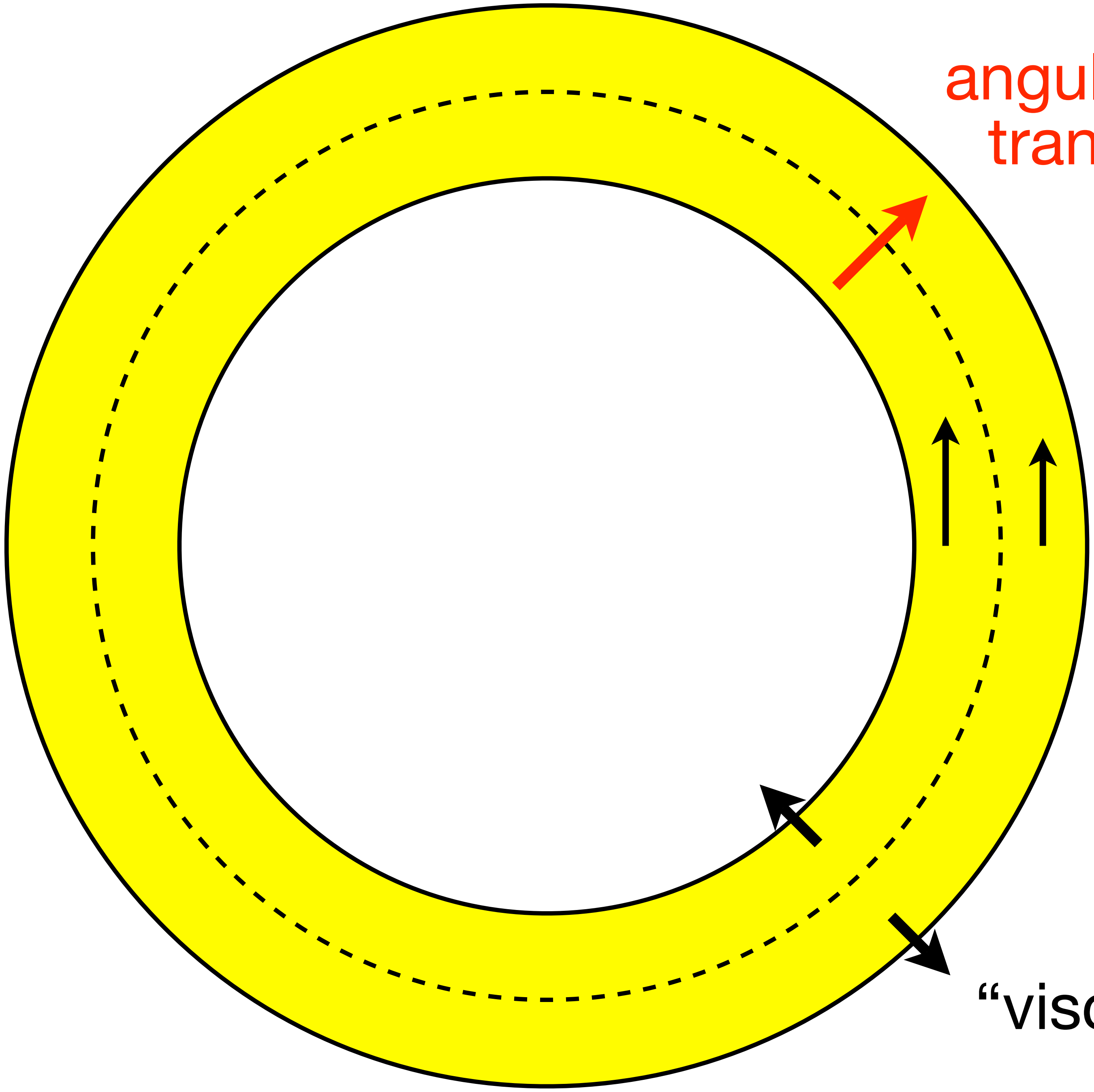
diffusion equation
for surface density

Evolution of an accretion disc

- Interpretation:

$$\frac{\mathrm{d}\Omega}{\mathrm{d}r} < 0$$

$$\frac{\mathrm{d}h}{\mathrm{d}r} > 0$$



angular momentum transport

orbital shear

“viscous” spreading

Analysis of the diffusion equation

- Inner boundary condition:
  - Important because spreading ring will reach $r = 0$ in finite time
  - Depends on nature of central object

- Star with negligible magnetic field:
  - Disc may extend down to stellar surface
  - Star usually rotates at only a fraction of Keplerian rate:

$$\Omega_* < \left(\frac{GM}{R_*^3}\right)^{1/2}$$

(star is supported mainly by pressure)

  - Angular velocity makes a rapid adjustment from Keplerian to stellar in a viscous boundary layer

Analysis of the diffusion equation

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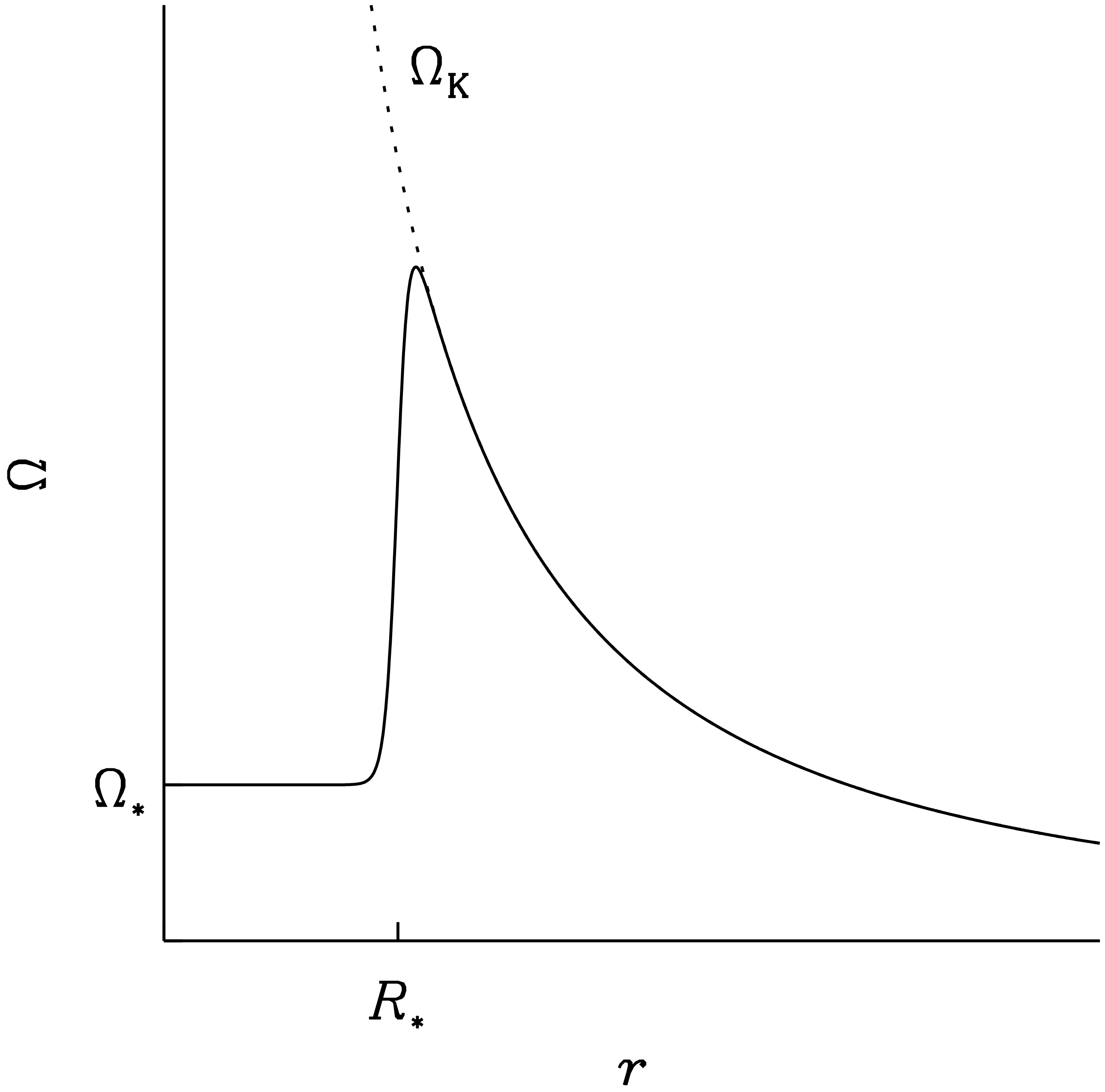
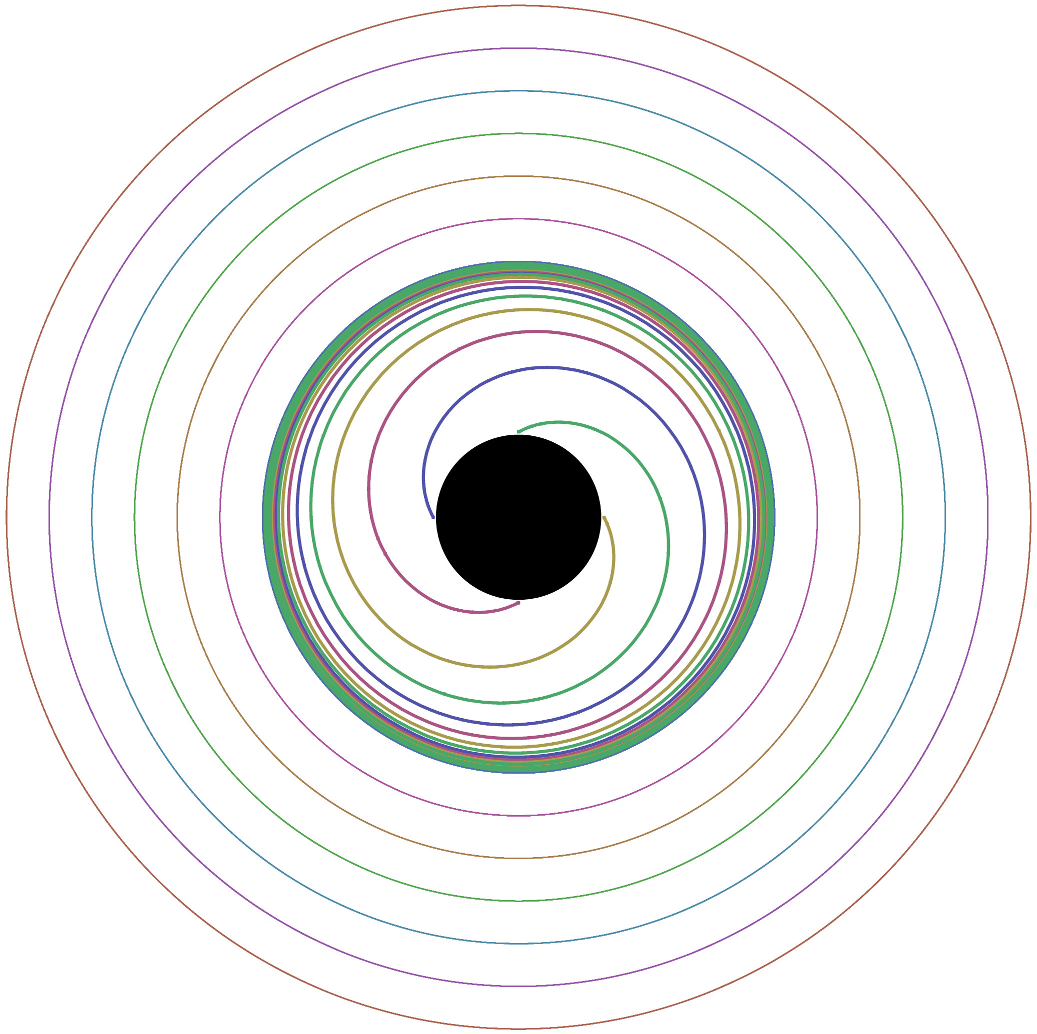


- Usual argument: $\dfrac{\mathrm{d}\Omega}{\mathrm{d}r} = 0$ at $r = r_{\mathrm{in}}\,(\approx R_*)$ so $\mathcal{G} = 0$ there

Analysis of the diffusion equation

- Black hole:
  - Circular orbits are unstable close to event horizon



Non-rotating black hole:

$$\Omega = \left(\frac{GM}{r^3}\right)^{1/2}$$

$$\kappa^2 = \Omega^2 \left(1 - \frac{6\,GM}{c^2 r}\right)$$

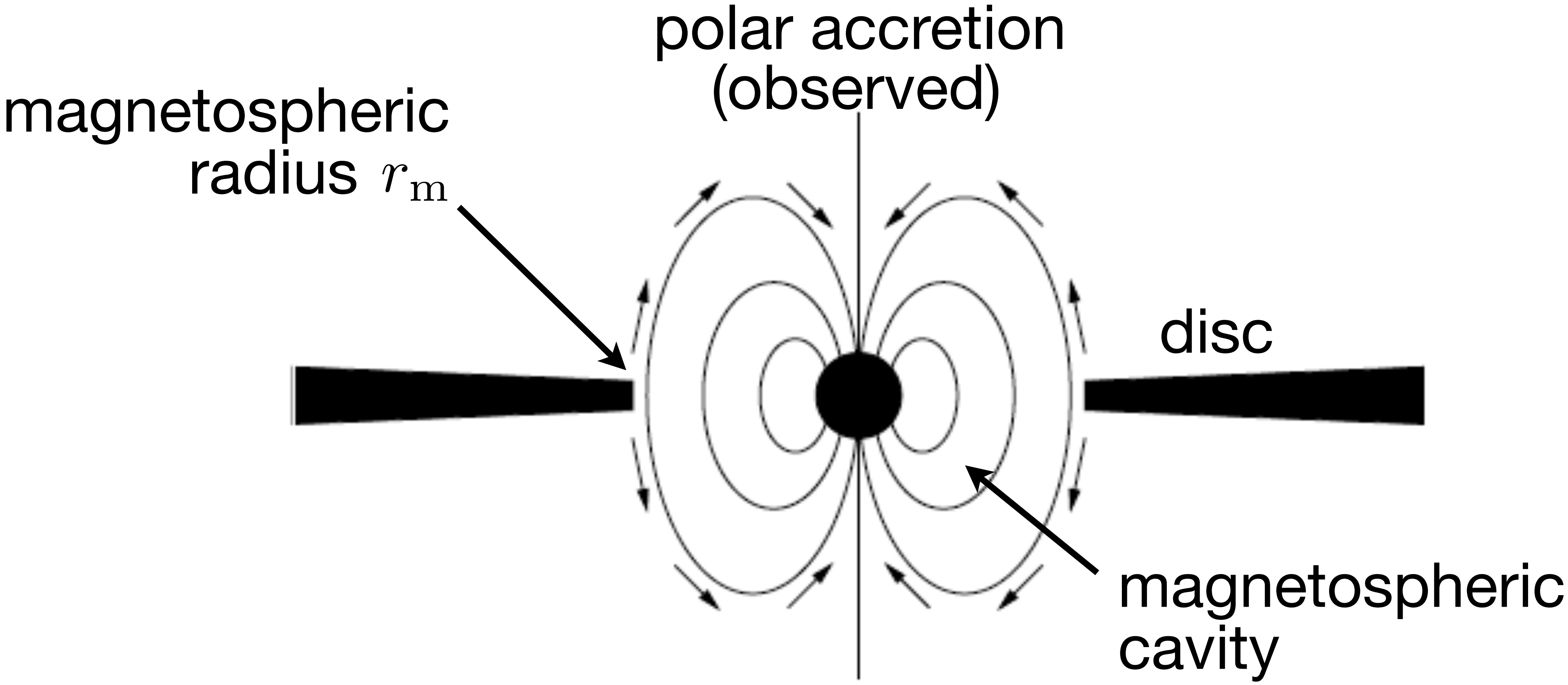
Orbits unstable for

$$r < \frac{6\,GM}{c^2} = 3\,r_\mathrm{S}$$

- Rapid inspiral from r_ms
- $\bar{u}_r \uparrow$ and $\Sigma \downarrow$ rapidly, so effectively $\mathcal{G} \approx 0$ at $r_\mathrm{in} \approx r_\mathrm{ms}$

Analysis of the diffusion equation

- Star with significant magnetic field (e.g. dipole):



- $\mathcal{G}$ not necessarily 0 at $r_{\mathrm{in}} \approx r_{\mathrm{m}}$
- Star can exert torque on disc
- Depends on magnetic coupling (complicated problem)

Analysis of the diffusion equation

- Mathematically, we may let $r_{\mathrm{in}} \to 0$
- To allow mass flux at origin but no torque:

$$\left.\begin{aligned} \mathcal{F} &\to \mathrm{cst} \\ \mathcal{G} &\to 0 \end{aligned}\right\} \quad \text{as} \quad r \to 0$$

- To allow torque at origin:

$$\mathcal{G} \to \mathrm{cst} \qquad \text{as} \quad r \to 0$$

- For a Keplerian disc,

$$\mathcal{F} \propto r\Sigma\bar{u}_r \propto r^{1/2}\frac{\partial}{\partial r}(r^{1/2}\bar{\nu}\Sigma)$$
$$\mathcal{G} \propto r^{1/2}\bar{\nu}\Sigma$$

- First case (no torque): $\bar{\nu}\Sigma \to \mathrm{cst} \quad \text{as} \quad r \to 0$
- Second case (torque): $r^{1/2}\bar{\nu}\Sigma \to \mathrm{cst} \quad \text{as} \quad r \to 0$

Analysis of the diffusion equation

- Consideration of angular momentum transport processes and local vertical structure of disc (later) leads to

$$\bar{\nu} = \bar{\nu}(r, \Sigma) \qquad \text{(e.g. double power law)}$$

- If $\bar{\nu} = \bar{\nu}(r)$ only, diffusion equation is linear
- If $\bar{\nu} = \bar{\nu}(r, \Sigma)$, diffusion equation is nonlinear

- Usually solve as initial-value problem or look at steady solutions
- Linear case can be treated using Green's function $G(r, s, t)$:
- Solution of diffusion equation with initial condition

$$\Sigma(r, 0) = \delta(r - s) \qquad \text{(i.e. very narrow ring)}$$

- Then solution for any initial condition $\Sigma_0(r)$ is

$$\Sigma(r, t) = \int_0^\infty G(r, s, t)\, \Sigma_0(s)\, \mathrm{d}s$$

Analysis of the diffusion equation

- Can calculate $G(r,s,t)$ in terms of Bessel functions for any power law $\bar{\nu} \propto r^p$ and any boundary conditions
- Classic example: (Lynden-Bell & Pringle 1974)

$$\bar{\nu} = \text{cst}$$

$\mathcal{G} = 0$ inner BC at $r = 0$

no outer boundary

modified Bessel function

$$G(r,s,t) = \frac{r^{-1/4} s^{5/4}}{6\bar{\nu}t} \exp\left[-\frac{r^2+s^2}{12\bar{\nu}t}\right] I_{1/4}\left(\frac{rs}{6\bar{\nu}t}\right)$$

(omit derivation)

- Becomes more asymmetrical as t increases
- As $t \to \infty$, all mass accreted by central object and all angular momentum carried to $r = \infty$ by negligible mass

Analysis of the diffusion equation

- Easier example: suppose $\bar{\nu} = Ar$ with $A = \mathrm{const}$:
- Let $y = \left(\frac{4r}{3A}\right)^{1/2}$ and $g = r^{1/2}\bar{\nu}\Sigma = Ar^{3/2}\Sigma$ to obtain

$$\frac{\partial g}{\partial t} = \frac{\partial^2 g}{\partial y^2}$$

classical diffusion equation

- Spreading-ring solution with zero torque ($g = 0$) at centre:

$$g \propto t^{-1/2}\left\{\exp\left[-\frac{(y-y_0)^2}{4t}\right] - \exp\left[-\frac{(y+y_0)^2}{4t}\right]\right\}$$

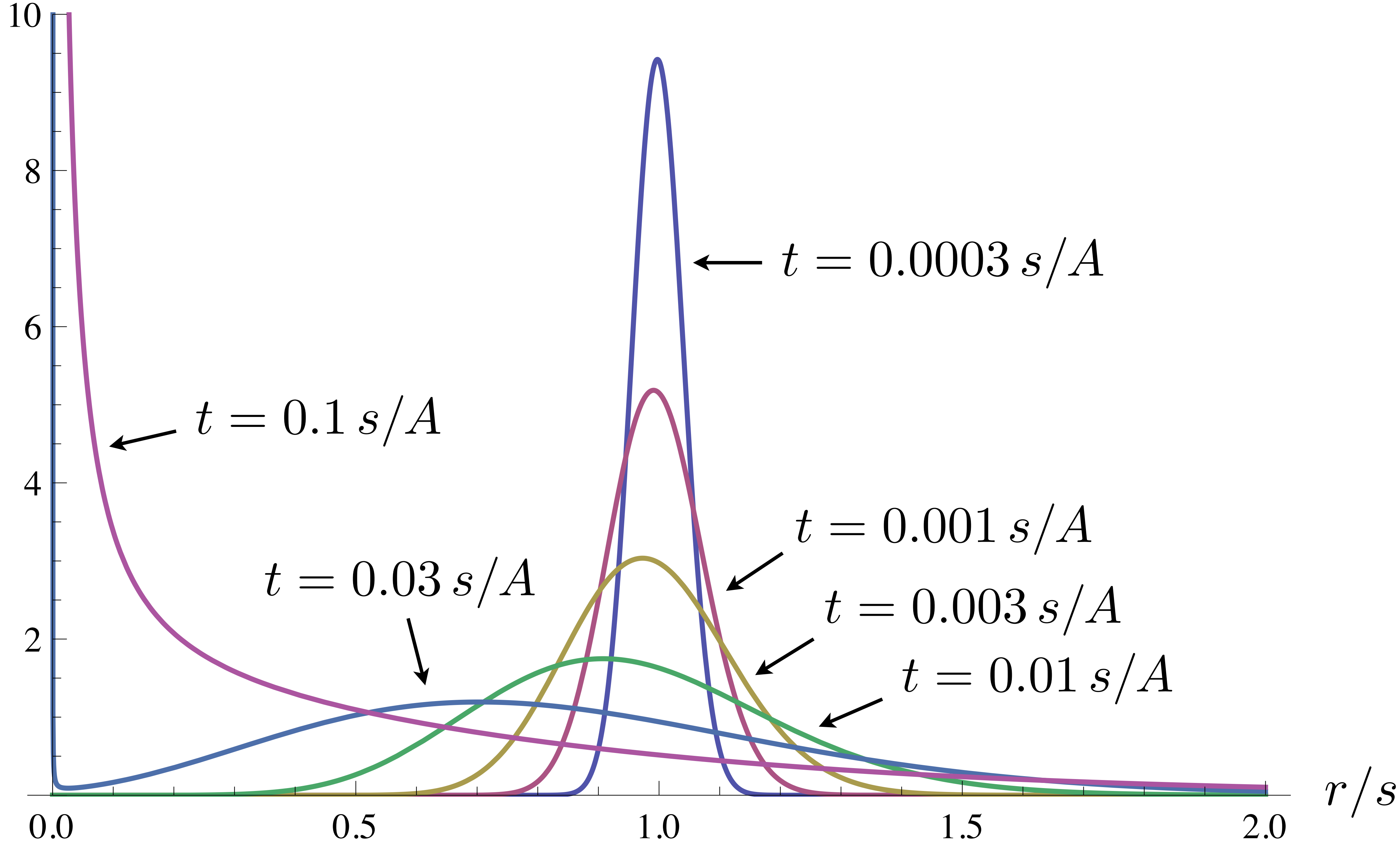
- Thus

$$G(r,s,t) = \frac{r^{-3/2}t^{-1/2}}{(3\pi A)^{1/2}}\exp\left[-\frac{(r+s)}{3At}\right]\sinh\left[\frac{2(rs)^{1/2}}{3At}\right]$$

- Remaining mass $\propto \int_0^\infty G(r,s,t)\,r\,\mathrm{d}r = \mathrm{erf}\left[\left(\frac{s}{3At}\right)^{1/2}\right]$

Analysis of the diffusion equation

$$G(r,s,t) = \frac{r^{-3/2}t^{-1/2}}{(3\pi A)^{1/2}} \exp\left[-\frac{(r+s)}{3At}\right] \sinh\left[\frac{2(rs)^{1/2}}{3At}\right]$$

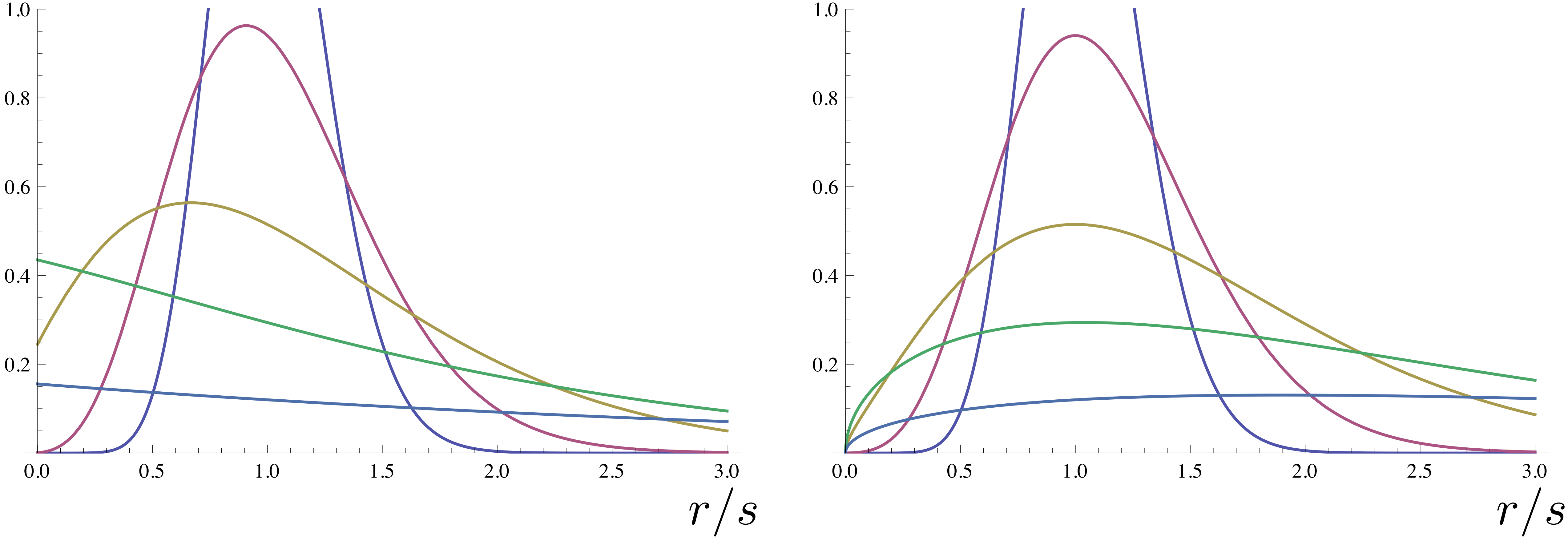


Analysis of the diffusion equation

$$G(r,s,t) = \frac{r^{-3/2}t^{-1/2}}{(3\pi A)^{1/2}} \exp\left[-\frac{(r+s)}{3At}\right] \sinh\left[\frac{2(rs)^{1/2}}{3At}\right]$$

$Gr \propto$ mass / unit radius

$Gr^{3/2} \propto$ ang mom / unit radius



$$t = \{0.01, 0.03, 0.1, 0.3, 1\}\, s/A$$

Analysis of the diffusion equation

- Nonlinear case with $\bar{\nu}\Sigma \propto r^p \Sigma^q$ and $r_{\rm in} \to 0$:
- No intrinsic length-scale
- Special algebraic similarity solutions
- Generally attract solutions of initial-value problem
- But if $q < 0$, viscous instability occurs (see later)

Analysis of the diffusion equation

- Steady discs

$$\mathcal{F} = \mathrm{cst} = -\dot{M}$$

$\dot{M} =$ mass accretion rate
(but always neglect slow increase of M)

$$\mathcal{F}\frac{\mathrm{d}h}{\mathrm{d}r} + \frac{\mathrm{d}\mathcal{G}}{\mathrm{d}r} = 0 \qquad \Rightarrow \qquad -\dot{M}h + \mathcal{G} = \mathrm{cst}$$

- If $\mathcal{G} = 0$ at $r = r_{\mathrm{in}}$, solution is $\mathcal{G} = \dot{M}(h - h_{\mathrm{in}})$
- In Keplerian case (recall $\mathcal{G} = -2\pi\bar{\nu}\Sigma r^3\frac{\mathrm{d}\Omega}{\mathrm{d}r}$)

$$\bar{\nu}\Sigma = \frac{\dot{M}}{3\pi}\left[1 - \left(\frac{r_{\mathrm{in}}}{r}\right)^{1/2}\right]$$

- If we know $\bar{\nu}(r, \Sigma)$, we can solve for $\Sigma(r)$

Analysis of the diffusion equation

- Other solutions:
  - Non-accreting disc: $\mathcal{F} = 0, \ \mathcal{G} = \mathrm{cst}$
  - Decretion disc: $\mathcal{F} > 0$

  } require torque from central object
- Note that $\mathcal{F} < 0$ requires $\dfrac{\mathrm{d}\mathcal{G}}{\mathrm{d}r} > 0$
- Energy balance in steady disc:
  - Energy dissipation rate per unit volume $= T_{r\phi}\, r\dfrac{\mathrm{d}\Omega}{\mathrm{d}r} = \mu \left(r\dfrac{\mathrm{d}\Omega}{\mathrm{d}r} \right)^2$
  - Energy emission rate per unit area (from each face of disc)

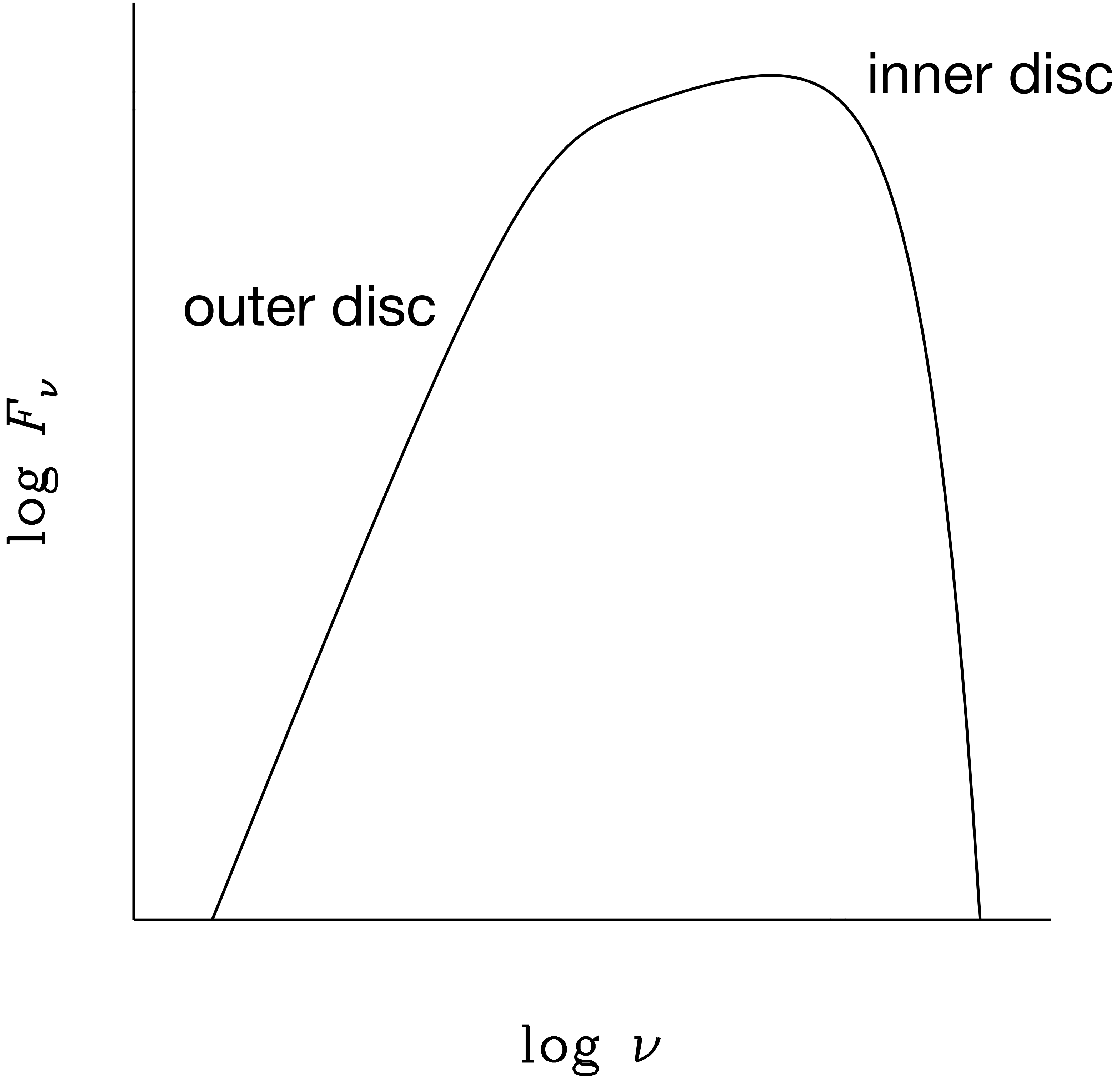
$$= \frac{1}{2}\bar{\nu}\Sigma \left(r\frac{\mathrm{d}\Omega}{\mathrm{d}r} \right)^2$$

  - Effective blackbody temperature $T_{\mathrm{eff}}(r)$ given by

$$\sigma T_{\mathrm{eff}}^4 = \frac{9}{8}\bar{\nu}\Sigma\Omega^2 = \frac{3GM\dot{M}}{8\pi r^3}\left[1 - \left(\frac{r_{\mathrm{in}}}{r}\right)^{1/2}\right]$$

Analysis of the diffusion equation

- Typical (theoretical) spectrum of emitted radiation:



Analysis of the diffusion equation

- Total luminosity of disc $(r_{\rm in} < r < \infty)$

$$L_{\rm disc} = \int_{r_{\rm in}}^{\infty} \frac{3GM\dot{M}}{8\pi r^3} \left[1 - \left(\frac{r_{\rm in}}{r} \right)^{1/2} \right] 2\pi r \, {\rm d}r = \frac{1}{2} \frac{GM\dot{M}}{r_{\rm in}}$$

$=$ rate of release of orbital binding energy $(\infty \to r_{\rm in})$

$= \frac{1}{2}$ rate of release of potential energy

(remaining kinetic energy released in boundary layer if $\Omega_* = 0$)

Vertical disc structure

- Vertical hydrostatic equilibrium
  - Dominant balance (for non-self-gravitating gas disc)

$$0 = -\rho \frac{\partial \Phi}{\partial z} - \frac{\partial p}{\partial z}$$

  - Expand potential

$$\Phi(r, z) = \Phi(r, 0) + \frac{1}{2} \Phi_{zz}(r, 0) z^2 + \cdots$$

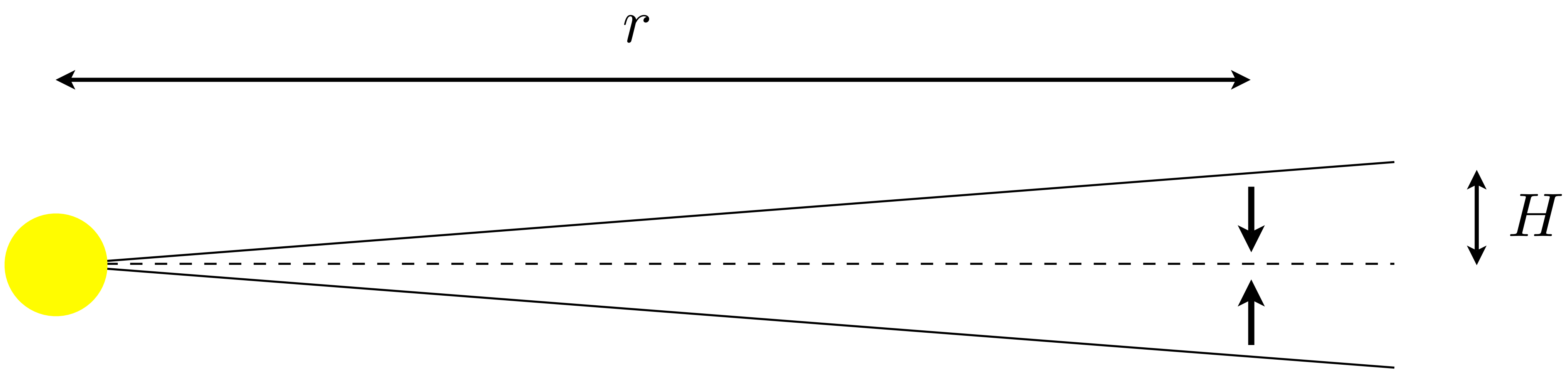
  - So for a thin disc

$$\frac{\partial \Phi}{\partial z} \approx \Omega_z^2 z \qquad (\Omega_z = \Omega \text{ for Keplerian disc})$$

  - Equation of vertical hydrostatic equilibrium

$$\frac{\partial p}{\partial z} \approx -\rho \Omega_z^2 z$$

Vertical disc structure



- Disc supported against gravity by pressure in vertical direction
- Centrifugal support in radial direction

Vertical disc structure

- Order-of-magnitude estimates and time-scales
  - Simple scaling arguments ($\sim$), omitting factors of order unity
  - Important dimensionless parameter:

$$\frac{H}{r} \qquad \text{(angular semi-thickness, aspect ratio)}$$

  - For a thin disc: $\dfrac{H}{r} \ll 1$
  - H may represent: $\left\{ \begin{array}{l} \text{true semi-thickness} \\ \text{height of photosphere above midplane} \\ \text{measure of density scale-height} \end{array} \right.$
  - From vertical hydrostatic equilibrium:

$$\frac{p}{H} \sim \rho \Omega^2 H \quad \Rightarrow \quad c_\mathrm{s} \sim \Omega H$$

$$c_\mathrm{s} = \left(\frac{p}{\rho} \right)^{1/2}$$

isothermal
sound speed

Vertical disc structure

- (Effective) viscosity:
  - Dimensional argument: $[\mu] = ML^{-1}T^{-1} = \left[\frac{p}{\Omega}\right]$
  - Write $\mu = \frac{\alpha p}{\Omega}$ in terms of dimensionless viscosity α ("alpha viscosity prescription", Shakura & Sunyaev 1973)
  - (Mean) kinematic viscosity $\bar{\nu} \sim \frac{\mu}{\rho} \sim \frac{\alpha c_{\mathrm{s}}^2}{\Omega} \sim \alpha c_{\mathrm{s}} H$
  - Analogous to kinetic theory of molecular transport: $\nu \sim v\ell$ for mean speed v and mean free path ℓ
  - Molecular viscosity usually negligible for astrophysical discs
  - Similar estimate applies to effective "eddy viscosity" of turbulence of mean turbulent speed v and correlation length ℓ
  - Assuming that $v \lesssim c_{\mathrm{s}}$ (subsonic turbulence) and $\ell \lesssim H$, expect that $\alpha \lesssim 1$

Vertical disc structure

- Stress in a Keplerian disc: $T_{r\phi} = \mu r \frac{\mathrm{d}\Omega}{\mathrm{d}r} = -\frac{3}{2}\alpha p$
- Alternative version of alpha prescription: $T_{r\phi} = -\alpha p$
- Whatever process gives rise to stress, scales with local pressure
- Reasonable assumption for local processes (see later)

Vertical disc structure

- Three important characteristic time-scales:
  - Dynamical time-scale:

$$t_{\rm dyn} \sim \frac{1}{\Omega} \quad \text{(orbital motion)} \qquad \sim \frac{H}{c_{\rm s}} \quad \text{(hydrostatic)}$$

  - Viscous time-scale:

$$t_{\rm visc} \sim \frac{r^2}{\bar{\nu}} \sim \alpha^{-1} \left(\frac{H}{r}\right)^{-2} t_{\rm dyn} \qquad \text{(radial motion)}$$

  - Thermal time-scale:

$$t_{\rm th} \sim \frac{pH}{\bar{\nu}\Sigma\Omega^2} \sim \frac{c_{\rm s}^2}{\bar{\nu}\Omega^2} \sim \frac{H^2}{\bar{\nu}} \sim \alpha^{-1} t_{\rm dyn} \qquad \text{(thermal balance)}$$

  - For a thin disc with $\alpha < 1$: $t_{\rm dyn} < t_{\rm th} \ll t_{\rm visc}$
  - All three time-scales usually increase rapidly with r

Vertical disc structure

- Mach number of orbital motion:

$$\mathrm{Ma} \sim \frac{r\Omega}{c_\mathrm{s}} \sim \left(\frac{H}{r}\right)^{-1}$$

- Characteristic radial velocity due to viscosity:

$$|\bar{u}_r| \sim \frac{\bar{\nu}}{r} \sim \alpha \left(\frac{H}{r}\right) c_\mathrm{s}$$

- For a thin disc,

$$|\bar{u}_r| \ll c_\mathrm{s} \ll r\Omega$$

orbital motion: highly supersonic

accretion flow: highly subsonic

Vertical disc structure

- Corrections to orbital motion:
  - Contribution of radial pressure gradient:

$$\frac{\partial p}{\partial r}\Big/\rho r\Omega^2 \sim \frac{c_{\mathrm{s}}^2}{r^2\Omega^2} \sim \left(\frac{H}{r}\right)^2$$

  - Vertical variations of radial gravitational acceleration $\sim \left(\frac{H}{r}\right)^2$
  - Other terms, e.g. $\rho u_r \frac{\partial u_r}{\partial r}$, are smaller
  - Conclude that $u_\phi = r\Omega \left[1 + O\left(\frac{H}{r}\right)^2\right]$

    i.e. fluid velocity close to orbital velocity of test particle

- In general, thin-disc approximations involve fractional errors $O\left(\frac{H}{r}\right)^2$ and a formal asymptotic treatment is possible

Vertical disc structure

Typical values of H/r :

- protoplanetary discs : 0.05 – 0.1
- binary stars : 0.01 – 0.02
- active galactic nuclei : 0.001
- planetary rings : 0.0000001

Vertical disc structure

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- Barotropic models:
- Vertical hydrostatic equilibrium:

$$\frac{\partial p}{\partial z} = -\rho\Omega_z^2 z$$

- Can be solved if pressure is a known function of density (physical arguments, or just for analytical convenience)
- Vertically isothermal model:

$p = c_\mathrm{s}^2\rho$ $\qquad$ c_s independent of z

- Solution:

$$\rho(r,z) = \rho_0(r,0)\,\mathrm{e}^{-z^2/2H^2}$$

$H = \dfrac{c_\mathrm{s}}{\Omega_z}$ $\qquad$ isothermal scale height

Vertical disc structure

- Polytropic model:

$$p = K\rho^{1+1/n} \qquad K,\ n \text{ independent of } z$$

- Introduce (pseudo-)enthalpy:

$$w = \int \frac{\mathrm{d}p}{\rho} = (n+1)K\rho^{1/n}$$

$$\frac{\partial w}{\partial z} = -\Omega_z^2 z$$

- Solution: $w = \frac{1}{2}\Omega_z^2(H^2 - z^2)$ $\qquad H =$ true semi-thickness

$$\rho(r,z) = \rho(r,0)\left(1 - \frac{z^2}{H^2}\right)^n \qquad \text{(vacuum above } z = H \text{)}$$

$$p(r,z) = p(r,0)\left(1 - \frac{z^2}{H^2}\right)^{n+1}$$

Vertical disc structure

- Radiative models:
- Energy dissipated by viscosity carried away by radiation
- Radiative diffusion (optically thick disc):

$$\boldsymbol{F} = -\frac{16\sigma T^3}{3\kappa\rho}\boldsymbol{\nabla}T \qquad \kappa = \text{Rosseland mean opacity}$$

- Dominant balance in thermal energy equation (thin disc):

$$0 = \mu\left(r\frac{\mathrm{d}\Omega}{\mathrm{d}r}\right)^2 - \frac{\partial F_z}{\partial z}$$

- Contributions from F_r and from radial advection are smaller by $O(H/r)^2$

Vertical disc structure

- Vertical structure of radiative Keplerian disc:

$$\frac{\partial p}{\partial z} = -\rho\Omega^2 z$$

$$\frac{\partial F}{\partial z} = \frac{9}{4}\mu\Omega^2$$

$$F = -\frac{16\sigma T^3}{3\kappa\rho}\frac{\partial T}{\partial z}$$

together with:

- equation of state, e.g. $p = \dfrac{k\rho T}{\mu_{\mathrm{m}} m_{\mathrm{p}}} + \dfrac{4\sigma T^4}{3c}$ (ideal gas + radiation)
- viscosity prescription
- opacity function $\kappa(\rho, T)$
- boundary conditions, e.g. $\rho = p = T = 0$ at $z = H$
  (or match to atmopsheric model)

- Analogous to radial structure of a star

Vertical disc structure

- Opacity often approximated by a power law, e.g. for ionized discs:

$$\kappa = \mathrm{const} \approx 0.33\,\mathrm{cm}^2\,\mathrm{g}^{-1}$$

Thomson opacity
electron scattering
hotter regions

$$\kappa = C_\kappa \rho T^{-7/2}$$

$$C_\kappa \approx 4.5 \times 10^{24}\,\mathrm{cm}^5\,\mathrm{g}^{-2}\,\mathrm{K}^{7/2}$$

Kramers opacity
free-free / bound-free
cooler regions

- Cooler discs: dust, molecules, ...

Vertical disc structure

- Keplerian disc, alpha viscosity, gas pressure, Thomson opacity:

$$\frac{\partial p}{\partial z} = -\rho\Omega^2 z$$

$$\frac{\partial F}{\partial z} = \frac{9}{4}\alpha p\Omega$$

$$F = -\frac{16\sigma T^3}{3\kappa\rho}\frac{\partial T}{\partial z}$$

$$p = \frac{k\rho T}{\mu_{\mathrm{m}} m_{\mathrm{p}}}$$

$$\Sigma = \int_{-H}^{H} \rho \, \mathrm{d}z$$

Vertical disc structure

- Order-of-magnitude treatment:

$$\frac{\partial p}{\partial z} = -\rho\Omega^2 z \qquad \frac{p}{H} \sim \rho\Omega^2 H$$

$$\frac{\partial F}{\partial z} = \frac{9}{4}\alpha p\Omega \qquad \frac{F}{H} \sim \alpha p\Omega$$

$$F = -\frac{16\sigma T^3}{3\kappa\rho}\frac{\partial T}{\partial z} \qquad F \sim \frac{\sigma T^3}{\kappa\rho}\frac{T}{H}$$

$$p = \frac{k\rho T}{\mu_\mathrm{m} m_\mathrm{p}} \qquad p \sim \frac{k\rho T}{\mu_\mathrm{m} m_\mathrm{p}}$$

$$\Sigma = \int_{-H}^{H} \rho\,\mathrm{d}z \qquad \Sigma \sim \rho H$$

- Algebraic solution:

$$H \sim \alpha^{1/6}\Sigma^{1/3}\Omega^{-5/6}\left(\frac{\mu_\mathrm{m} m_\mathrm{p}}{k}\right)^{-2/3}\left(\frac{\sigma}{\kappa}\right)^{-1/6}$$

Vertical disc structure

- Viscosity:

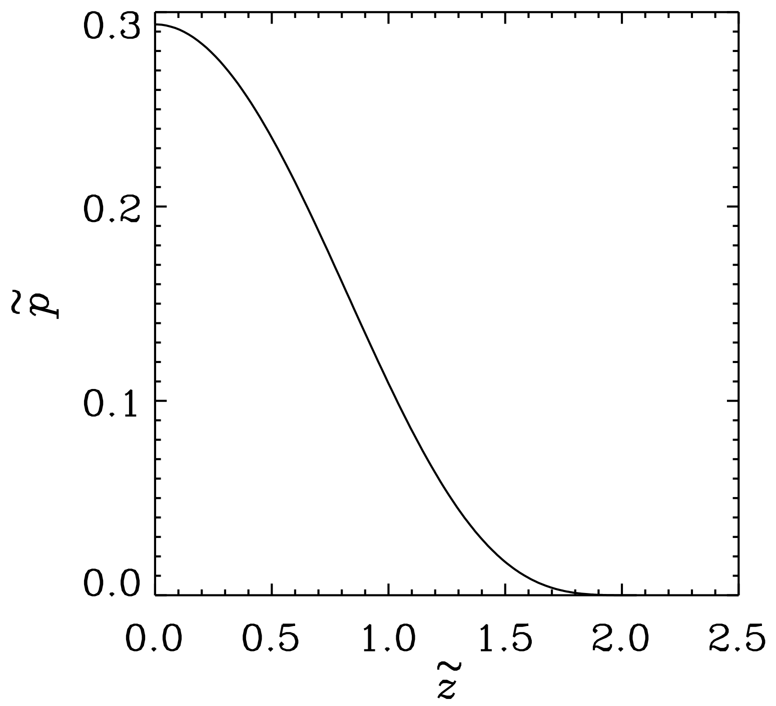
$$\bar{\nu} \sim \alpha c_{\rm s} H \sim \alpha \Omega H^2 \sim \alpha^{4/3} \Sigma^{2/3} \Omega^{-2/3} \left(\frac{\mu_{\rm m} m_{\rm p}}{k}\right)^{-4/3} \left(\frac{\sigma}{\kappa}\right)^{-1/3}$$

- Thus $\bar{\nu} \propto r\Sigma^{2/3}$

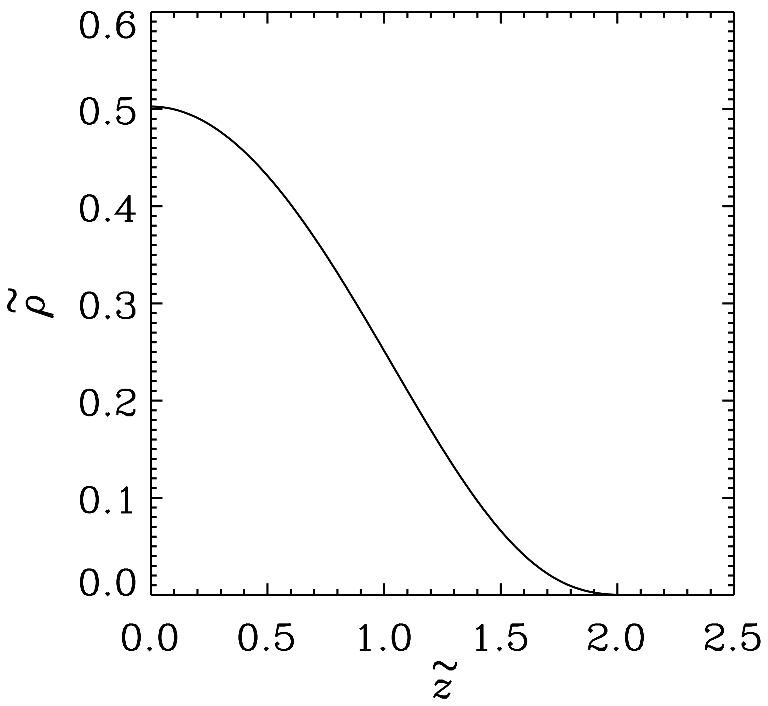
- Full treatment:
  - Use order-of-magnitude treatment as a dimensional analysis
  - It defines characteristic units $U_H, U_\rho, U_p, U_T,$ etc.
  - Then write $z = U_H\,\tilde{z},\ \rho = U_\rho\,\tilde{\rho}(\tilde{z}),$ etc. to obtain a system of dimensionless ODEs with no free parameters
  - Solve numerically
  - Find $\bar{\nu} = Ar\Sigma^{2/3}$ with a precise coefficient A

- Power-law constitutive relations give power-law viscosity

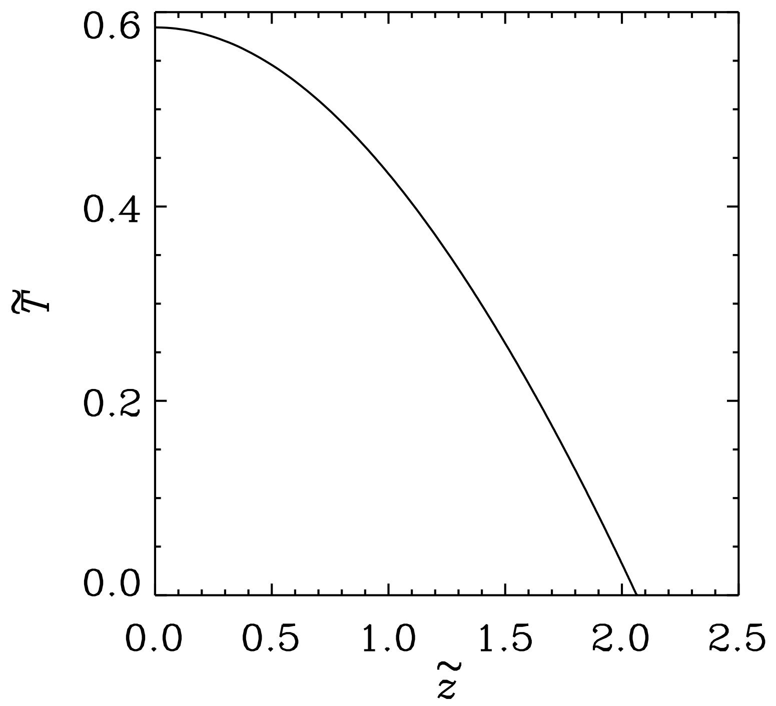
Vertical disc structure



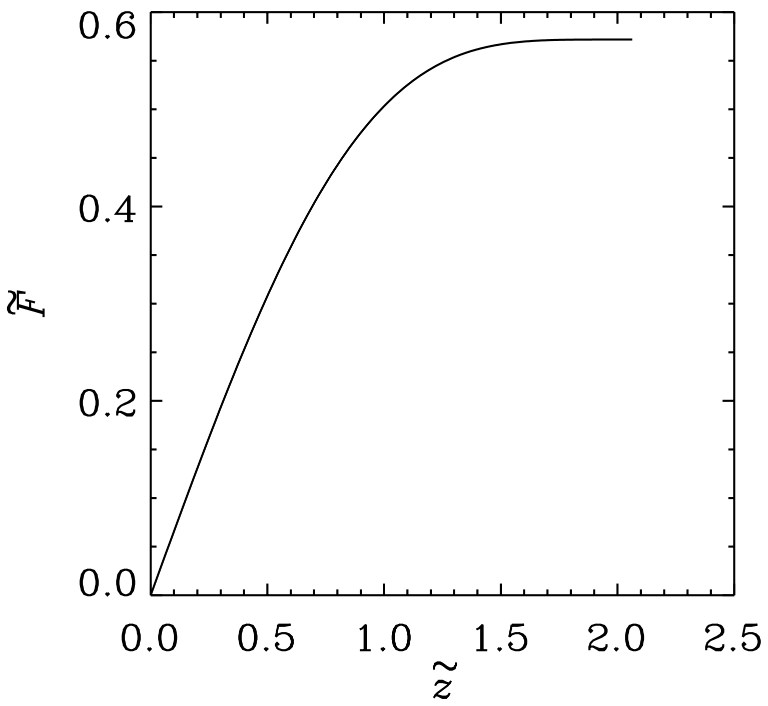
pressure



density



temperature



radiative flux

similar to

polytropic models

$$\rho \propto (H^2 - z^2)^n$$

$$p \propto (H^2 - z^2)^{n+1}$$

$$T \propto (H^2 - z^2)$$

Thermal and viscous stability

- Viscous stability:
- Nonlinear diffusion equation for Keplerian disc:

$$\frac{\partial\Sigma}{\partial t}=\frac{3}{r}\frac{\partial}{\partial r}\left[r^{1/2}\frac{\partial}{\partial r}(r^{1/2}\bar{\nu}\Sigma)\right] \qquad \bar{\nu}=\bar{\nu}(r,\Sigma)$$

- Linearize about any given solution $\Sigma_0(r,t)$:

$$\Sigma(r,t)=\Sigma_0(r,t)+\Sigma'(r,t) \qquad |\Sigma'|\ll\Sigma_0$$

$$(\bar{\nu}\Sigma)'=\frac{\partial(\bar{\nu}\Sigma)}{\partial\Sigma}\Sigma'=q\bar{\nu}\Sigma' \qquad q=\frac{\partial\ln(\bar{\nu}\Sigma)}{\partial\ln\Sigma}$$

- Linearized diffusion equation:

$$\frac{\partial\Sigma'}{\partial t}=\frac{3}{r}\frac{\partial}{\partial r}\left[r^{1/2}\frac{\partial}{\partial r}(r^{1/2}q\bar{\nu}\Sigma')\right]$$

- Unstable for $q<0$: rapid growth on short length-scales

Thermal and viscous stability

- Thermal stability:
- So far, assumed a balance between heating and cooling:

$$\frac{9}{4}\bar{\nu}\Sigma\Omega^2 = \mathcal{H} = \mathcal{C} = 2F^+$$

- Relax this assumption, but assume that $\alpha \ll 1$ so that

$$t_{\rm dyn} \ll t_{\rm th} \ll t_{\rm visc}$$

- Consider behaviour on the timescale $t_{\rm th}$:
  - disc is hydrostatic
  - surface density does not evolve
- By solving equations of vertical structure except thermal balance, can calculate $\mathcal{H}$ and $\mathcal{C}$ as functions of $(\Sigma, \bar{\nu}\Sigma)$
- In fact $\mathcal{H}$ depends only on $\bar{\nu}\Sigma$
- Equation of thermal balance defines a curve in the $(\Sigma, \bar{\nu}\Sigma)$ plane

Thermal and viscous stability

- Infinitesimal perturbations:

$$\mathrm{d}\mathcal{H} = \frac{\mathrm{d}\mathcal{H}}{\mathrm{d}(\bar{\nu}\Sigma)}\,\mathrm{d}(\bar{\nu}\Sigma) \qquad \mathrm{d}\mathcal{C} = \frac{\partial\mathcal{C}}{\partial\Sigma}\,\mathrm{d}\Sigma + \frac{\partial\mathcal{C}}{\partial(\bar{\nu}\Sigma)}\,\mathrm{d}(\bar{\nu}\Sigma)$$

- Along the equilibrium curve, $\mathrm{d}\mathcal{H} = \mathrm{d}\mathcal{C}$ and $\mathrm{d}(\bar{\nu}\Sigma) = q\bar{\nu}\,\mathrm{d}\Sigma$

$$\Rightarrow \quad \frac{\mathrm{d}\mathcal{H}}{\mathrm{d}(\bar{\nu}\Sigma)} = \frac{1}{q\bar{\nu}}\frac{\partial\mathcal{C}}{\partial\Sigma} + \frac{\partial\mathcal{C}}{\partial(\bar{\nu}\Sigma)} \qquad q = \frac{\partial\ln(\bar{\nu}\Sigma)}{\partial\ln\Sigma}$$

- Thermal energy content of disc per unit area $\sim pH \sim (\Omega/\alpha)\bar{\nu}\Sigma$
- If some heat is added, $\bar{\nu}\Sigma$ increases but Σ is fixed on t_{th}
- Unstable if excess heating exceeds excess cooling, i.e. if

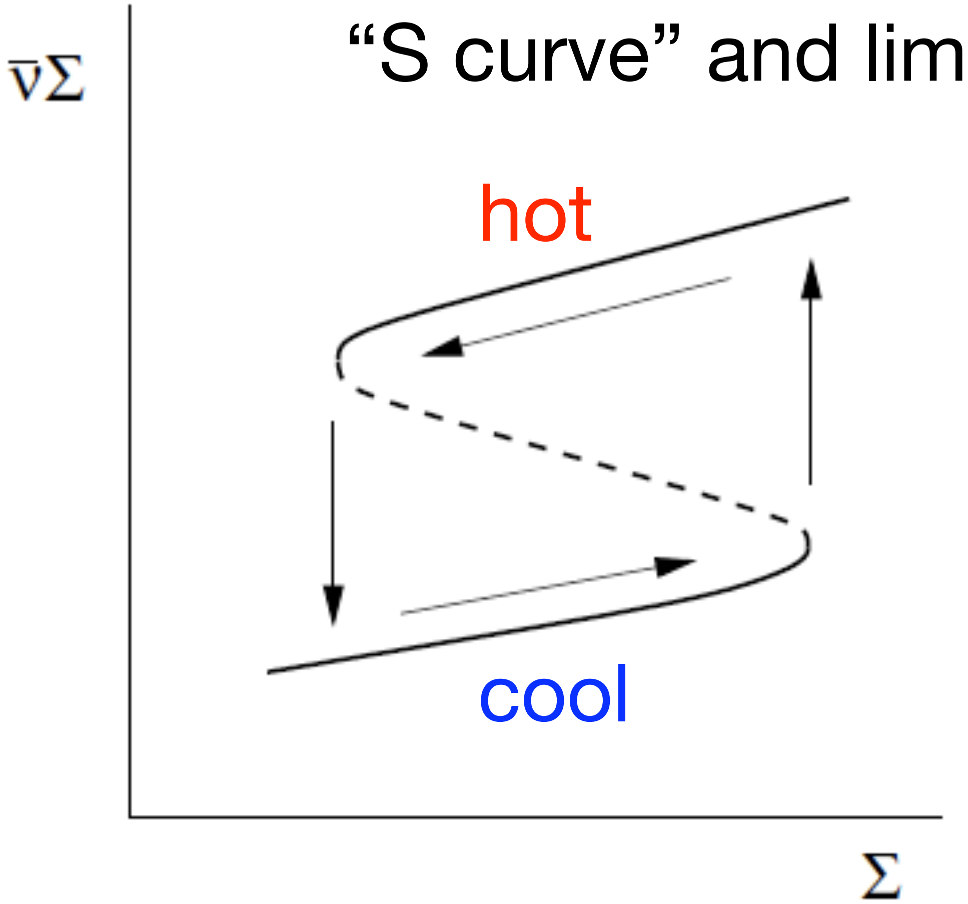
$$\frac{\mathrm{d}\mathcal{H}}{\mathrm{d}(\bar{\nu}\Sigma)} > \frac{\mathrm{d}\mathcal{C}}{\mathrm{d}(\bar{\nu}\Sigma)} \qquad \text{i.e.} \qquad \frac{1}{q\bar{\nu}}\frac{\partial\mathcal{C}}{\partial\Sigma} > 0$$

- In practice $\partial\mathcal{C}/\partial\Sigma < 0$ (because, at fixed $\bar{\nu}\Sigma$, $\Sigma \propto 1/\bar{\nu} \propto 1/(\alpha T)$)
  so thermal instability occurs (like viscous instability) when $q < 0$

Thermal and viscous stability

- Outbursts:
- Radiative disc with gas pressure and Thomson opacity has $\bar{\nu}\Sigma \propto r\Sigma^{5/3}$ and is viscously and thermally stable
- For cooler discs undergoing H ionization, instability can occur

"S curve" and limit cycle



for steady accretion:

$$\bar{\nu}\Sigma = \frac{\dot{M}}{3\pi}\left[1 - \left(\frac{r_{\rm in}}{r}\right)^{1/2}\right] \approx \frac{\dot{M}}{3\pi}$$

- Explains outbursts in many cataclysmic variables, X-ray binaries and other systems