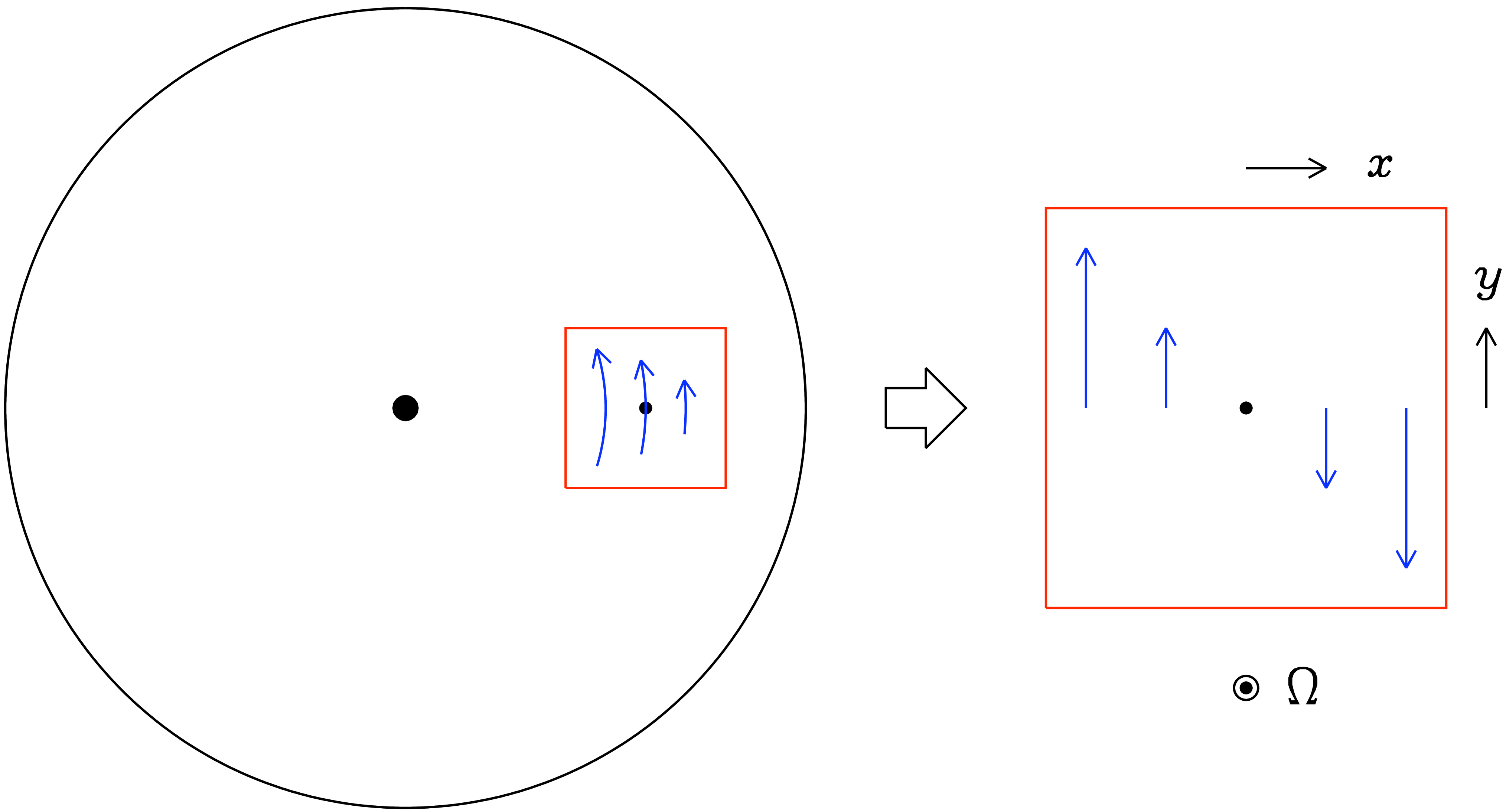
Local approximation

- Shearing sheet / local approximation (Goldreich & Lynden-Bell 1965)
- Local model of a differentially rotating disc



Local approximation

- Consider an orbiting reference point with cylindrical coordinates

$$(r, \phi, z) = (r_0, \phi_0 + \Omega_0 t, 0) \qquad \Omega_0 = \Omega(r_0)$$

- Use as origin of a local Cartesian coordinate system (x, y, z)

$$x = r - r_0 \qquad \text{radial}$$

$$y = r_0(\phi - \phi_0 - \Omega_0 t) \qquad \text{azimuthal}$$

$$z = z \qquad \text{vertical}$$

- Orbital motion appears locally as a uniform rotation $\boldsymbol{\Omega}_0 = \Omega_0\, \boldsymbol{e}_z$ plus a linear shear flow $\boldsymbol{u}_0 = -S_0 x\, \boldsymbol{e}_y$

$$S = -r\frac{\mathrm{d}\Omega}{\mathrm{d}r} \qquad \text{rate of orbital shear}$$

Local approximation

- Effective potential in rotating frame (different from previous effective potential under $h = \mathrm{cst}$) expanded to second order in x and z

$$
\begin{aligned}
&= \Phi(r,z) - \tfrac{1}{2}\Omega_0^2 r^2 \\
&= \Phi(r_0,0) + \cancel{\Phi_{,r}(r_0,0)x} + \tfrac{1}{2}\Phi_{,rr}(r_0,0)x^2 + \tfrac{1}{2}\Phi_{,zz}(r_0,0)z^2 \\
&\quad - \tfrac{1}{2}\Omega_0^2(r_0^2 + \cancel{2r_0x} + x^2) \\
&= \cancel{\mathrm{cst}} + \tfrac{1}{2}[\partial_r(r\Omega^2) - \Omega^2]_0 x^2 + \tfrac{1}{2}\Omega_{z0}^2 z^2 \\
&= -\Omega_0 S_0 x^2 + \tfrac{1}{2}\Omega_{z0}^2 z^2
\end{aligned}
$$

Local approximation

- Particle dynamics in local approximation

$$\ddot{x} - 2\Omega_0 \dot{y} = 2\Omega_0 S_0 x$$

$$\ddot{y} + 2\Omega_0 \dot{x} = 0$$

$$\ddot{z} = -\Omega_{z0}^2 z$$

(Keplerian case
$S_0 = \frac{3}{2}\Omega_0$
$\Omega_{z0} = \Omega_0$
→ "Hill's equations")
(without satellite)

- Simple orbital motion:

$$x = \mathrm{cst}$$

$$\dot{y} = -S_0 x$$

Coriolis force balances effective potential gradient

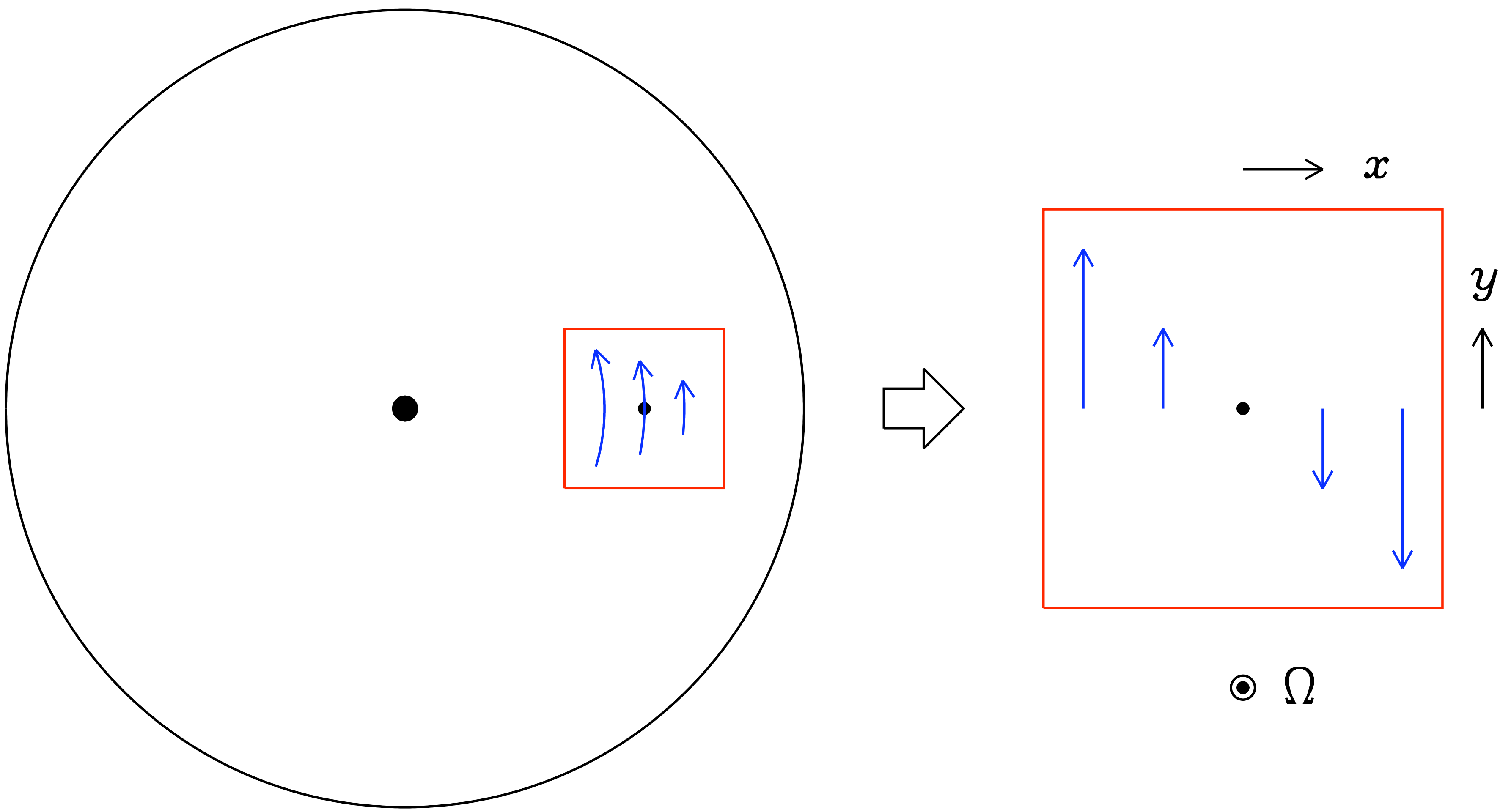
- General solution involves horizontal and vertical oscillations
- Canonical y momentum (per unit mass):

$$p_y = \dot{y} + 2\Omega_0 x = \mathrm{cst}$$

  - Plays role of specific angular momentum in local approximation
  - Has uniform gradient in simple orbital motion: $p_y = (2\Omega_0 - S_0)x$

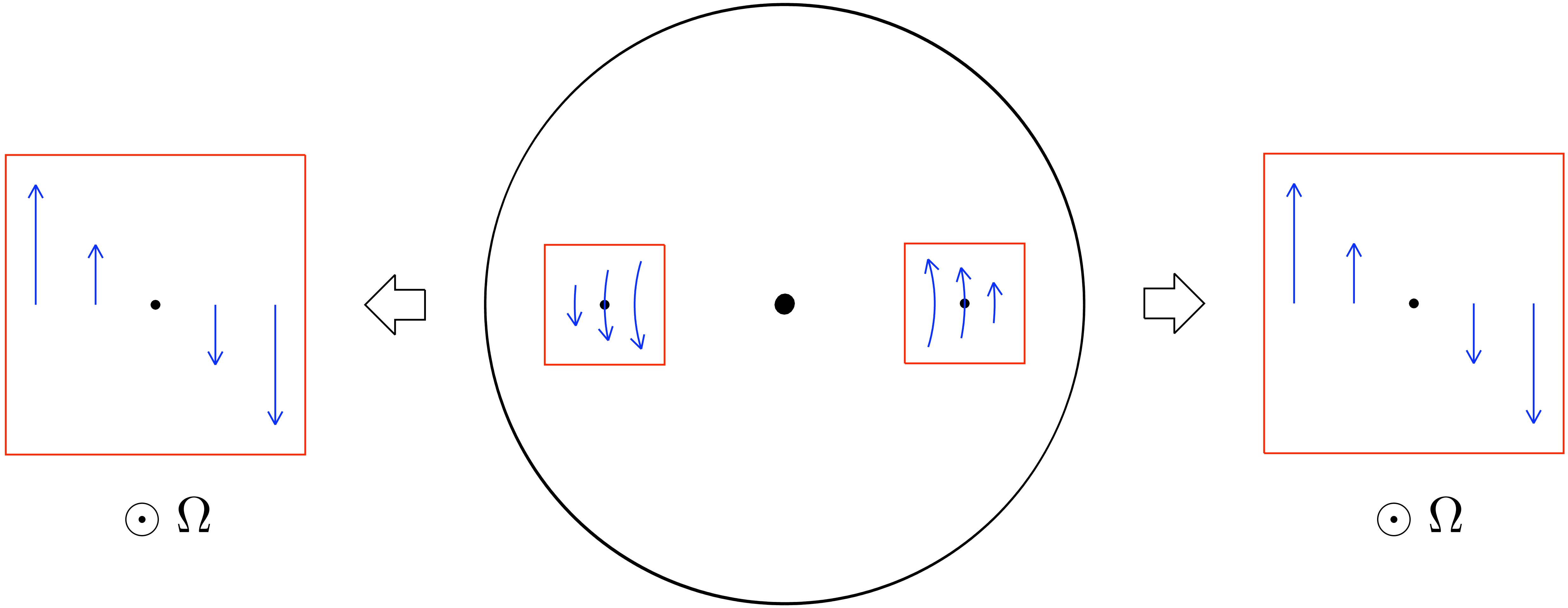
Local approximation

- Symmetries of local approximation:
  (higher than those of original disc!)
  - Spatial homogeneity (horizontally):
    every point in xy plane is equivalent (up to Galilean boost)
  - Rotation by π about z axis



Local approximation

Rotational symmetry



- Direction to central object cannot be determined
- No accretion flow therefore expected
- Local model knows about Ω (and S) but not about r

Local approximation

- Boundary conditions of shearing sheet
  - Horizontally unbounded or apply (modified) periodic boundary conditions
  - Vertical structure:
    - Ignore z completely (2D shearing sheet)
    - Neglect vertical gravity: homogeneous in z
    - Include vertical gravity: isothermal, radiative, etc. models

Incompressible fluid dynamics in shearing sheet

- Homogeneous incompressible fluid
- 3D system, unbounded or periodic in x, y, z
- Uniform kinematic viscosity ν [discuss its role]

$$\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u} \cdot \boldsymbol{\nabla} \boldsymbol{u} + 2\boldsymbol{\Omega}_0 \times \boldsymbol{u} = -\boldsymbol{\nabla}\Phi - \frac{1}{\rho}\boldsymbol{\nabla} p + \nu\nabla^2 \boldsymbol{u}$$

$$\boldsymbol{\nabla} \cdot \boldsymbol{u} = 0$$

neglect (balanced
by pressure gradient)

- Effective potential $\Phi = -\Omega_0 S_0 x^2 + \cancel{\tfrac{1}{2}\Omega_{z0}^2 z^2}$
- Basic state:

$$\boldsymbol{u} = \boldsymbol{u}_0 = -S_0 x\, \boldsymbol{e}_y$$

$$p = p_0 = \mathrm{cst}$$

- Uniform viscous stress, but no divergence and so no accretion flow

Incompressible fluid dynamics in shearing sheet

- Perturbations (not necessarily small):

$$\boldsymbol{u} = \boldsymbol{u}_0 + \boldsymbol{v}(x, y, z, t)$$

$$p = p_0 + p'(x, y, z, t)$$

$$\frac{\partial \boldsymbol{v}}{\partial t} + \boldsymbol{u}_0 \cdot \boldsymbol{\nabla}\boldsymbol{v} + \boldsymbol{v} \cdot \boldsymbol{\nabla}\boldsymbol{u}_0 + \boldsymbol{v} \cdot \boldsymbol{\nabla}\boldsymbol{v} + 2\boldsymbol{\Omega}_0 \times \boldsymbol{v} = -\frac{1}{\rho}\boldsymbol{\nabla}p' + \nu\nabla^2\boldsymbol{v}$$

$$\boldsymbol{\nabla} \cdot \boldsymbol{v} = 0$$

- Now drop the subscript 0 on Ω_0 and S_0 and let $\psi = p'/\rho$:

$$\left(\frac{\partial}{\partial t} - Sx\frac{\partial}{\partial y} + \boldsymbol{v} \cdot \boldsymbol{\nabla}\right) v_x - 2\Omega v_y = -\frac{\partial \psi}{\partial x} + \nu\nabla^2 v_x$$

$$\left(\frac{\partial}{\partial t} - Sx\frac{\partial}{\partial y} + \boldsymbol{v} \cdot \boldsymbol{\nabla}\right) v_y + (2\Omega - S)v_x = -\frac{\partial \psi}{\partial y} + \nu\nabla^2 v_y$$

$$\left(\frac{\partial}{\partial t} - Sx\frac{\partial}{\partial y} + \boldsymbol{v} \cdot \boldsymbol{\nabla}\right) v_z = -\frac{\partial \psi}{\partial z} + \nu\nabla^2 v_z$$

Incompressible fluid dynamics in shearing sheet

- Shearing waves (after Kelvin / Thomson 1887):
- Consider a plane-wave disturbance of the form

$$\boldsymbol{v}(\boldsymbol{x},t) = \mathrm{Re}\left\{\tilde{\boldsymbol{v}}(t)\exp[\mathrm{i}\boldsymbol{k}(t)\cdot\boldsymbol{x}]\right\}$$

$$\psi(\boldsymbol{x},t) = \mathrm{Re}\left\{\tilde{\psi}(t)\exp[\mathrm{i}\boldsymbol{k}(t)\cdot\boldsymbol{x}]\right\}$$

time-dependent wavevector

- Then

$$\left(\frac{\partial}{\partial t} - Sx\frac{\partial}{\partial y}\right)\boldsymbol{v} = \mathrm{Re}\left\{\left[\frac{\mathrm{d}\tilde{\boldsymbol{v}}}{\mathrm{d}t} + \left(\mathrm{i}\frac{\mathrm{d}\boldsymbol{k}}{\mathrm{d}t}\cdot\boldsymbol{x} - Sx\,\mathrm{i}k_y\right)\tilde{\boldsymbol{v}}\right]\exp[\mathrm{i}\boldsymbol{k}(t)\cdot\boldsymbol{x}]\right\}$$

- If we choose

$$\frac{\mathrm{d}\boldsymbol{k}}{\mathrm{d}t} = Sk_y\,\boldsymbol{e}_x$$

then two terms cancel and we are left with $\dfrac{\mathrm{d}\tilde{\boldsymbol{v}}}{\mathrm{d}t}$

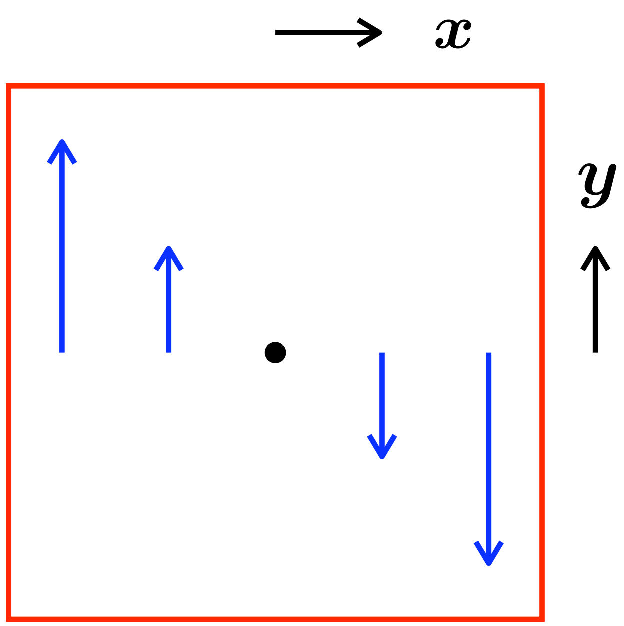
- This means

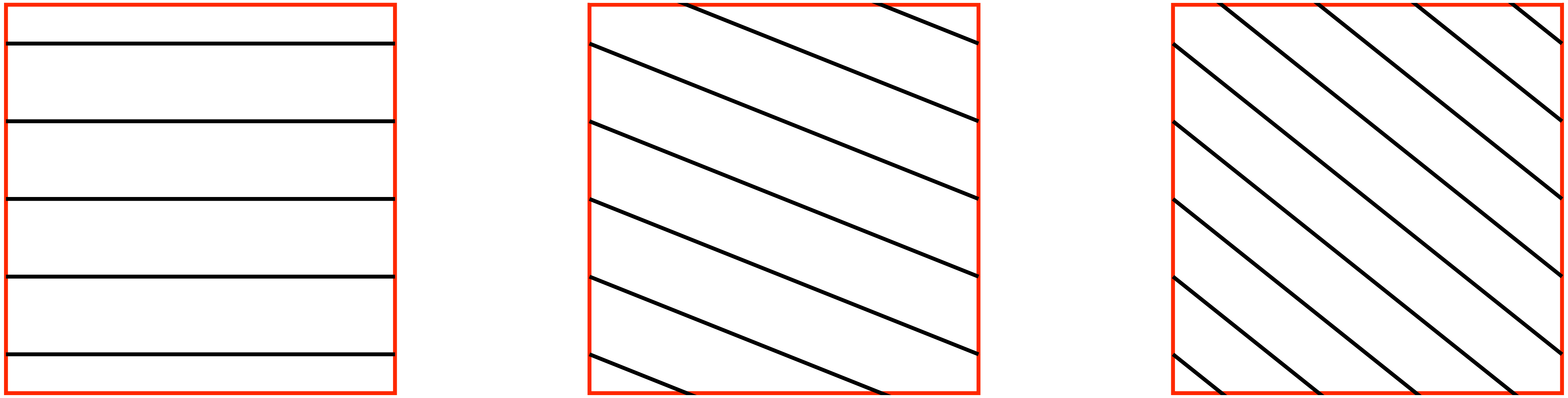
$$k_x = k_{x0} + Sk_y t \qquad k_y = \mathrm{cst} \qquad k_z = \mathrm{cst}$$

Incompressible fluid dynamics in shearing sheet

$$k_x = k_{x0} + Sk_y t \qquad k_y = \mathrm{cst} \qquad k_z = \mathrm{cst}$$

- Tilting / shearing of wavefronts:

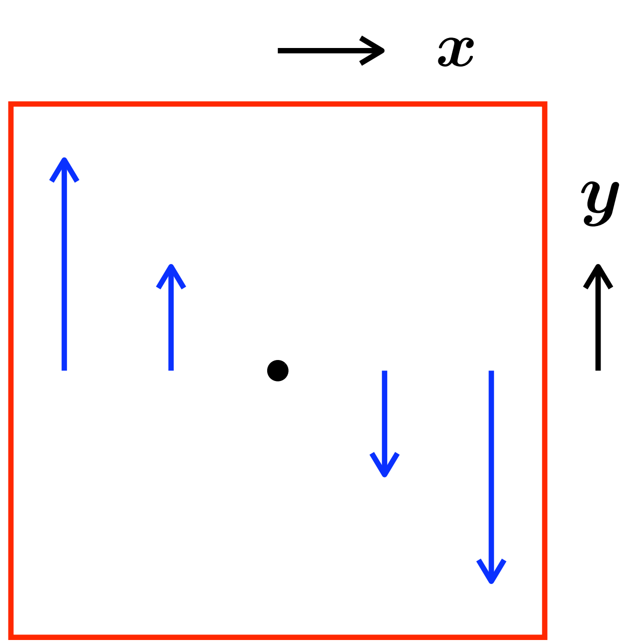


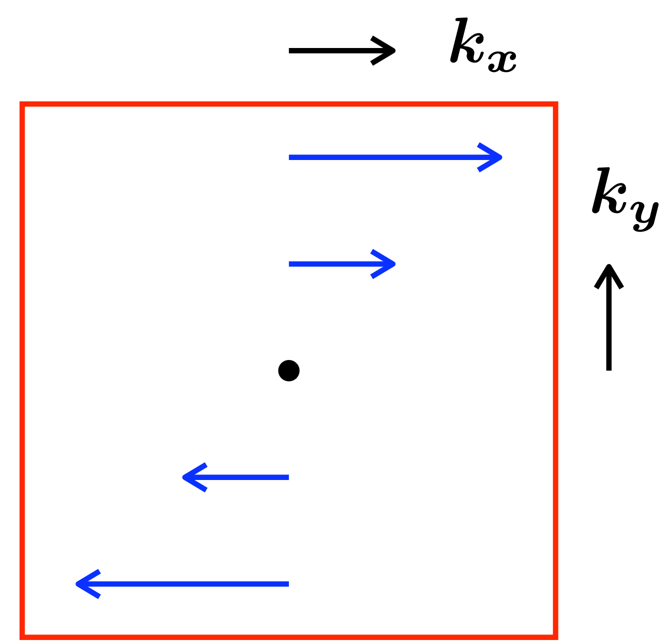


Incompressible fluid dynamics in shearing sheet

$$k_x = k_{x0} + S k_y t \qquad k_y = \mathrm{cst} \qquad k_z = \mathrm{cst}$$

- Dual shear flow in Fourier space:





Incompressible fluid dynamics in shearing sheet

- Furthermore

$$\begin{aligned} \boldsymbol{v}\cdot\boldsymbol{\nabla}\boldsymbol{v} &= \mathrm{Re}\left[\tilde{\boldsymbol{v}}\,\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}}\right]\cdot\boldsymbol{\nabla}\,\mathrm{Re}\left[\tilde{\boldsymbol{v}}\,\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}}\right] \\ &= \mathrm{Re}\left[\boldsymbol{k}\cdot\tilde{\boldsymbol{v}}\,\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}}\right]\mathrm{Re}\left[\mathrm{i}\tilde{\boldsymbol{v}}\,\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}}\right] \\ &= 0 \end{aligned}$$

because $\boldsymbol{\nabla}\cdot\boldsymbol{v} = 0 \quad \Rightarrow \quad \mathrm{i}\boldsymbol{k}\cdot\tilde{\boldsymbol{v}} = 0$

- Special result for incompressible fluid
- Nonlinearity doesn’t vanish for a superposition of shearing waves

Incompressible fluid dynamics in shearing sheet

- Amplitude equations for shearing waves:

$$\begin{aligned}
\frac{\mathrm{d}\tilde{v}_x}{\mathrm{d}t} - 2\Omega\tilde{v}_y &= -\mathrm{i}k_x\tilde{\psi} - \nu k^2\tilde{v}_x \\
\frac{\mathrm{d}\tilde{v}_y}{\mathrm{d}t} + (2\Omega - S)\tilde{v}_x &= -\mathrm{i}k_y\tilde{\psi} - \nu k^2\tilde{v}_y \qquad\qquad k^2 = |\boldsymbol{k}|^2 \\
\frac{\mathrm{d}\tilde{v}_z}{\mathrm{d}t} &= -\mathrm{i}k_z\tilde{\psi} - \nu k^2\tilde{v}_z
\end{aligned}$$

$$\mathrm{i}\boldsymbol{k}\cdot\tilde{\boldsymbol{v}} = 0$$

- Viscous terms are taken care of by a viscous decay factor

$$\begin{aligned}
E_\nu(t) &= \exp\left(-\int \nu k^2\,\mathrm{d}t\right) \\
&= \exp\left\{-\nu\left[(k_{x0}^2 + k_y^2 + k_z^2)t + Sk_{x0}k_y t^2 + \tfrac{1}{3}S^2k_y^2t^3\right]\right\}
\end{aligned}$$

- Faster than exponential decay if $k_y \neq 0$

Incompressible fluid dynamics in shearing sheet

- Write $\tilde{\boldsymbol{v}} = E_\nu(t)\hat{\boldsymbol{v}}(t), \ \tilde{\psi} = E_\nu(t)\hat{\psi}(t)$ to obtain inviscid problem

$$\begin{aligned}
\frac{\mathrm{d}\hat{v}_x}{\mathrm{d}t} - 2\Omega\hat{v}_y \qquad &= -\mathrm{i}k_x\hat{\psi} \\
\frac{\mathrm{d}\hat{v}_y}{\mathrm{d}t} + (2\Omega - S)\hat{v}_x &= -\mathrm{i}k_y\hat{\psi} \\
\frac{\mathrm{d}\hat{v}_z}{\mathrm{d}t} \qquad &= -\mathrm{i}k_z\hat{\psi}
\end{aligned}$$

$$\mathrm{i}\boldsymbol{k}\cdot\hat{\boldsymbol{v}} = 0$$

- Eliminate variables in favour of $\hat{v}_x$ to obtain (after algebra)

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}(k^2\hat{v}_x) + \kappa^2 k_z^2\hat{v}_x = 0$$

$$\kappa^2 = 2\Omega(2\Omega - S)$$

square of epicyclic frequency
in local approximation

Incompressible fluid dynamics in shearing sheet

- Analysis of axisymmetric/unsheared waves $(k_y = 0)$:

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}(k^2\hat{v}_x) + \kappa^2 k_z^2 \hat{v}_x = 0$$

- Constant coefficients, so exponential / sinusoidal solutions
- Inviscid case:
  - Oscillations (inertial waves) if $\kappa^2 > 0$
  - Exponential growth if $\kappa^2 < 0$
- With viscosity, include factor $E_\nu = \exp(-\nu k^2 t)$:
  - Damped oscillations if $\kappa^2 > 0$
  - Unstable to sufficiently long wavelengths if $\kappa^2 < 0$

Incompressible fluid dynamics in shearing sheet

- Analysis of non-axisymmetric/sheared waves $(k_y \neq 0)$:

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}(k^2\hat{v}_x) + \kappa^2 k_z^2 \hat{v}_x = 0$$

- Non-constant coefficients; solutions involve Legendre functions
- Asymptotic behaviour as $t \to \infty$:

$$k^2 \sim k_x^2 \sim S^2 k_y^2 t^2$$

- ODE has regular singular point at $t = \infty$:

$$\hat{v}_x \propto t^\sigma \qquad\qquad (\hat{v}_y \propto t^{\sigma+1},\ \hat{v}_z \propto t^{\sigma+1},\ \hat{\psi} \propto t^\sigma)$$

- Indicial equation:

$$(\sigma+2)(\sigma+1)S^2k_y^2 + \kappa^2 k_z^2 = 0$$

$$\sigma = -\frac{3}{2} \pm \left(\frac{1}{4} - \frac{\kappa^2 k_z^2}{S^2 k_y^2}\right)^{1/2}$$

Incompressible fluid dynamics in shearing sheet

$$\hat{v}_x \propto t^{\sigma} \qquad (\hat{v}_y \propto t^{\sigma+1},\ \hat{v}_z \propto t^{\sigma+1},\ \hat{\psi} \propto t^{\sigma})$$

$$\sigma = -\frac{3}{2} \pm \left(\frac{1}{4} - \frac{\kappa^2 k_z^2}{S^2 k_y^2}\right)^{1/2}$$

- Three cases to consider:
  - $\kappa^2 > (k_y^2/k_z^2)(S^2/4):\ \sigma = -\frac{3}{2} + \text{imaginary}\ :\quad |\hat{\boldsymbol{v}}|^2 \propto t^{-1} \to 0$
  - $0 < \kappa^2 < (k_y^2/k_z^2)(S^2/4):\ \sigma < -1:\quad |\hat{\boldsymbol{v}}|^2 \propto t^{2(\sigma+1)} \to 0$
  - $\kappa^2 < 0$: one root has $\sigma > -1$: $|\hat{\boldsymbol{v}}|^2 \propto t^{2(\sigma+1)} \to \infty$
- Therefore inviscid solutions decay when $\kappa^2 > 0$ but grow (in energy norm) when $\kappa^2 < 0$
- When $\nu > 0$, viscous decay factor E_ν kills off any algebraic growth

Incompressible fluid dynamics in shearing sheet

- Special case of non-rotating shear flow (plane Couette flow)

$$\frac{\mathrm{d}\hat{v}_x}{\mathrm{d}t} = -\mathrm{i}k_x\hat{\psi}$$

$$\frac{\mathrm{d}\hat{v}_y}{\mathrm{d}t} - S\hat{v}_x = -\mathrm{i}k_y\hat{\psi}$$

$$\frac{\mathrm{d}\hat{v}_z}{\mathrm{d}t} = -\mathrm{i}k_z\hat{\psi}$$

$$\mathrm{i}\boldsymbol{k}\cdot\hat{\boldsymbol{v}} = 0$$

- Eliminate variables in favour of $\hat{v}_x$ to obtain (after algebra)

$$\frac{\mathrm{d}}{\mathrm{d}t}(k^2\hat{v}_x) = 0$$

Incompressible fluid dynamics in shearing sheet

- Generic non-axisymmetric disturbances $(k_y \neq 0)$:

$$\hat{v}_x \propto k^{-2}\,, \quad \hat{\psi} \propto k^{-4}$$

  - As $t \to \infty$:

$$\hat{v}_x \propto t^{-2} \quad \hat{v}_y \to \mathrm{cst}, \quad \hat{v}_z \to \mathrm{cst}$$

- Generic axisymmetric disturbances $(k_y = 0)$:

$$\hat{\psi} = 0 \quad \hat{v}_x = \mathrm{cst}\,, \quad \hat{v}_z = \mathrm{cst}\,, \quad \mathrm{d}\hat{v}_y/\mathrm{d}t = S\hat{v}_x$$

  - Algebraic growth tempered by viscous decay
  - Kinetic energy grows by a factor $O(\mathrm{Re})^2$ in a time $O(\mathrm{Re})$
  - Reynolds number $\mathrm{Re} = S/\nu k^2$
  - This mechanism plays an essential role in the transition to turbulence in non-rotating shear flows but is suppressed in rotating shear flows because of inertial oscillations

Incompressible fluid dynamics in shearing sheet

- Summary:
  - rotating shear flow is linearly stable when $\kappa^2 > 0$
  - rotating shear flow is linearly unstable when $\kappa^2 < 0$
- Agrees with stability of circular test-particle orbits
- Agrees with Rayleigh's criterion for the linear stability of a cylindrical shear flow $\boldsymbol{u} = r\Omega(r)\,\boldsymbol{e}_\phi$ to axisymmetric perturbations
- The case $\kappa^2 = 0$ (either non-rotating shear flow or one with uniform specific angular momentum) is marginally Rayleigh-stable and allows algebraic growth in the absence of viscosity
- [Discussion of laboratory experiments and numerical simulations]

Incompressible fluid dynamics in shearing sheet

- 2D incompressible dynamics

$$\left(\frac{\partial}{\partial t} - Sx\frac{\partial}{\partial y} + \boldsymbol{v}\cdot\boldsymbol{\nabla}\right)v_x - 2\Omega v_y = -\frac{\partial\psi}{\partial x} + \nu\nabla^2 v_x$$

$$\left(\frac{\partial}{\partial t} - Sx\frac{\partial}{\partial y} + \boldsymbol{v}\cdot\boldsymbol{\nabla}\right)v_y + (2\Omega - S)v_x = -\frac{\partial\psi}{\partial y} + \nu\nabla^2 v_y$$

$$\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} = 0$$

- Introduce streamfunction $\chi(x,y,t)$: $v_x = \dfrac{\partial\chi}{\partial y},\ v_y = -\dfrac{\partial\chi}{\partial x}$
- Instantaneous streamlines are curves $\chi = \mathrm{cst}$
- Vorticity perturbation

$$\boldsymbol{\nabla}\times\boldsymbol{v} = \left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right)\boldsymbol{e}_z = (-\nabla^2\chi)\boldsymbol{e}_z = \zeta\,\boldsymbol{e}_z$$

Incompressible fluid dynamics in shearing sheet

- Curl of equation of motion (to eliminate pressure):

$$\left(\frac{\partial}{\partial t} - Sx\frac{\partial}{\partial y} + \boldsymbol{v}\cdot\boldsymbol{\nabla}\right)\zeta - S\cancel{\frac{\partial v_y}{\partial y}} + \cancel{\frac{\partial v_x}{\partial x}\frac{\partial v_y}{\partial x}} + \cancel{\frac{\partial v_y}{\partial x}\frac{\partial v_y}{\partial y}}$$

$$\cancel{+(2\Omega - S)\frac{\partial v_x}{\partial x}} - \cancel{\frac{\partial v_x}{\partial y}\frac{\partial v_x}{\partial x}} - \cancel{\frac{\partial v_y}{\partial y}\frac{\partial v_x}{\partial y}} + \cancel{2\Omega\frac{\partial v_y}{\partial y}} = \nu\nabla^2\zeta$$

- Can also be written using Jacobian:

$$\left(\frac{\partial}{\partial t} - Sx\frac{\partial}{\partial y}\right)\zeta = \frac{\partial(\chi,\zeta)}{\partial(x,y)} + \nu\nabla^2\zeta$$

- Solve in conjunction with Poisson equation $\nabla^2\chi = -\zeta$
- Total absolute vorticity is $(2\Omega - S + \zeta)\,\boldsymbol{e}_z$
- Coriolis force drops out of 2D incompressible dynamics!
- Too constrained to allow epicyclic motion / inertial oscillations
- Pure vortex dynamics with background shear

Incompressible fluid dynamics in shearing sheet

$$\left(\frac{\partial}{\partial t} - Sx\frac{\partial}{\partial y} + \boldsymbol{v}\cdot\boldsymbol{\nabla}\right)\zeta = \nu\nabla^2\zeta$$

- Multiply by ζ to obtain enstrophy equation

$$\begin{aligned}\left(\frac{\partial}{\partial t} - Sx\frac{\partial}{\partial y} + \boldsymbol{v}\cdot\boldsymbol{\nabla}\right)\left(\frac{1}{2}\zeta^2\right) &= \nu\zeta\nabla^2\zeta \\ &= \boldsymbol{\nabla}\cdot(\nu\zeta\boldsymbol{\nabla}\zeta) - \nu|\boldsymbol{\nabla}\zeta|^2\end{aligned}$$

- With suitable boundary conditions,

$$\frac{\mathrm{d}}{\mathrm{d}t}\int\frac{1}{2}\zeta^2\,\mathrm{d}A = -\int\nu|\boldsymbol{\nabla}\zeta|^2\,\mathrm{d}A$$

  so enstrophy decays

- To maintain vorticity perturbations in the presence of viscosity requires baroclinic or 3D effects or other source terms

Incompressible fluid dynamics in shearing sheet

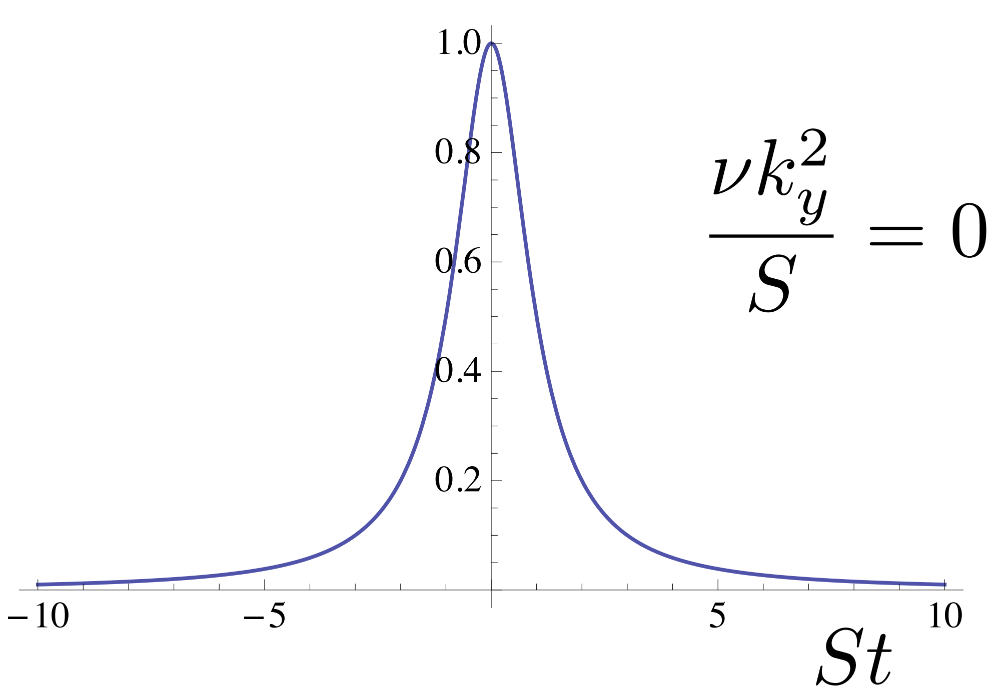
$$\left(\frac{\partial}{\partial t} - Sx\frac{\partial}{\partial y} + \boldsymbol{v}\cdot\boldsymbol{\nabla}\right)\zeta = \nu\nabla^2\zeta$$

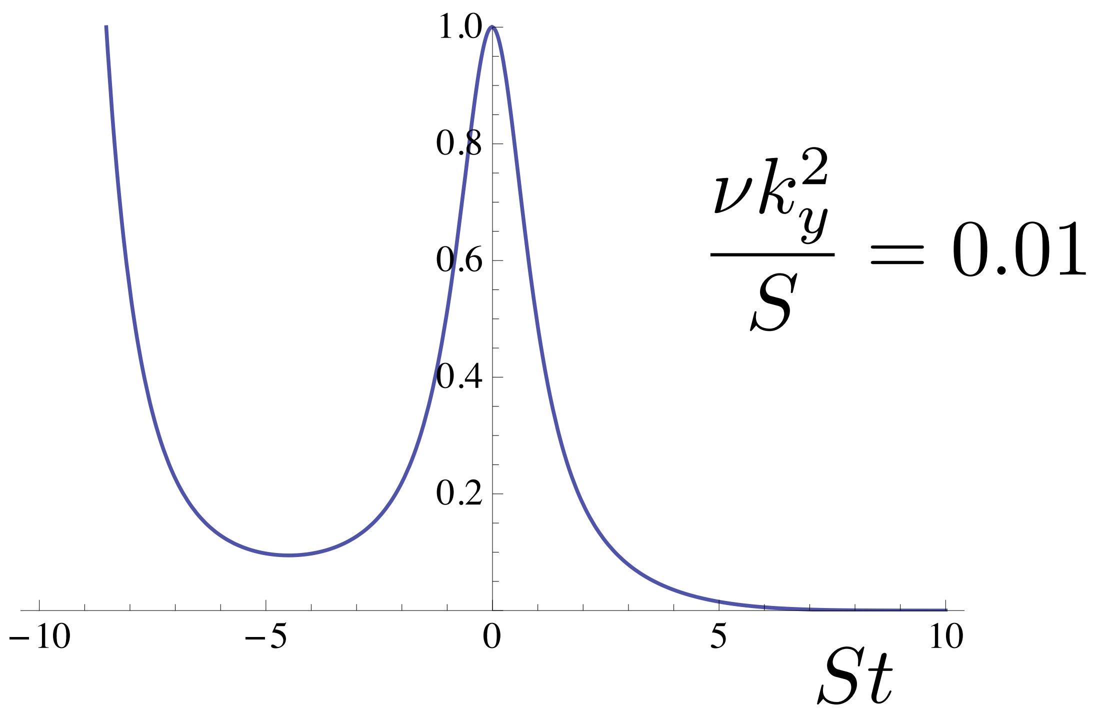
- Shearing-wave solutions $\zeta(\boldsymbol{x},t) = \mathrm{Re}\left\{\tilde{\zeta}(t)\exp[\mathrm{i}\boldsymbol{k}(t)\cdot\boldsymbol{x}]\right\}$:

$$\frac{\mathrm{d}\tilde{\zeta}}{\mathrm{d}t} = -\nu k^2\tilde{\zeta}$$ (nonlinear term vanishes)

$$\tilde{\zeta} \propto E_\nu(t)$$

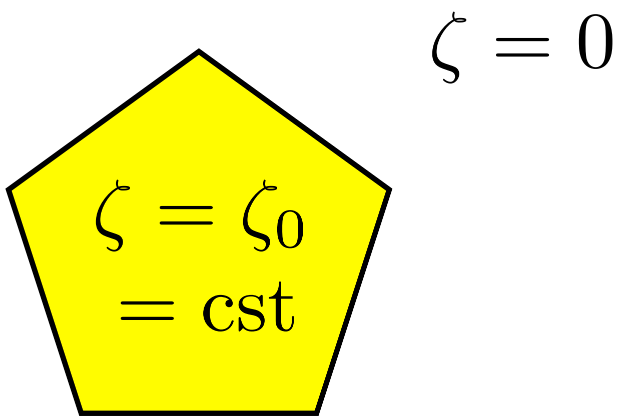
- Kinetic energy $\propto |\tilde{\boldsymbol{v}}|^2 \propto k^{-2}|\tilde{\zeta}|^2 \propto k^{-2}E_\nu^2$





Incompressible fluid dynamics in shearing sheet

- Elliptical vortex patches
- Set $\nu = 0$. Can vorticity resist shear (nonlinear effect)?
- Vortex patch: contour dynamics:



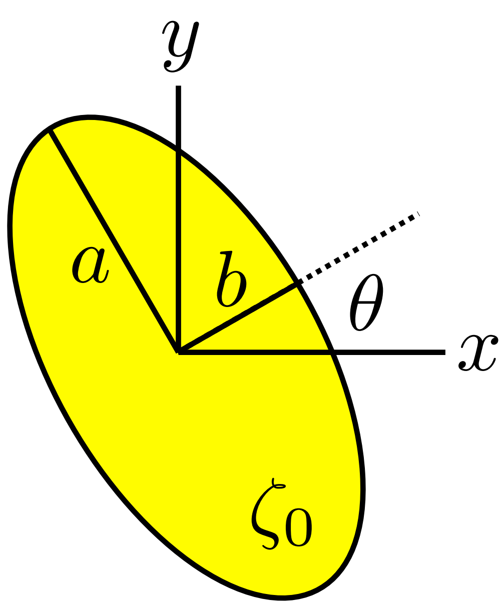
$$\frac{\mathrm{D}\zeta}{\mathrm{D}t} = 0$$

$\zeta \;\rightarrow\; \boldsymbol{v} \;\rightarrow\;$ advection of contour (by $\boldsymbol{u} = \boldsymbol{v} - Sx\,\boldsymbol{e}_y$)

- Do steady solutions exist?

Incompressible fluid dynamics in shearing sheet

- Elliptical vortex patch



- Kirchhoff: $\boldsymbol{v}$ induced by ζ_0 causes ellipse to rotate

  with angular velocity $\dot{\theta} = \dfrac{ab\,\zeta_0}{(a+b)^2}$

- Shear $\boldsymbol{u}_0 = -Sx\,\boldsymbol{e}_y$ deforms the ellipse according to

$$\frac{\dot{a}}{a} = -\frac{\dot{b}}{b} = S\sin\theta\cos\theta \qquad \dot{\theta} = \frac{S(b^2\cos^2\theta - a^2\sin^2\theta)}{a^2 - b^2}$$

Incompressible fluid dynamics in shearing sheet

- Combine effects:

$$\frac{\dot{a}}{a} = -\frac{\dot{b}}{b} = S \sin\theta \cos\theta$$

$$\dot{\theta} = \frac{S(b^2 \cos^2\theta - a^2 \sin^2\theta)}{a^2 - b^2} + \frac{ab\,\zeta_0}{(a+b)^2}$$

- Area πab is conserved. Rewrite in terms of aspect ratio $r = \dfrac{a}{b}$:

$$\frac{\dot{r}}{r} = 2S \sin\theta \cos\theta$$

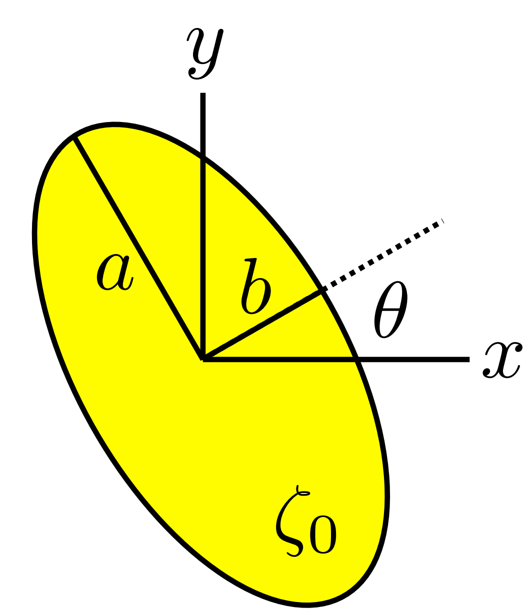
$$\dot{\theta} = \frac{S(\cos^2\theta - r^2 \sin^2\theta)}{r^2 - 1} + \frac{r\,\zeta_0}{(r+1)^2}$$

- 2D autonomous dynamical system
- Chaplygin (1899); Moore & Saffman (1971); Kida (1981)
- Note that ζ_0 is the vorticity perturbation relative to the background

Incompressible fluid dynamics in shearing sheet

$$\frac{\dot{r}}{r} = 2S\sin\theta\cos\theta$$

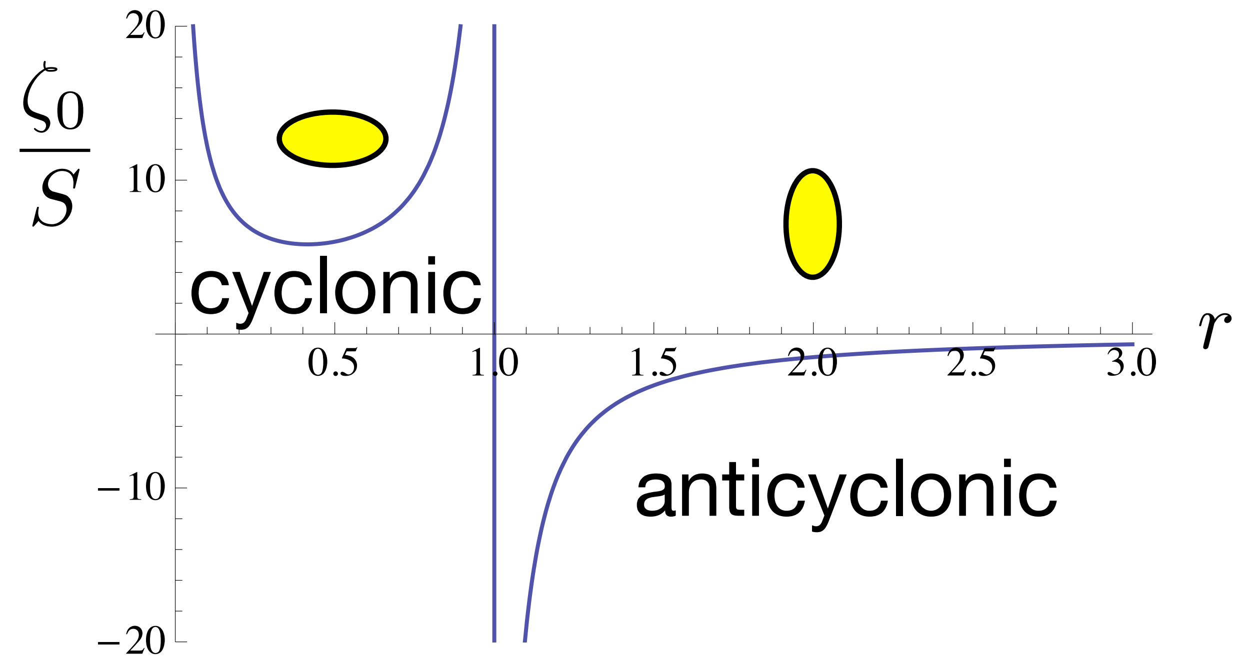
$$\dot{\theta} = \frac{S(\cos^2\theta - r^2\sin^2\theta)}{r^2-1} + \frac{r\,\zeta_0}{(r+1)^2}$$



- Fixed points:

$\theta = 0$ without loss of generality (let $r < 1$ if need be)

$$\frac{S}{r^2-1} + \frac{r\,\zeta_0}{(r+1)^2} = 0 \qquad \Rightarrow \qquad \frac{\zeta_0}{S} = -\frac{(r+1)}{r(r-1)}$$



Incompressible fluid dynamics in shearing sheet

- Stability of fixed point $\theta = 0$: linearized equations:

$$\dot{\delta r} = 2Sr\,\delta\theta$$

$$\dot{\delta\theta} = S\,\delta r\frac{\partial}{\partial r}\left[\frac{1}{r^2-1} + \frac{r}{(r+1)^2}\frac{\zeta_0}{S}\right] = S\,\delta r\frac{\partial f}{\partial r}$$

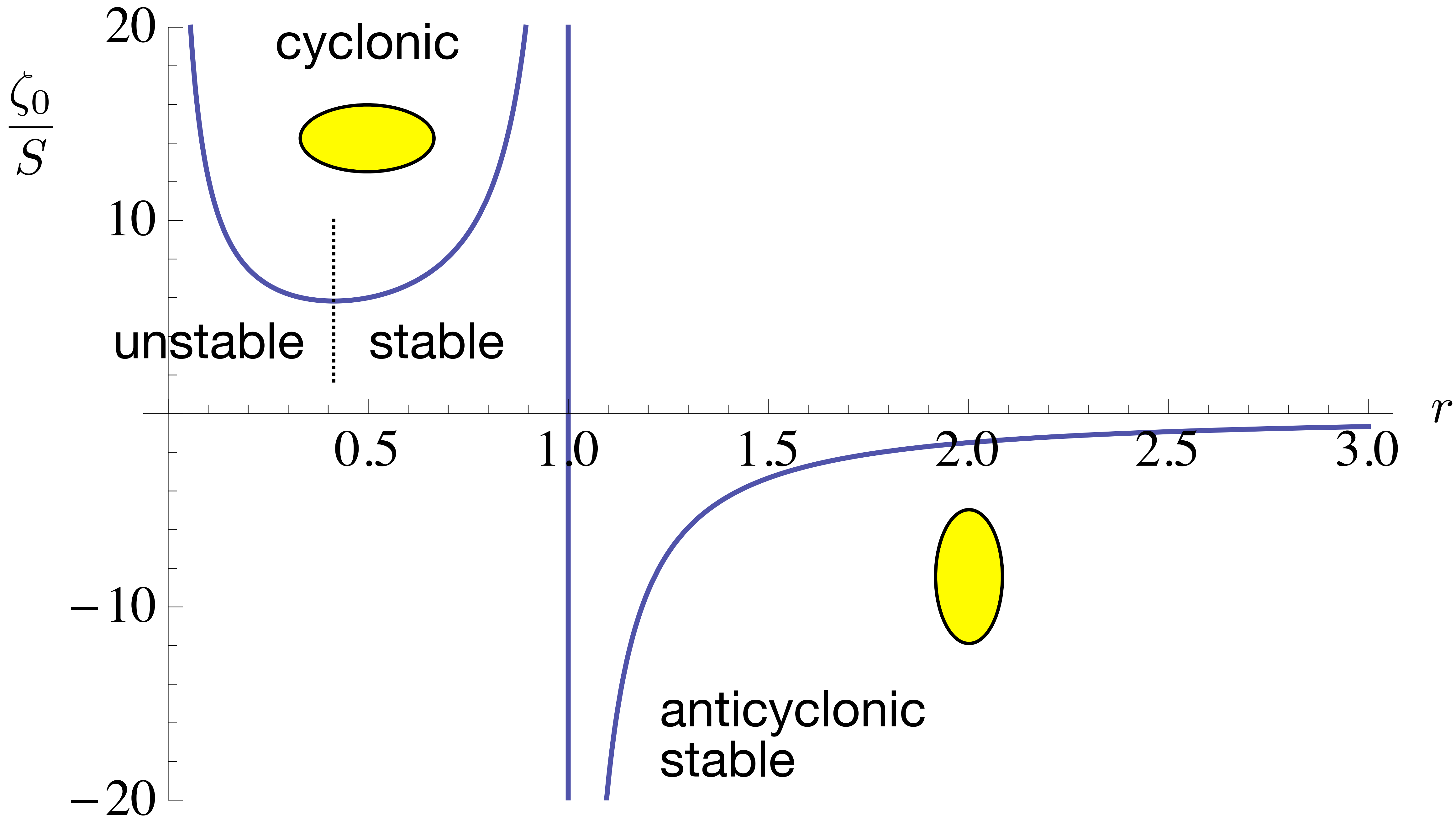
($f = 0$ at equilibrium)

$$\Rightarrow \quad \ddot{\delta r} = 2S^2 r\frac{\partial f}{\partial r}\,\delta r \qquad\qquad \frac{\partial f}{\partial r} = -\frac{(r^2+2r-1)}{r(r^2-1)^2}$$

- Unstable if $\dfrac{\partial f}{\partial r} > 0$, i.e. $r < \sqrt{2} - 1$
- Stable if $\dfrac{\partial f}{\partial r} < 0$, i.e. $r > \sqrt{2} - 1$

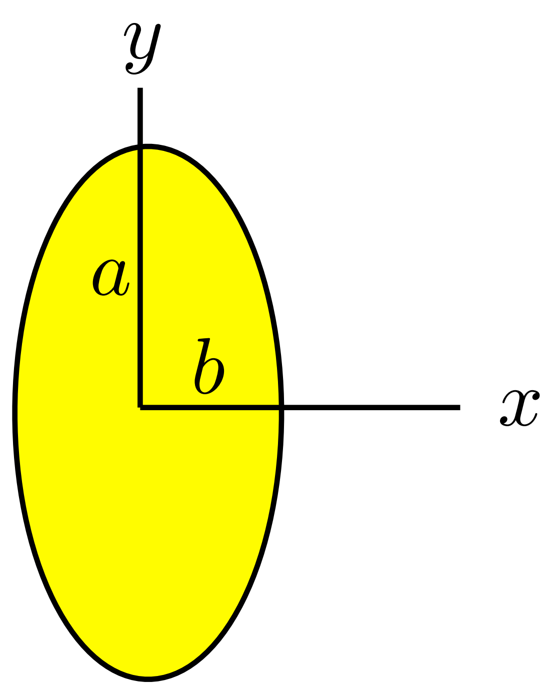
- Other instabilities exist, e.g. elliptical instability (3D)

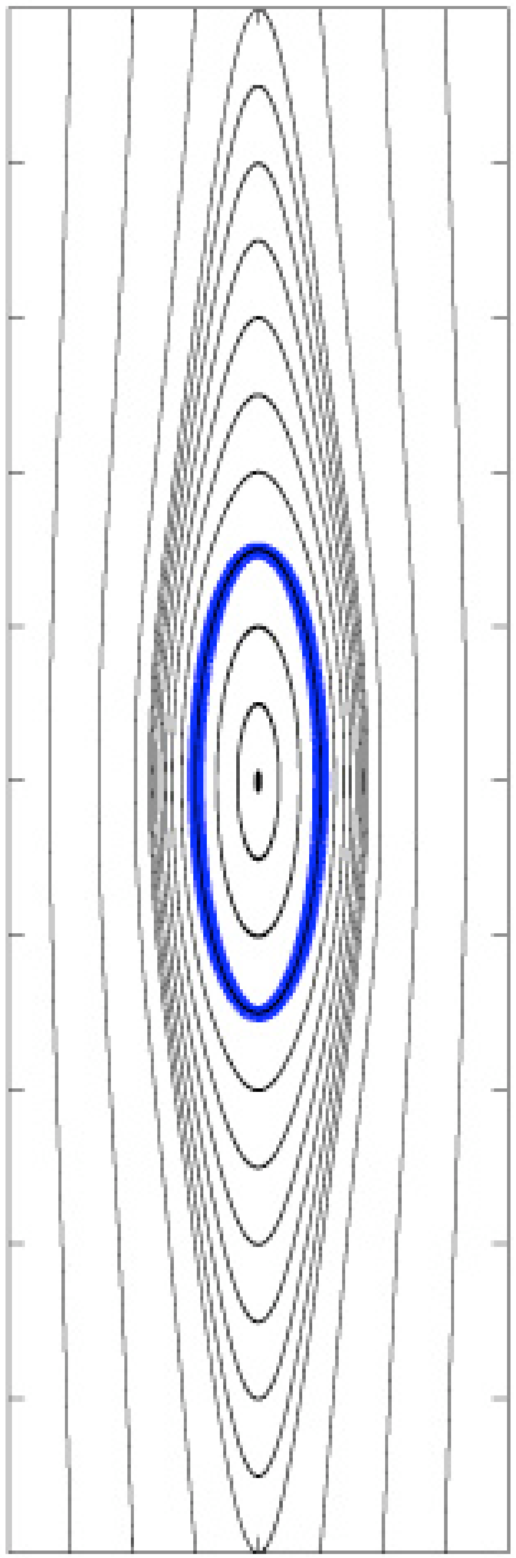
Incompressible fluid dynamics in shearing sheet



Incompressible fluid dynamics in shearing sheet

- Particle dynamics in core of steady elliptical vortex





- Total streamfunction
  (nested elliptical streamlines)

$$\propto \frac{x^2}{b^2} + \frac{y^2}{a^2}$$

$$\propto r^2 x^2 + y^2$$

so $\boldsymbol{u} \propto (y, -r^2 x)$

- Let $\boldsymbol{u} = A\left(\frac{y}{r}, -rx\right)$

$$(\boldsymbol{\nabla} \times \boldsymbol{u})_z = -A\left(\frac{1}{r} + r\right) = -S + \zeta_0 = -\frac{(r^2+1)}{r(r-1)} S$$

$$\Rightarrow \quad A = \frac{S}{r-1}$$

Incompressible fluid dynamics in shearing sheet

- Motion of particle subject to drag force:

$$\ddot{x} - 2\Omega\dot{y} = 2\Omega S x - \gamma(\dot{x} - u_x)$$

$$\ddot{y} + 2\Omega\dot{x} = -\gamma(\dot{y} - u_y)$$

$$\boldsymbol{u} = A\left(\frac{y}{r}, -rx\right)$$

$$A = \frac{S}{r-1}$$

- Linear system: solutions $x,\, y \propto \mathrm{e}^{\lambda t}$:

$$(\lambda^2 - 2\Omega S + \gamma\lambda)x = (2\Omega\lambda + \gamma A r^{-1})y$$

$$(\lambda^2 + \gamma\lambda)y = -(2\Omega\lambda + \gamma A r)x$$

$$(\lambda^2 - 2\Omega S + \gamma\lambda)(\lambda^2 + \gamma\lambda) + (2\Omega\lambda + \gamma A r^{-1})(2\Omega\lambda + \gamma A r) = 0$$

$$\lambda^4 + 2\gamma\lambda^3 + (4\Omega^2 - 2\Omega S + \gamma^2)\lambda^2 + \left[-2\Omega S + 2\Omega A(r + r^{-1})\right]\gamma\lambda + \gamma^2 A^2 = 0$$

$$\lambda^4 + 2\gamma\lambda^3 + (\kappa^2 + \gamma^2)\lambda^2 - 2\Omega\zeta_0\gamma\lambda + \gamma^2 A^2 = 0$$

Incompressible fluid dynamics in shearing sheet

$$\lambda^4 + 2\gamma\lambda^3 + (\kappa^2 + \gamma^2)\lambda^2 - 2\Omega\zeta_0\gamma\lambda + \gamma^2 A^2 = 0$$

- Limit of small γ (weak drag; large particles):
  - $\lambda \sim \pm \mathrm{i}\kappa + c_1\gamma + O(\gamma^2)$ $\qquad c_1 = -1 - \dfrac{\Omega\zeta_0}{\kappa^2}$
  - $\lambda \sim c_2\gamma + O(\gamma^2)$ $\qquad c_2 = \dfrac{\Omega\zeta_0}{\kappa^2} \pm \left(\dfrac{\Omega^2\zeta_0^2}{\kappa^4} - \dfrac{A^2}{\kappa^2}\right)^{1/2}$
  - For stability (decay to centre), require

    $-\kappa^2 < \Omega\zeta_0 < 0$ (must be anticyclonic)
- Limit of large γ (strong drag; small particles):
  - $\lambda \sim c_3\gamma + O(1)$ $\qquad c_3 = -1$
  - $\lambda \sim \pm \mathrm{i}A + c_4\gamma^{-1} + O(\gamma^{-2})$ $\qquad c_4 = \Omega\zeta_0 + A^2$
  - For stability (decay to centre), require

    $\Omega\zeta_0 < -A^2$

Incompressible fluid dynamics in shearing sheet

$$\lambda^4 + 2\gamma\lambda^3 + (\kappa^2 + \gamma^2)\lambda^2 - 2\Omega\zeta_0\gamma\lambda + \gamma^2 A^2 = 0$$

- For general γ, when does marginal stability occur?
  - $\lambda = 0$: never
  - $\lambda = -\mathrm{i}\omega,\ \omega \in \mathbf{R},\ \omega \neq 0$:

$$\omega^4 - (\kappa^2 + \gamma^2)\omega^2 + \gamma^2 A^2 = 0$$

$$2\gamma\omega^3 + 2\Omega\zeta_0\gamma\omega = 0$$

$$\Rightarrow\ \omega^2 = -\Omega\zeta_0\ (> 0)$$ (must be anticyclonic)

$$(\Omega\zeta_0)^2 + (\kappa^2 + \gamma^2)\Omega\zeta_0 + \gamma^2 A^2 = 0$$

- LHS is negative for all γ, so all particles decay to centre, if

$$A^2 < -\Omega\zeta_0 < \kappa^2$$ (agrees with two limits considered)

$$\frac{S^2}{(r-1)^2} < \frac{(r+1)\Omega S}{r(r-1)} < 2\Omega(2\Omega - S)$$

Incompressible fluid dynamics in shearing sheet

$$\frac{S^2}{(r-1)^2} < \frac{(r+1)\Omega S}{r(r-1)} < 2\Omega(2\Omega - S)$$

- Keplerian disc:

$$\frac{9}{4}\frac{1}{(r-1)^2} < \frac{3}{2}\frac{(r+1)}{r(r-1)} < 1$$

$$\Rightarrow \quad r > 3$$

2D compressible dynamics in shearing sheet

- 2D compressible sheet: inviscid, self-gravitating
- Surface density $\Sigma(x, y, t)$
- 2D pressure $P(x, y, t)$
  - Relate to vertically integrated quantities $\int \rho \, \mathrm{d}z, \ \int p \, \mathrm{d}z$

    but only a model, not derivable exactly from 3D equations
- Basic equations:

$$\frac{\partial \Sigma}{\partial t} + \boldsymbol{\nabla} \cdot (\Sigma \boldsymbol{u}) = 0 \qquad\qquad \Phi = -\Omega S x^2$$

$$\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u} \cdot \boldsymbol{\nabla} \boldsymbol{u} + 2\boldsymbol{\Omega} \times \boldsymbol{u} = -\boldsymbol{\nabla} \Phi - \boldsymbol{\nabla} \Phi_{\mathrm{d,m}} - \frac{1}{\Sigma} \boldsymbol{\nabla} P$$

- Disc potential $\Phi_{\mathrm{d}}(x, y, z, t)$ satisfies $\nabla^2 \Phi_{\mathrm{d}} = 4\pi G \Sigma \, \delta(z)$
- Then evaluate in midplane: $\Phi_{\mathrm{d,m}}(x, y, t) = \Phi_{\mathrm{d}}(x, y, 0, t)$
- Assume barotropic relation $P = P(\Sigma)$ for simplicity

2D compressible dynamics in shearing sheet

- Solve Poisson's equation in Fourier domain:

$$\tilde{\Sigma}(k_x, k_y, t) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \Sigma(x, y, t)\, \mathrm{e}^{-\mathrm{i}k_x x - \mathrm{i}k_y y}\, \mathrm{d}x\, \mathrm{d}y$$

etc.

$$\nabla^2 \Phi_\mathrm{d} = 4\pi G \Sigma\, \delta(z)$$

$$\Rightarrow \left(-k^2 + \frac{\partial^2}{\partial z^2}\right) \tilde{\Phi}_\mathrm{d} = 4\pi G \tilde{\Sigma}\, \delta(z) \qquad k = (k_x^2 + k_y^2)^{1/2}$$

$$\Rightarrow \tilde{\Phi}_\mathrm{d} = -\frac{2\pi G \tilde{\Sigma}}{k}\, \mathrm{e}^{-k|z|} \quad (k \neq 0)$$

so that $\left[\frac{\partial \tilde{\Phi}_\mathrm{d}}{\partial z}\right]_{0-}^{0+} = 4\pi G \tilde{\Sigma}$

$$\Rightarrow \tilde{\Phi}_\mathrm{d,m} = -\frac{2\pi G \tilde{\Sigma}}{k}$$

- $k = 0$ component gives no horizontal force anyway

2D compressible dynamics in shearing sheet

- Conservation of potential vorticity / "vortensity":

$$\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u}\cdot\boldsymbol{\nabla}\boldsymbol{u} + 2\boldsymbol{\Omega}\times\boldsymbol{u} = -\boldsymbol{\nabla}\Phi - \boldsymbol{\nabla}\Phi_{\mathrm{d,m}} - \frac{1}{\Sigma}\boldsymbol{\nabla}P$$

- Use identity $(\boldsymbol{\nabla}\times\boldsymbol{u})\times\boldsymbol{u} = \boldsymbol{u}\cdot\boldsymbol{\nabla}\boldsymbol{u} - \boldsymbol{\nabla}(\tfrac{1}{2}|\boldsymbol{u}|^2)$:

$$\frac{\partial \boldsymbol{u}}{\partial t} + [(2\boldsymbol{\Omega} + \boldsymbol{\nabla}\times\boldsymbol{u})\times\boldsymbol{u}] = \boldsymbol{\nabla}(\cdots) \qquad \text{since } P = P(\Sigma)$$

$$\frac{\partial}{\partial t}(\boldsymbol{\nabla}\times\boldsymbol{u}) + \boldsymbol{\nabla}\times[(2\boldsymbol{\Omega} + \boldsymbol{\nabla}\times\boldsymbol{u})\times\boldsymbol{u}] = \boldsymbol{0}$$

- Use identity $\boldsymbol{\nabla}\times(\boldsymbol{A}\times\boldsymbol{B}) = \boldsymbol{B}\cdot\boldsymbol{\nabla}\boldsymbol{A} - \boldsymbol{A}\cdot\boldsymbol{\nabla}\boldsymbol{B} + \boldsymbol{A}(\boldsymbol{\nabla}\cdot\boldsymbol{B}) - \boldsymbol{B}(\boldsymbol{\nabla}\cdot\boldsymbol{A})$:

$$\left(\frac{\partial}{\partial t} + \boldsymbol{u}\cdot\boldsymbol{\nabla}\right)(2\boldsymbol{\Omega} + \boldsymbol{\nabla}\times\boldsymbol{u}) = -(2\boldsymbol{\Omega} + \boldsymbol{\nabla}\times\boldsymbol{u})(\boldsymbol{\nabla}\cdot\boldsymbol{u}) \quad \text{since 2D}$$

$$= (2\boldsymbol{\Omega} + \boldsymbol{\nabla}\times\boldsymbol{u})\frac{1}{\Sigma}\left(\frac{\partial}{\partial t} + \boldsymbol{u}\cdot\boldsymbol{\nabla}\right)\Sigma$$

$$\Rightarrow \quad \frac{\mathrm{D}q}{\mathrm{D}t} = 0 \qquad \text{where } q = \frac{2\Omega + (\boldsymbol{\nabla}\times\boldsymbol{u})_z}{\Sigma}$$

2D compressible dynamics in shearing sheet

- Conservation of potential vorticity / "vortensity":

$$\frac{\mathrm{D}q}{\mathrm{D}t} = 0 \qquad \text{where} \quad q = \frac{2\Omega + (\boldsymbol{\nabla} \times \boldsymbol{u})_z}{\Sigma}$$

- Recall $\boldsymbol{u} = -Sx\,\boldsymbol{e}_y + \boldsymbol{v}$:

$$\left(\frac{\partial}{\partial t} - Sx\,\frac{\partial}{\partial y} + \boldsymbol{v} \cdot \boldsymbol{\nabla}\right) q = 0 \qquad\qquad q = \frac{2\Omega - S + (\boldsymbol{\nabla} \times \boldsymbol{v})_z}{\Sigma}$$

- Unlike incompressible 2D case, vortex dynamics not the whole story
- Vortical disturbances are coupled to acoustic ones

2D compressible dynamics in shearing sheet

- Linear stability of uniform 2D self-gravitating sheet

$$\Phi = -\Omega S x^2$$

$$\frac{\partial \Sigma}{\partial t} + \boldsymbol{\nabla} \cdot (\Sigma \boldsymbol{u}) = 0$$

$$\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u} \cdot \boldsymbol{\nabla} \boldsymbol{u} + 2\boldsymbol{\Omega} \times \boldsymbol{u} = -\boldsymbol{\nabla}\Phi - \boldsymbol{\nabla}\Phi_{\mathrm{d,m}} - \frac{1}{\Sigma}\boldsymbol{\nabla} P$$

$$\nabla^2 \Phi_{\mathrm{d}} = 4\pi G \Sigma\, \delta(z)$$

- Basic state: $\Sigma = \mathrm{cst}, \quad \boldsymbol{u} = -Sx\, \boldsymbol{e}_y$
- Linearized equations for perturbations Σ', $\boldsymbol{v}$, etc.:

$$P' = v_{\mathrm{s}}^2 \Sigma'$$

$$\left(\frac{\partial}{\partial t} - Sx \frac{\partial}{\partial y}\right) \Sigma' + \Sigma \boldsymbol{\nabla} \cdot \boldsymbol{v} = 0$$

$$\left(\frac{\partial}{\partial t} - Sx \frac{\partial}{\partial y}\right) \boldsymbol{v} - S v_x\, \boldsymbol{e}_y + 2\boldsymbol{\Omega} \times \boldsymbol{v} = -\boldsymbol{\nabla}\Phi'_{\mathrm{d,m}} - \frac{1}{\Sigma}\boldsymbol{\nabla} P'$$

$$\nabla^2 \Phi'_{\mathrm{d}} = 4\pi G \Sigma'\, \delta(z)$$

sound speed v_{s}

2D compressible dynamics in shearing sheet

- Solutions are shearing waves:

$$\Sigma'(\boldsymbol{x},t) = \mathrm{Re}\left\{\tilde{\Sigma}'(t)\exp[\mathrm{i}\boldsymbol{k}(t)\cdot\boldsymbol{x}]\right\} \qquad \text{etc.}$$

- Amplitude equations:

$$\frac{\mathrm{d}\tilde{\Sigma}'}{\mathrm{d}t} + \Sigma\,\mathrm{i}\boldsymbol{k}\cdot\tilde{\boldsymbol{v}} = 0$$

$$\frac{\mathrm{d}\tilde{v}_x}{\mathrm{d}t} - 2\Omega\tilde{v}_y = -\mathrm{i}k_x\left(\tilde{\Phi}'_{\mathrm{d,m}} + v_{\mathrm{s}}^2\frac{\tilde{\Sigma}'}{\Sigma}\right)$$

$$\frac{\mathrm{d}\tilde{v}_y}{\mathrm{d}t} + (2\Omega - S)\tilde{v}_x = -\mathrm{i}k_y\left(\tilde{\Phi}'_{\mathrm{d,m}} + v_{\mathrm{s}}^2\frac{\tilde{\Sigma}'}{\Sigma}\right)$$

$$\tilde{\Phi}'_{\mathrm{d,m}} = -\frac{2\pi G\tilde{\Sigma}'}{k}$$

- Vortensity perturbation $\tilde{q}' = \dfrac{\mathrm{i}k_x\tilde{v}_y - \mathrm{i}k_y\tilde{v}_x}{\Sigma} - \dfrac{(2\Omega - S)\tilde{\Sigma}'}{\Sigma^2}$

  satisfies $\dfrac{\mathrm{d}\tilde{q}'}{\mathrm{d}t} = 0$ as expected [exercise]

2D compressible dynamics in shearing sheet

- Consider axisymmetric waves: $k_y = 0, \ k_x = \mathrm{cst}, \ k = |k_x|$
- Amplitudes $\propto \mathrm{e}^{-\mathrm{i}\omega t}$

$$-\mathrm{i}\omega\tilde{\Sigma}' + \Sigma\,\mathrm{i}k_x\tilde{v}_x = 0$$

$$-\mathrm{i}\omega\tilde{v}_x - 2\Omega\tilde{v}_y = -\mathrm{i}k_x\left(v_\mathrm{s}^2 - \frac{2\pi G\Sigma}{|k_x|}\right)\frac{\tilde{\Sigma}'}{\Sigma}$$

$$-\mathrm{i}\omega\tilde{v}_y + (2\Omega - S)\tilde{v}_x = 0$$

- Multiply second equation by $\mathrm{i}\omega$ and eliminate $\tilde{\Sigma}'$ and $\tilde{v}_y$:

$$\omega^2\tilde{v}_x - 2\Omega(2\Omega - S)\tilde{v}_x = k_x^2\left(v_\mathrm{s}^2 - \frac{2\pi G\Sigma}{|k_x|}\right)\tilde{v}_x$$

- Deduce dispersion relation for “density waves”:

$$\omega^2 = \kappa^2 - 2\pi G\Sigma|k_x| + v_\mathrm{s}^2 k_x^2$$

- Also vortical solution $\omega = 0, \ \tilde{v}_x = 0$: zonal flow / geostrophic flow

2D compressible dynamics in shearing sheet

- Dispersion relation for density waves:

$$\omega^2 = \kappa^2 - 2\pi G\Sigma|k_x| + v_{\mathrm{s}}^2 k_x^2$$

inertial — self-gravity — acoustic (restoring forces)

self-gravity (destabilizing)

- "Acoustic-inertial waves"
- Disc is unstable to axisymmetric disturbances if $\omega^2 < 0$ for some k_x
- ω^2 is minimized with respect to $|k_x|$ when

$$0 = -2\pi G\Sigma + 2\, v_{\mathrm{s}}^2 |k_x| \qquad \Rightarrow \quad |k_x| = \frac{\pi G\Sigma}{v_{\mathrm{s}}^2}$$

$$(\omega^2)_{\mathrm{min}} = \kappa^2 - \frac{(\pi G\Sigma)^2}{v_{\mathrm{s}}^2} = \kappa^2 \left(1 - \frac{1}{Q^2}\right)$$

- Gravitational instability if $Q < 1$, where $Q = \dfrac{v_{\mathrm{s}}\kappa}{\pi G\Sigma}$ (Toomre stability parameter)

2D compressible dynamics in shearing sheet

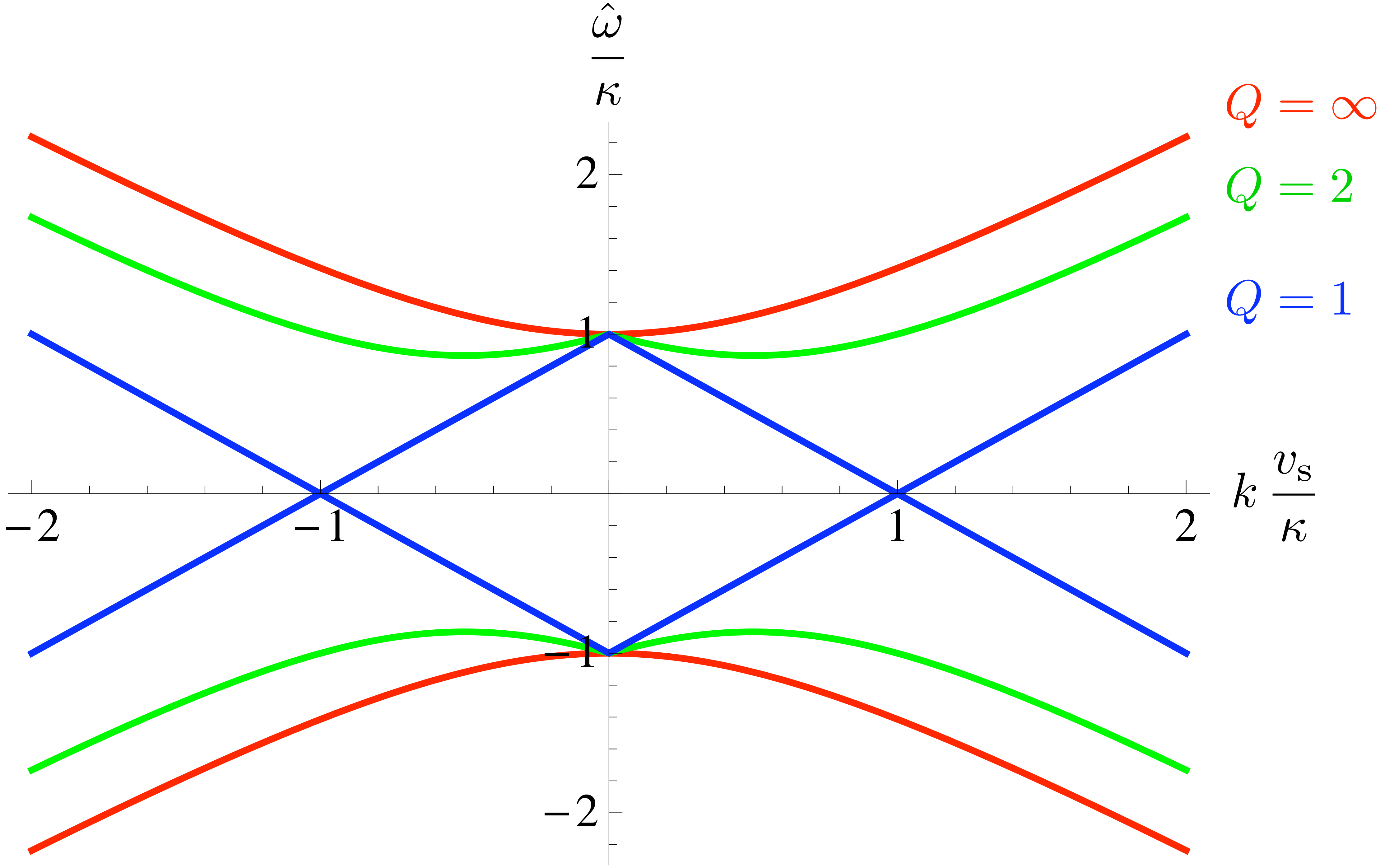
- Gravitational instability if $Q < 1$, where $Q = \dfrac{v_\mathrm{s}\kappa}{\pi G\Sigma}$
- Toomre stability parameter Q:
  - An inverse measure of self-gravity
  - A measure of temperature

$$Q = \frac{v_\mathrm{s}\kappa}{\pi G\Sigma}$$

product of stabilizing effects
(short and long scales)

destabilizing effect

2D compressible dynamics in shearing sheet



$$Q = \frac{v_{\rm s}\kappa}{\pi G \Sigma}$$

2D compressible dynamics in shearing sheet

- Occurrence of gravitational instability:
  - If $Q < 1$, disc tends to form rings (axisymmetric instability, exponential growth)
  - If $1 < Q \lesssim 1.5$, disc tends to form spiral waves or clumps (non-axisymmetric instability, transient growth)
- Since $Q \propto v_\mathrm{s} \propto T^{1/2}$, thermostatic regulation is possible: instability → motion → dissipation (shock/viscous) → heating
- Two possible outcomes of gravitational instability:
  - Fragmentation: formation of gravitationally bound objects (clumps...moonlets / planets / stars)
  - Gravitational turbulence: sustained activity of non-axisymmetric density waves (e.g. "self-gravity wakes" in Saturn's rings)
- Efficient cooling promotes fragmentation, or enhances the efficiency of gravitational turbulence, since cooling balances viscous heating

Satellite–disc interaction

- Common problem:
  - Orbiting companion, e.g. on circular orbit within disc
  - Gravitational (rather than hydrodynamic) interaction with disc
  - Perturbs orbital motion and excites waves
  - Calculate exchanges of energy and angular momentum
  - Determine orbital evolution of satellite (migration, etc.)

Satellite–disc interaction

- Test particle dynamics in xy plane, in local approximation (fluid dynamics more difficult, but results are similar in some ways)

$$\ddot{x} - 2\Omega\dot{y} = 2\Omega S x - \frac{\partial\Psi}{\partial x}$$

$$\ddot{y} + 2\Omega\dot{x} = -\frac{\partial\Psi}{\partial y}$$

- Satellite on circular orbit at reference radius $(x_\mathrm{s} = y_\mathrm{s} = 0)$:

$$\Psi = -GM_\mathrm{s}(x^2 + y^2)^{-1/2}$$

Satellite–disc interaction

$$\ddot{x} - 2\Omega\dot{y} = 2\Omega S x - \frac{\partial\Psi}{\partial x}$$

$$\ddot{y} + 2\Omega\dot{x} = -\frac{\partial\Psi}{\partial y}$$

- General solution in absence of satellite potential:

$$\dddot{x} = -4\Omega^2\dot{x} + 2\Omega S\dot{x} = -\kappa^2\dot{x}$$

$$\Rightarrow \quad x = x_0 + A_\mathrm{r}\cos\kappa t + A_\mathrm{i}\sin\kappa t = x_0 + \mathrm{Re}\left[A\,\mathrm{e}^{-\mathrm{i}\kappa t}\right]$$

$$y = y_0 - Sx_0 t - \frac{2\Omega}{\kappa}\,\mathrm{Re}\left[\mathrm{i}A\,\mathrm{e}^{-\mathrm{i}\kappa t}\right]$$

- Guiding centre $(x_0, y_0 - Sx_0 t)$
- Complex epicyclic amplitude $A = A_\mathrm{r} + \mathrm{i}A_\mathrm{i}$

Satellite–disc interaction

- To express "orbital elements" in terms of position and velocity:

$$x = x_0 + \mathrm{Re}\left[A\,\mathrm{e}^{-\mathrm{i}\kappa t}\right]$$
$$\dot{x} = \mathrm{Re}\left[-\mathrm{i}\kappa A\,\mathrm{e}^{-\mathrm{i}\kappa t}\right] = \kappa\,\mathrm{Im}\left[A\,\mathrm{e}^{-\mathrm{i}\kappa t}\right]$$
$$\ddot{x} = -\kappa^2\,\mathrm{Re}\left[A\,\mathrm{e}^{-\mathrm{i}\kappa t}\right]$$

$$\Rightarrow \quad A\,\mathrm{e}^{-\mathrm{i}\kappa t} = -\frac{\ddot{x}}{\kappa^2} + \frac{\mathrm{i}\dot{x}}{\kappa}$$

$$\Rightarrow \quad A = \left[-\frac{2\Omega}{\kappa^2}(\dot{y} + Sx) + \frac{\mathrm{i}\dot{x}}{\kappa}\right]\mathrm{e}^{\mathrm{i}\kappa t}$$

$$x_0 = x + \frac{\ddot{x}}{\kappa^2} = x + \frac{2\Omega}{\kappa^2}(\dot{y} + Sx) = \frac{2\Omega}{\kappa^2}(\dot{y} + 2\Omega x)$$

Satellite–disc interaction

- Canonical y momentum (per unit mass):

$$p_y = \dot{y} + 2\Omega x = \frac{\kappa^2}{2\Omega} x_0 = \mathrm{cst}$$

- Energy (per unit mass):

$$\varepsilon = \frac{1}{2}(\dot{x}^2 + \dot{y}^2) - \Omega S x^2$$

- Use $\kappa^2 |A|^2 = \dot{x}^2 + \frac{4\Omega^2}{\kappa^2}(\dot{y} + Sx)^2$:

$$\begin{aligned}
\varepsilon &= \frac{1}{2}\kappa^2 |A|^2 - \frac{2\Omega^2}{\kappa^2}(\dot{y} + Sx)^2 + \frac{1}{2}\dot{y}^2 - \Omega S x^2 \\
&= \frac{1}{2}\kappa^2 |A|^2 - \frac{\Omega S}{\kappa^2}(\dot{y} + 2\Omega x)^2 \\
&= \frac{1}{2}\kappa^2 |A|^2 - \frac{\Omega S}{\kappa^2} p_y^2 = \mathrm{cst}
\end{aligned}$$

Satellite–disc interaction

- In the presence of a satellite potential:

$$\dot{p}_y = -\frac{\partial\Psi}{\partial y}$$

$$\varepsilon + \Psi = \mathrm{cst}$$

$$\begin{aligned}\dot{A} &= \left[-\frac{2\Omega}{\kappa^2}(\ddot{y} + S\dot{x}) + \frac{\mathrm{i}\ddot{x}}{\kappa} - \frac{2\mathrm{i}\Omega}{\kappa}(\dot{y} + Sx) - \dot{x}\right]\mathrm{e}^{\mathrm{i}\kappa t} \\ &= \left[-\frac{2\Omega}{\kappa^2}(\ddot{y} + 2\Omega\dot{x}) + \frac{\mathrm{i}}{\kappa}(\ddot{x} - 2\Omega\dot{y} - 2\Omega Sx)\right]\mathrm{e}^{\mathrm{i}\kappa t} \\ &= \left(\frac{2\Omega}{\kappa^2}\frac{\partial\Psi}{\partial y} - \frac{\mathrm{i}}{\kappa}\frac{\partial\Psi}{\partial x}\right)\mathrm{e}^{\mathrm{i}\kappa t}\end{aligned}$$

Satellite–disc interaction

- Consider the unperturbed "circular" orbit ($A=0$)

$$x = x_0 = \mathrm{cst}$$

$$y = -Sx_0t$$

- Calculate ΔA in linear approximation:

$$\dot{A} = \left(\frac{2\Omega}{\kappa^2}\frac{\partial\Psi}{\partial y} - \frac{\mathrm{i}}{\kappa}\frac{\partial\Psi}{\partial x}\right)\mathrm{e}^{\mathrm{i}\kappa t} \qquad \Psi = -GM_\mathrm{s}(x^2+y^2)^{-1/2}$$

$$= GM_\mathrm{s}(x^2+y^2)^{-3/2}\left(\frac{2\Omega y}{\kappa^2} - \frac{\mathrm{i}x}{\kappa}\right)\mathrm{e}^{\mathrm{i}\kappa t}$$

$$\approx -\mathrm{i}\frac{GM_\mathrm{s}}{\kappa x_0^2}(1+S^2t^2)^{-3/2}\left(1 - \mathrm{i}\frac{2\Omega}{\kappa}St\right)\mathrm{e}^{\mathrm{i}\kappa t}$$

$$\Delta A = \int_{-\infty}^{\infty}\dot{A}\,\mathrm{d}t$$

$$= -\mathrm{i}\frac{GM_\mathrm{s}}{\kappa x_0^2}\int_{-\infty}^{\infty}(1+S^2t^2)^{-3/2}\left(\cos\kappa t + \frac{2\Omega}{\kappa}St\sin\kappa t\right)\mathrm{d}t$$

Satellite–disc interaction

$$\Delta A = -\mathrm{i}\frac{GM_\mathrm{s}}{\kappa x_0^2}\int_{-\infty}^{\infty}(1+S^2t^2)^{-3/2}\left(\cos\kappa t + \frac{2\Omega}{\kappa}St\sin\kappa t\right)\mathrm{d}t$$

- Let $f(k) = \int_{-\infty}^{\infty}(1+x^2)^{-3/2}\cos kx\,\mathrm{d}x = 2kK_1(k) \quad (k>0)$

  modified Bessel function

- Then

$$\Delta A = -\mathrm{i}C\frac{GM_\mathrm{s}}{\kappa S x_0^2} \qquad\qquad C = f\left(\frac{\kappa}{S}\right) - \frac{2\Omega}{\kappa}f'\left(\frac{\kappa}{S}\right)$$

- For Keplerian orbits ($\kappa/S = 2/3$), $\quad C \approx 3.359$
- So encounter with satellite excites an epicyclic oscillation at first order

Satellite–disc interaction

- Long before and after the encounter, $\Psi \to 0$
- Since $\varepsilon + \Psi$ is exactly conserved, $\Delta\varepsilon = 0$ in the encounter
- But $\varepsilon = \dfrac{1}{2}\kappa^2|A|^2 - \dfrac{\Omega S}{\kappa^2}p_y^2$, so $\Delta(p_y^2) = \dfrac{\kappa^4}{2\Omega S}\Delta(|A|^2)$
- Assume a "circular" orbit before the encounter:

$$A = 0, \quad p_y = \frac{\kappa^2}{2\Omega}x_0$$

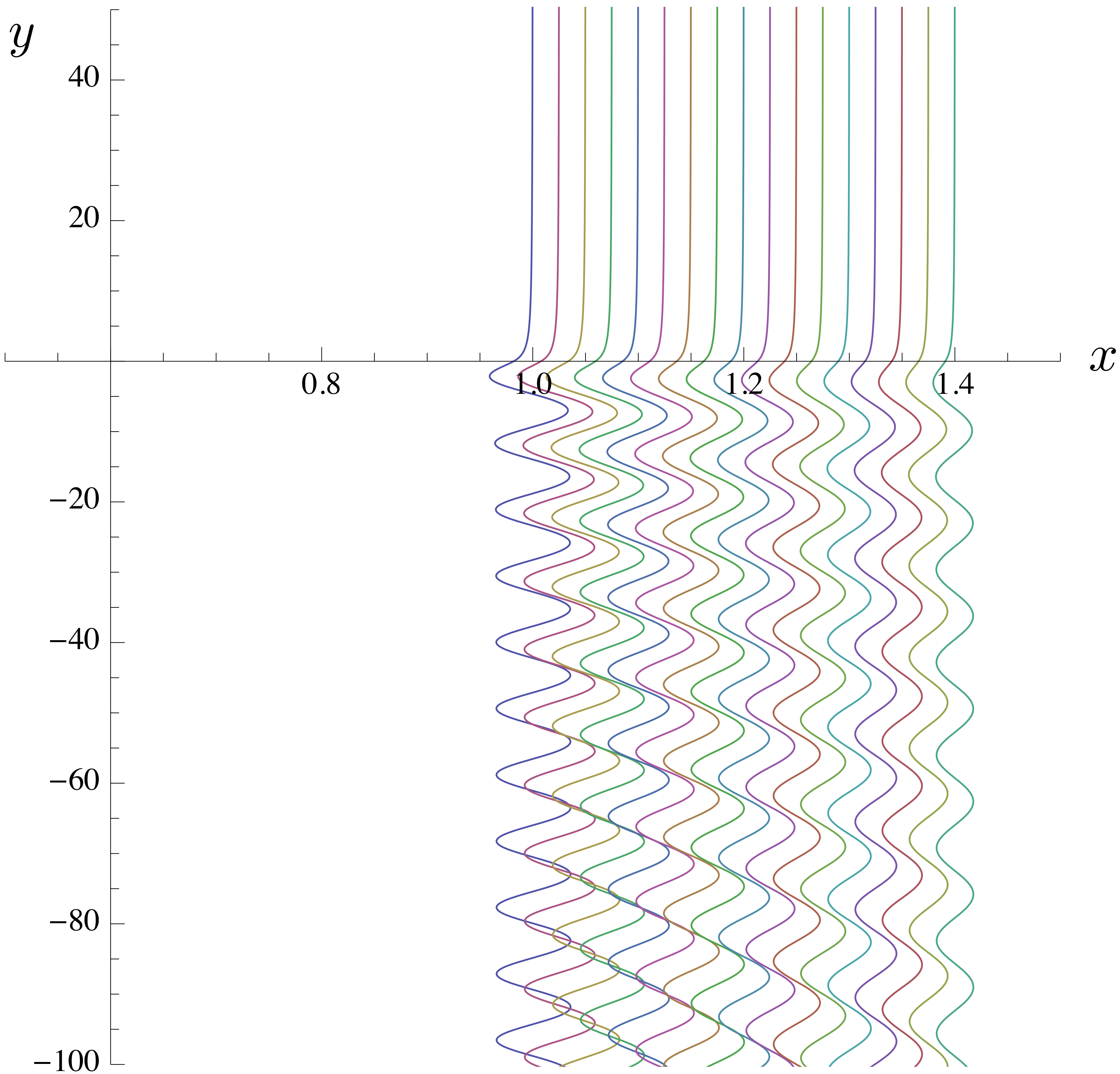
- Then, after the encounter:

$$A \approx -\mathrm{i}C\frac{GM_\mathrm{s}}{\kappa S x_0^2}, \quad p_y^2 \approx \frac{\kappa^4}{4\Omega^2}x_0^2 + \frac{\kappa^4}{2\Omega S}\left(C\frac{GM_\mathrm{s}}{\kappa S x_0^2}\right)^2$$

$$\Rightarrow \quad p_y \approx \frac{\kappa^2}{2\Omega}x_0 + \underbrace{\frac{(CGM_\mathrm{s})^2}{2S^3x_0^5}}_{\Delta p_y \text{ correct to second order}}$$

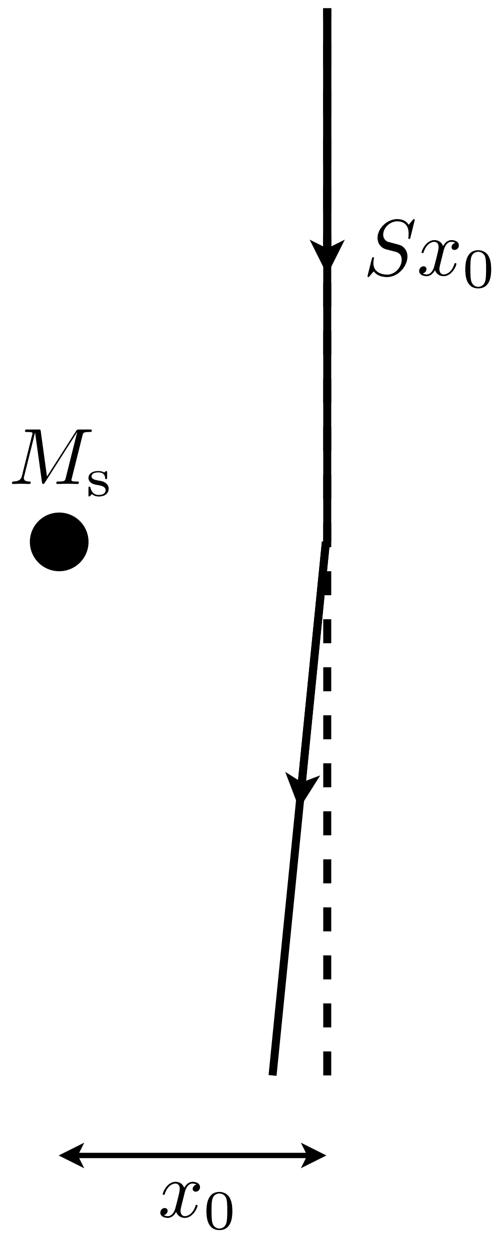
- Irreversibility / dissipation implicit in assuming circular initial orbit

Satellite–disc interaction



Satellite–disc interaction

- Simplified version: "impulse approximation"



$$\Delta v_\perp \approx \frac{GM_\mathrm{s}}{x_0^2}\frac{1}{S}$$

$$\Delta(v_\perp^2) + \Delta(v_\parallel^2) = 0 \quad \text{(energy)}$$

$$\left(\frac{GM_\mathrm{s}}{Sx_0^2}\right)^2 + 2Sx_0\Delta v_\parallel \approx 0$$

$$\Delta v_\parallel \approx -\frac{(GM_\mathrm{s})^2}{2S^3x_0^5} \qquad \text{(lacks } C^2 \text{ factor)}$$

Satellite–disc interaction

- y force on disc per unit x at location x :

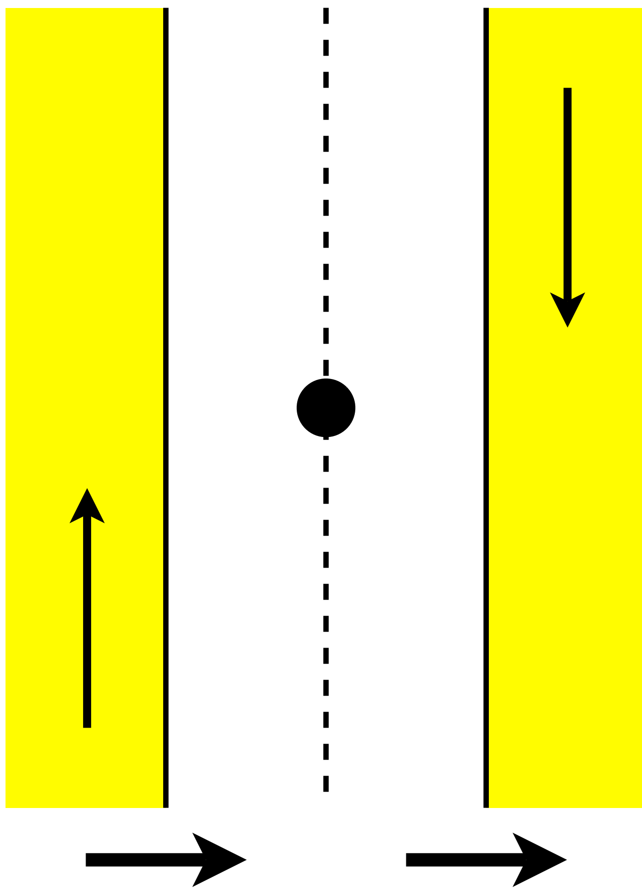
$$= \frac{(CGM_{\rm s})^2}{2S^3x^5}\,\underbrace{\Sigma}_{\text{surface density}}\,\underbrace{|Sx|}_{\text{encounter rate}}$$

$$\propto x^{-4}\,\mathrm{sgn}(x)$$

- Torque per unit radius is the same $\times r_0$
- Satellite experiences an equal an opposite torque
- Effect is of second order in $M_{\rm s}$
- Similar result for density waves (response of a fluid disc)
- x^{-4} divergence is moderated within $|x| \lesssim H$ (or Hill radius)

Satellite–disc interaction

- Gravitational interaction is "repulsive"!



transfer of p_y (or angular momentum)

- One-sided torque leads to gap opening if $M_{\rm s}$ large enough and ν small enough
- Asymmetry leads to net torque on satellite and to migration (usually inwards)

Satellite–disc interaction

- Now include periodic nature of y coordinate $(L_y = 2\pi r_0)$:

$$\dot{A} = \left(\frac{2\Omega}{\kappa^2}\frac{\partial\Psi}{\partial y} - \frac{\mathrm{i}}{\kappa}\frac{\partial\Psi}{\partial x}\right)\mathrm{e}^{\mathrm{i}\kappa t}$$

$$= F(t)\,\mathrm{e}^{\mathrm{i}\kappa t}$$

$$= \sum_{n=-\infty}^{\infty} f_n\,\mathrm{e}^{-\mathrm{i}n\omega t}\,\mathrm{e}^{\mathrm{i}\kappa t}$$

$$T = \frac{2\pi r_0}{S|x_0|}$$

$$\omega = \frac{2\pi}{T} = \frac{S|x_0|}{r_0}$$

- Add damping of epicyclic motion:

$$\dot{A} = \sum_{n=-\infty}^{\infty} f_n\,\mathrm{e}^{-\mathrm{i}n\omega t}\,\mathrm{e}^{\mathrm{i}\kappa t} - \gamma A$$

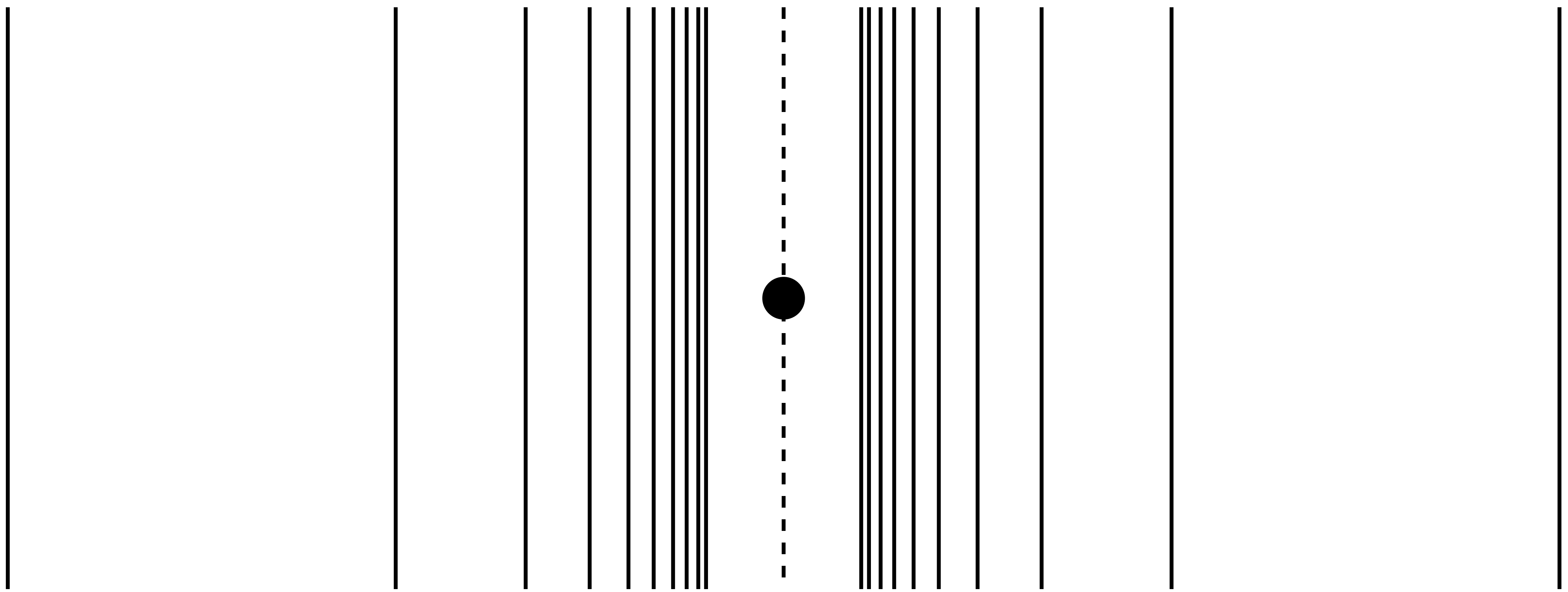
- Long-term response:

$$A = \sum_{n=-\infty}^{\infty} \frac{\mathrm{i}f_n\,\mathrm{e}^{-\mathrm{i}n\omega t}\,\mathrm{e}^{\mathrm{i}\kappa t}}{(n\omega-\kappa)+\mathrm{i}\gamma}$$

- “Lindblad resonances” where $\dfrac{x}{r_0} = \dfrac{1}{n}\dfrac{\kappa}{S}$, resolved by damping

Satellite–disc interaction

- Lindblad resonances: $\dfrac{x}{r_0} = \dfrac{1}{n}\dfrac{\kappa}{S}$



- In a Keplerian disc, LRs correspond to orbital commensurabilities

$$\frac{\Omega}{\Omega_0} = \frac{n}{n-1}$$

- In a fluid disc, density waves are launched there
  (wave emission resolves singularity in response)

Magnetorotational instability

- Homogeneous incompressible fluid
- Local approximation (shearing sheet / box)
- 3D system, unbounded or periodic in x, y, z
- Uniform kinematic viscosity ν and magnetic diffusivity η

$$\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u} \cdot \boldsymbol{\nabla} \boldsymbol{u} + 2\boldsymbol{\Omega} \times \boldsymbol{u} = -\boldsymbol{\nabla}\Phi - \frac{1}{\rho}\boldsymbol{\nabla}\Pi + \frac{1}{\mu_0 \rho}\boldsymbol{B} \cdot \boldsymbol{\nabla}\boldsymbol{B} + \nu\nabla^2 \boldsymbol{u}$$

$$\frac{\partial \boldsymbol{B}}{\partial t} + \boldsymbol{u} \cdot \boldsymbol{\nabla}\boldsymbol{B} = \boldsymbol{B} \cdot \boldsymbol{\nabla}\boldsymbol{u} + \eta\nabla^2 \boldsymbol{B} \qquad \Pi = p + \frac{|\boldsymbol{B}|^2}{2\mu_0}$$

$$\boldsymbol{\nabla} \cdot \boldsymbol{u} = 0 \qquad \boldsymbol{\nabla} \cdot \boldsymbol{B} = 0$$

neglect (balanced by pressure gradient)

- Effective potential $\Phi = -\Omega S x^2 + \cancel{\tfrac{1}{2}\Omega_z^2 z^2}$
- Basic state:

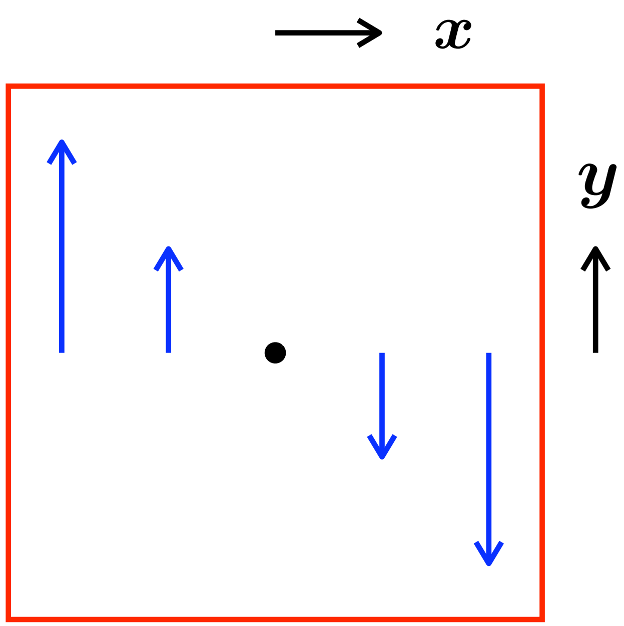
$$\boldsymbol{u} = \boldsymbol{u}_0 = -Sx\,\boldsymbol{e}_y \qquad \boldsymbol{B} = \boldsymbol{B}_0(t) \text{ with } \frac{\mathrm{d}\boldsymbol{B}_0}{\mathrm{d}t} = -SB_{x0}\,\boldsymbol{e}_y$$

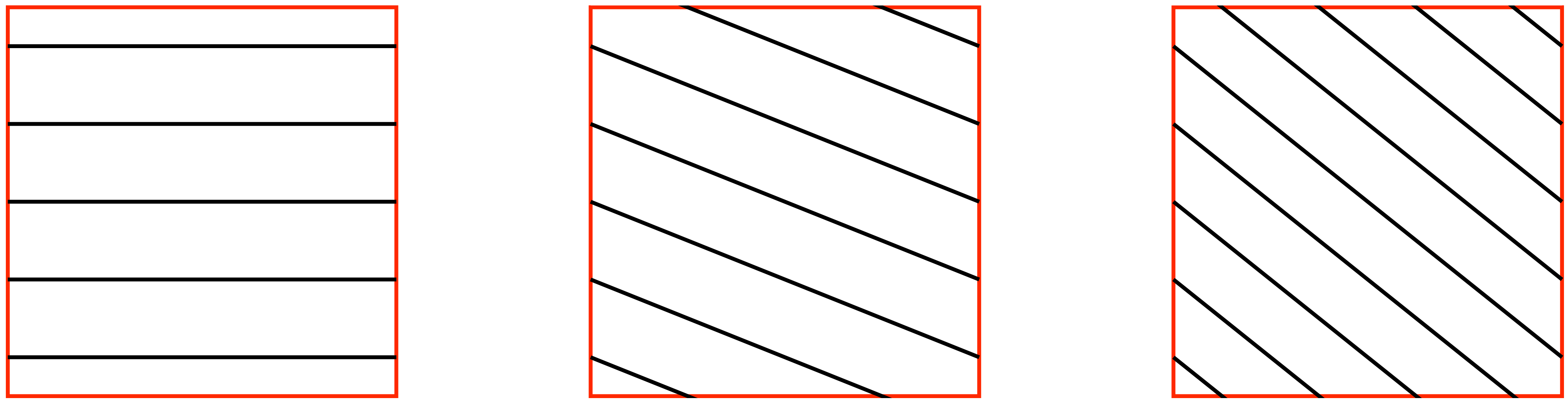
$$\Pi = \Pi_0 = \mathrm{cst} \qquad B_{x0} = \mathrm{cst} \qquad B_{y0} = \mathrm{cst} - SB_{x0}t \qquad B_{z0} = \mathrm{cst}$$

Magnetorotational instability

$$B_{x0} = \mathrm{cst} \qquad B_{y0} = \mathrm{cst} - SB_{x0}t \qquad B_{z0} = \mathrm{cst}$$

- Tilting / shearing of magnetic field:





Magnetorotational instability

- Perturbations in the form of shearing waves:

$$\boldsymbol{u} = \boldsymbol{u}_0 + \mathrm{Re}\,\{\tilde{\boldsymbol{v}}(t)\exp[\mathrm{i}\boldsymbol{k}(t)\cdot\boldsymbol{x}]\}$$

$$\boldsymbol{B} = \boldsymbol{B}_0 + (\mu_0\rho)^{-1/2}\mathrm{Re}\left\{\tilde{\boldsymbol{b}}(t)\exp[\mathrm{i}\boldsymbol{k}(t)\cdot\boldsymbol{x}]\right\}$$

$$\Pi = \Pi_0 + \rho\,\mathrm{Re}\left\{\tilde{\psi}(t)\exp[\mathrm{i}\boldsymbol{k}(t)\cdot\boldsymbol{x}]\right\} \qquad \text{with} \quad \frac{\mathrm{d}\boldsymbol{k}}{\mathrm{d}t} = Sk_y\,\boldsymbol{e}_x$$

- Nonlinear terms vanish because

$$\begin{aligned}\boldsymbol{v}\cdot\boldsymbol{\nabla}\boldsymbol{b} &= \mathrm{Re}\left[\tilde{\boldsymbol{v}}\,\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}}\right]\cdot\boldsymbol{\nabla}\,\mathrm{Re}\left[\tilde{\boldsymbol{b}}\,\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}}\right]\\ &= \mathrm{Re}\left[\boldsymbol{k}\cdot\tilde{\boldsymbol{v}}\,\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}}\right]\mathrm{Re}\left[\mathrm{i}\tilde{\boldsymbol{b}}\,\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}}\right]\\ &= 0\end{aligned}$$

because $\boldsymbol{\nabla}\cdot\boldsymbol{v} = 0 \quad \Rightarrow \quad \mathrm{i}\boldsymbol{k}\cdot\tilde{\boldsymbol{v}} = 0$

and similarly for $\boldsymbol{v}\cdot\boldsymbol{\nabla}\boldsymbol{v},\ \boldsymbol{b}\cdot\boldsymbol{\nabla}\boldsymbol{v},\ \boldsymbol{b}\cdot\boldsymbol{\nabla}\boldsymbol{b}$

Magnetorotational instability

- Amplitude equations:

$$
\begin{aligned}
\frac{\mathrm{d}\tilde{v}_x}{\mathrm{d}t} - 2\Omega\tilde{v}_y &= -\mathrm{i}k_x\tilde{\psi} + \mathrm{i}\omega_\mathrm{a}\tilde{b}_x - \nu k^2\tilde{v}_x \\
\frac{\mathrm{d}\tilde{v}_y}{\mathrm{d}t} + (2\Omega - S)\tilde{v}_x &= -\mathrm{i}k_y\tilde{\psi} + \mathrm{i}\omega_\mathrm{a}\tilde{b}_y - \nu k^2\tilde{v}_y \\
\frac{\mathrm{d}\tilde{v}_z}{\mathrm{d}t} &= -\mathrm{i}k_z\tilde{\psi} + \mathrm{i}\omega_\mathrm{a}\tilde{b}_z - \nu k^2\tilde{v}_z \\
\frac{\mathrm{d}\tilde{b}_x}{\mathrm{d}t} &= \mathrm{i}\omega_\mathrm{a}\tilde{v}_x - \eta k^2\tilde{b}_x \\
\frac{\mathrm{d}\tilde{b}_y}{\mathrm{d}t} &= -S\tilde{b}_x + \mathrm{i}\omega_\mathrm{a}\tilde{v}_y - \eta k^2\tilde{b}_y \\
\frac{\mathrm{d}\tilde{b}_z}{\mathrm{d}t} &= \mathrm{i}\omega_\mathrm{a}\tilde{v}_z - \eta k^2\tilde{b}_z \\
\mathrm{i}\boldsymbol{k}\cdot\tilde{\boldsymbol{v}} &= \mathrm{i}\boldsymbol{k}\cdot\tilde{\boldsymbol{b}} = 0
\end{aligned}
$$

- Alfvén frequency $\omega_\mathrm{a} = \boldsymbol{k}\cdot\boldsymbol{v}_\mathrm{a} = (\mu_0\rho)^{-1/2}\boldsymbol{k}\cdot\boldsymbol{B}_0$

Magnetorotational instability

- Alfvén frequency is constant:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{k}\cdot\boldsymbol{B}_0) &= \frac{\mathrm{d}\boldsymbol{k}}{\mathrm{d}t}\cdot\boldsymbol{B}_0+\boldsymbol{k}\cdot\frac{\mathrm{d}\boldsymbol{B}_0}{\mathrm{d}t}\\ &= Sk_y\,\boldsymbol{e}_x\cdot\boldsymbol{B}_0+\boldsymbol{k}\cdot(-SB_{x0}\,\boldsymbol{e}_y)\\ &= 0\end{aligned}$$

- Alfvén frequency measures the restoring effect of magnetic tension (amount of bending of field lines)

Magnetorotational instability

- General shearing waves require numerical solution
- Consider purely horizontal disturbances with a vertical wavevector:

$$k_x = k_y = 0 \qquad \tilde{v}_z = \tilde{b}_z = \tilde{\psi} = 0$$

- Amplitude equations have constant coefficients
- Solutions $\propto \mathrm{e}^{-\mathrm{i}\omega t}$, instability if $\mathrm{Im}(\omega) > 0$

$$\begin{aligned}
-\mathrm{i}\omega\tilde{v}_x - 2\Omega\tilde{v}_y &= \mathrm{i}\omega_\mathrm{a}\tilde{b}_x - \nu k^2\tilde{v}_x \\
-\mathrm{i}\omega\tilde{v}_y + (2\Omega - S)\tilde{v}_x &= \mathrm{i}\omega_\mathrm{a}\tilde{b}_y - \nu k^2\tilde{v}_y \\
-\mathrm{i}\omega\tilde{b}_x &= \mathrm{i}\omega_\mathrm{a}\tilde{v}_x - \eta k^2\tilde{b}_x \\
-\mathrm{i}\omega\tilde{b}_y &= -S\tilde{b}_x + \mathrm{i}\omega_\mathrm{a}\tilde{v}_y - \eta k^2\tilde{b}_y
\end{aligned}$$

- Set determinant to zero: magnetorotational dispersion relation

$$[(\omega + \mathrm{i}\nu k^2)(\omega + \mathrm{i}\eta k^2) - \omega_\mathrm{a}^2]^2 - \underbrace{2\Omega(2\Omega - S)}_{\kappa^2}(\omega + \mathrm{i}\eta k^2)^2 - 2\Omega S\omega_\mathrm{a}^2 = 0$$

Magnetorotational instability

$$[(\omega+\mathrm{i}\nu k^2)(\omega+\mathrm{i}\eta k^2)-\omega_\mathrm{a}^2]^2-2\Omega\underbrace{(2\Omega-S)}_{\kappa^2}(\omega+\mathrm{i}\eta k^2)^2-2\Omega S\omega_\mathrm{a}^2=0$$

- Case of zero magnetic field (or no bending of field, $\omega_\mathrm{a}=0$):

$$\omega=\pm\kappa-\mathrm{i}\nu k^2$$ (epicyclic oscillation with viscous damping)

Magnetorotational instability

$$[(\omega + \mathrm{i}\nu k^2)(\omega + \mathrm{i}\eta k^2) - \omega_\mathrm{a}^2]^2 - 2\Omega\underbrace{(2\Omega - S)}_{\kappa^2}(\omega + \mathrm{i}\eta k^2)^2 - 2\Omega S\omega_\mathrm{a}^2 = 0$$

- Case of ideal MHD ($\nu = \eta = 0$):

$$\omega^4 - (2\omega_\mathrm{a}^2 + \kappa^2)\omega^2 + \omega_\mathrm{a}^2(\omega_\mathrm{a}^2 - 2\Omega S) = 0$$

$$\Rightarrow \ \omega^2 = \omega_\mathrm{a}^2 + \frac{1}{2}\kappa^2\left[1 \pm \left(1 + \frac{16\omega_\mathrm{a}^2\Omega^2}{\kappa^4}\right)^{1/2}\right]$$

- Assume that $\kappa^2 > 0$, otherwise system is hydrodynamically unstable
- Both roots for ω^2 are real and at least one is positive
- Instability occurs if and only if product of roots < 0, i.e.

$$0 < \omega_\mathrm{a}^2 < 2\Omega S$$

(Chandrasekhar’s criterion for “magnetorotational instability / MRI”)
(Velikhov 1959; Chandrasekhar 1960; ... ; Balbus & Hawley 1991)

Magnetorotational instability

- Unstable root:

$$\omega^2 = \omega_\mathrm{a}^2 + \frac{1}{2}\kappa^2\left[1 - \left(1 + \frac{16\omega_\mathrm{a}^2\Omega^2}{\kappa^4}\right)^{1/2}\right]$$

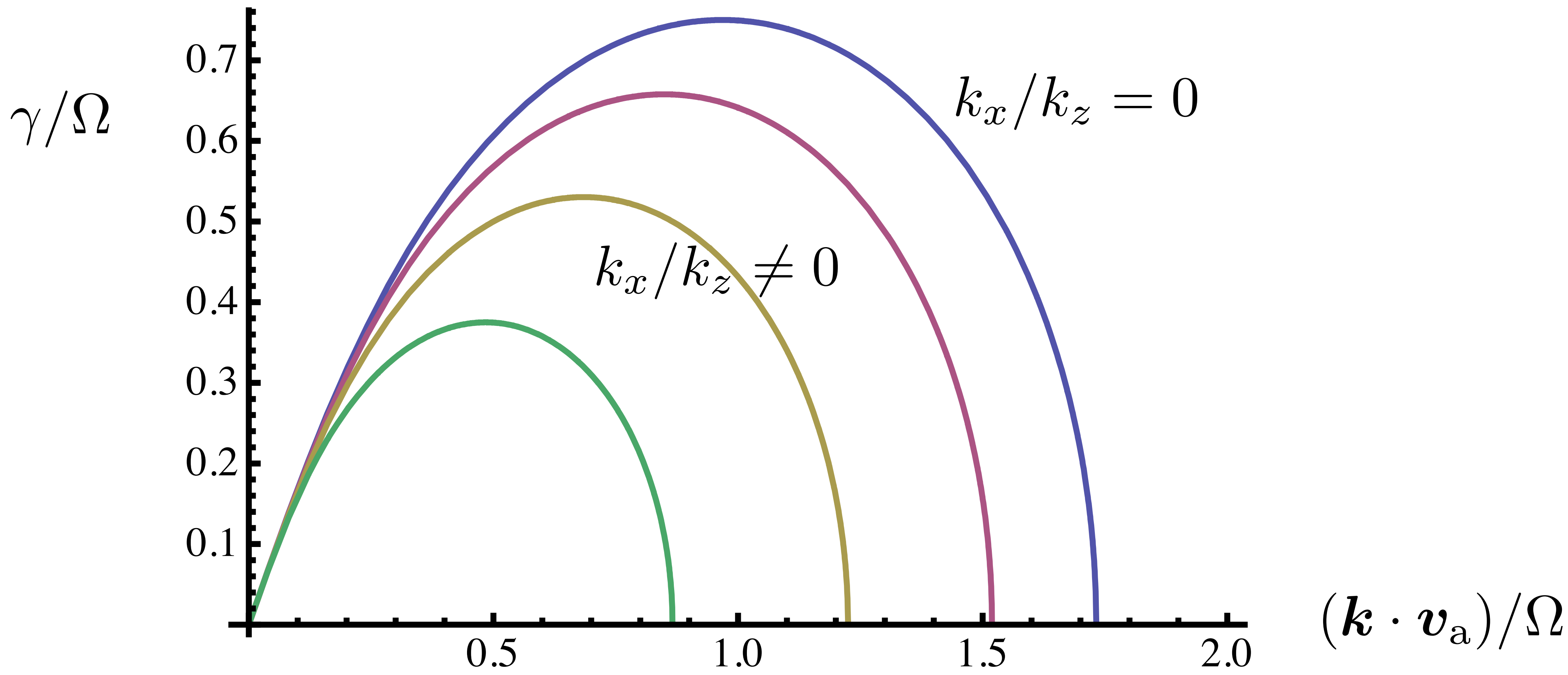
- Maximize growth rate with respect to k :

$$0 = \frac{\partial\omega^2}{\partial\omega_\mathrm{a}^2} = 1 - \frac{4\Omega^2}{\kappa^2}\left(1 + \frac{16\omega_\mathrm{a}^2\Omega^2}{\kappa^4}\right)^{-1/2} \qquad \Rightarrow \quad \omega_\mathrm{a}^2 = \Omega^2 - \frac{\kappa^4}{16\Omega^2}$$

$\Rightarrow \quad (\omega^2)_\mathrm{min} = -\dfrac{S^2}{4}$ so maximum growth rate is $\dfrac{S}{2}$

- Keplerian disc: energy grows by $\exp(3\pi) \approx 12392$ per orbit
- Optimal wavelength $2\pi\sqrt{\dfrac{16}{15}}\dfrac{v_{\mathrm{a}z}}{\Omega} \propto B_z$

Magnetorotational instability



Magnetorotational instability

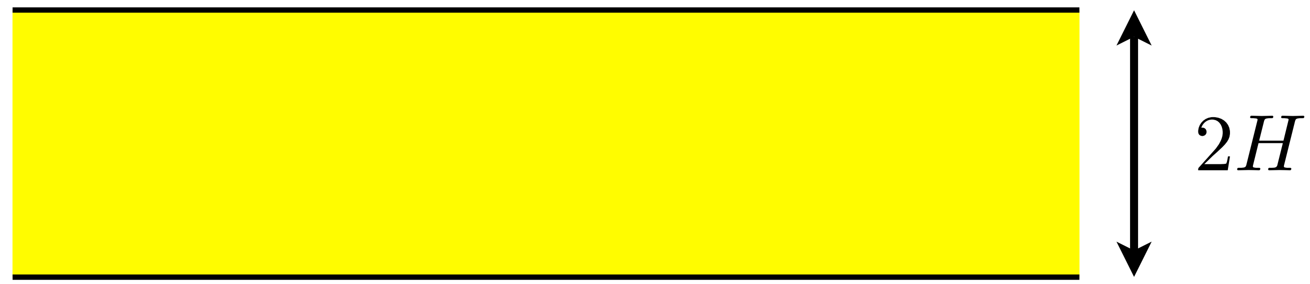
- As $B_z \to 0$ diffusion becomes more important
- Non-ideal MHD: if $\nu = \eta$ (for simplicity) then

$$\omega = \omega_{\text{ideal}} - \mathrm{i}\eta k^2 \qquad \text{(reduces growth rate)}$$

- If k can take any value then instability persists for small k

Magnetorotational instability

- Effect of vertical boundaries:



- Suppose $k = \dfrac{n\pi}{2H}, \quad n \in \mathbf{Z}$
- $n = 0$ mode gives no instability, so consider $n = 1$:
- Instability in ideal MHD when

$$0 < \omega_{\mathrm{a}}^2 < 2\Omega S \qquad \Rightarrow \; 0 < v_{\mathrm{a}} < \frac{2\sqrt{3}}{\pi} H\Omega \qquad \text{(Keplerian)}$$

- Diffusive damping rate of $n = 1$ mode $= \eta(\pi/2H)^2$
- Ideal growth rate $\sim \omega_{\mathrm{a}} = v_{\mathrm{a}}(\pi/2H)$
- Instability occurs for an intermediate range of field strengths,

roughly $\dfrac{\eta}{H} \lesssim v_{\mathrm{a}} \lesssim c_{\mathrm{s}}$

Magnetorotational instability

- Summary:
  - Hydrodynamic instability when

    $$2\Omega(2\Omega - S) < 0 \qquad \text{(Rayleigh)}$$

  - Magnetohydrodynamic instability (weak field, ideal MHD) when

    $$-2\Omega S < 0 \qquad \text{(Chandrasekhar)}$$

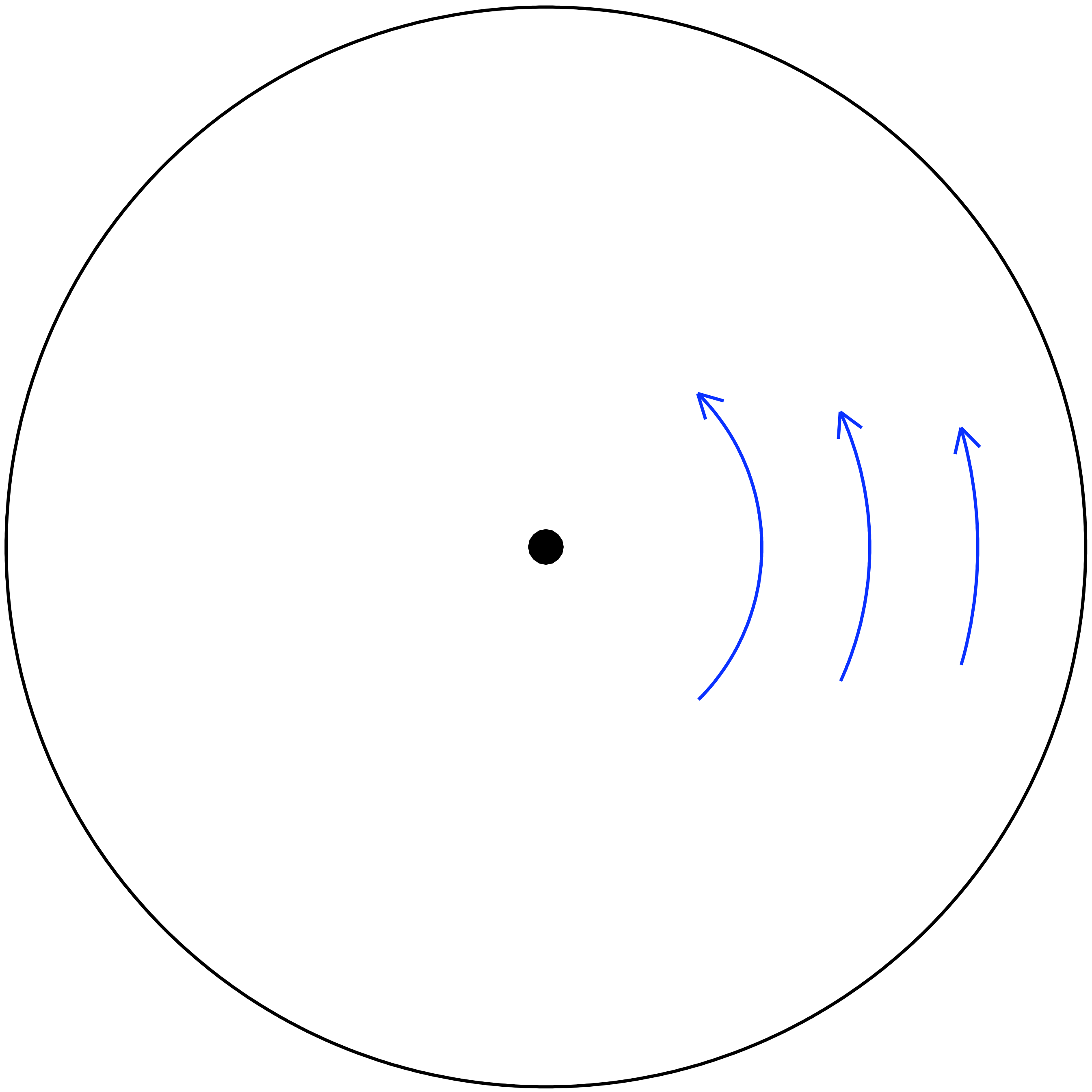
  - Paradox of $|\boldsymbol{B}| \to 0$ resolved by going to non-ideal MHD
  - In cylindrical geometry:

    $$\frac{\mathrm{d}}{\mathrm{d}r}(r^2|\Omega|) < 0 \ \text{(Rayleigh)} \ \text{versus} \ \frac{\mathrm{d}}{\mathrm{d}r}|\Omega| < 0 \ \text{(MRI)}$$

  - Usual situation in astrophysical discs:
    Rayleigh-stable but MRI-unstable

Magnetorotational instability

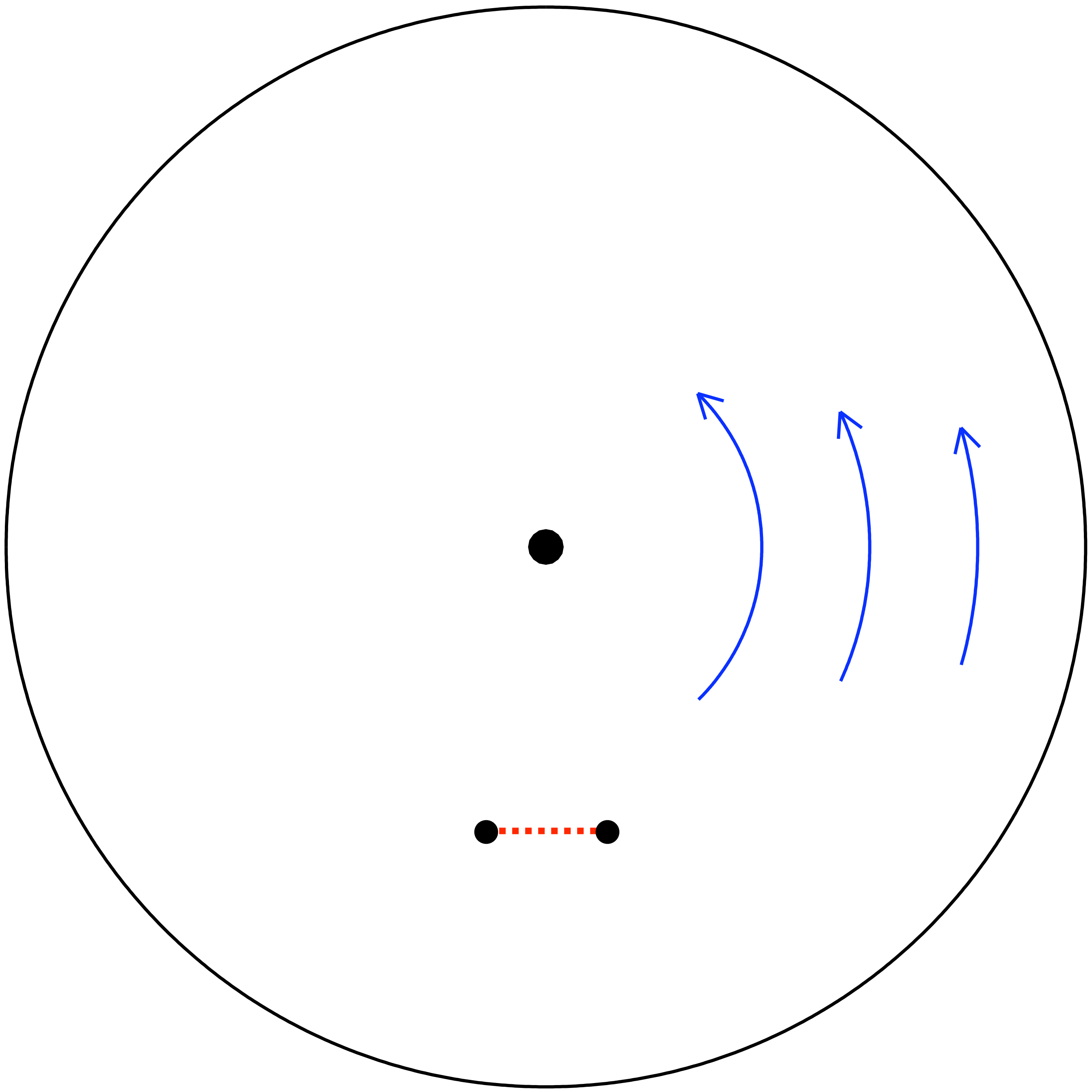
- Physical interpretation / mechanical analogy:



$$\frac{\mathrm{d}\Omega}{\mathrm{d}r} < 0$$

$$\frac{\mathrm{d}(r^2\Omega)}{\mathrm{d}r} > 0$$

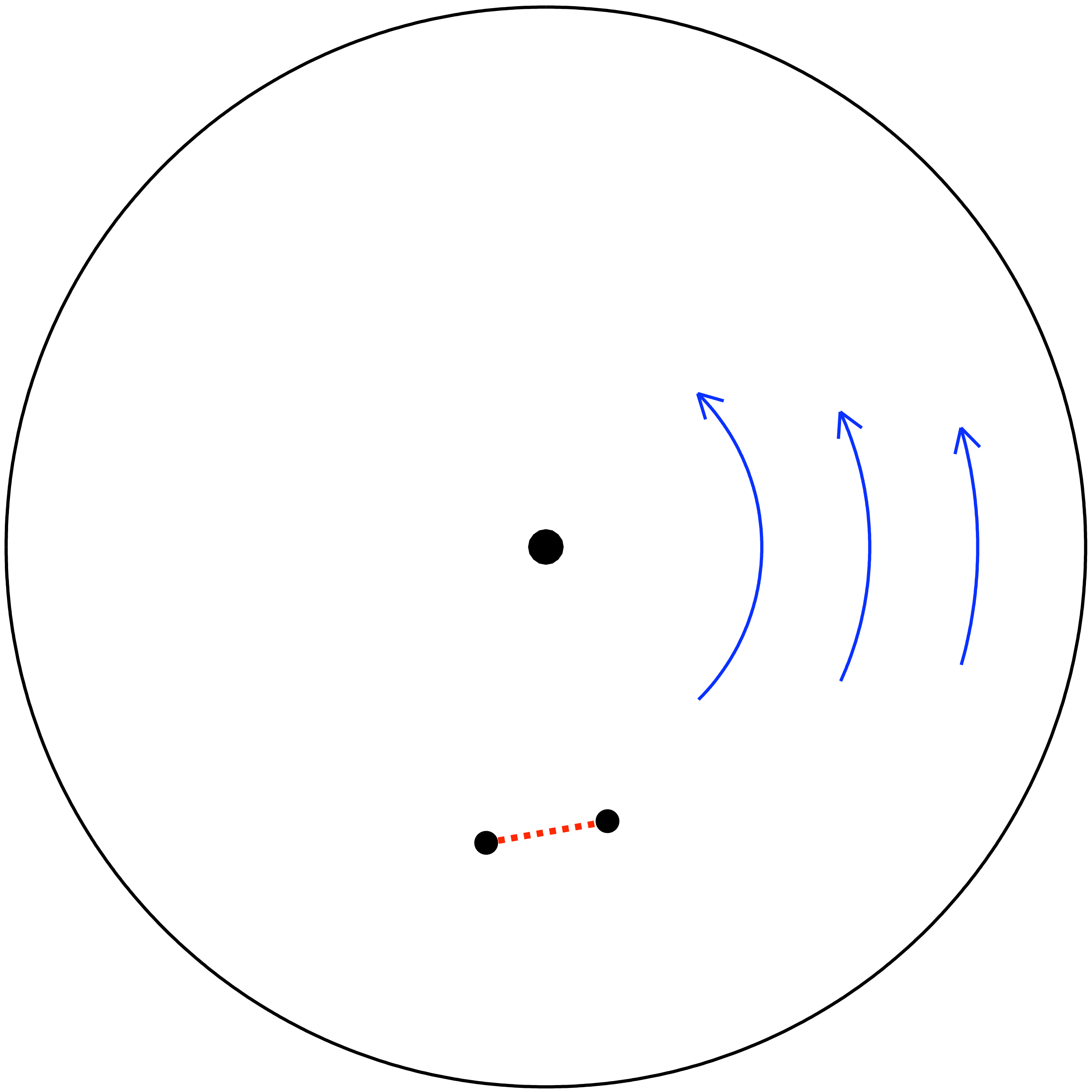
Magnetorotational instability



$$\frac{\mathrm{d}\Omega}{\mathrm{d}r} < 0$$

$$\frac{\mathrm{d}(r^2\Omega)}{\mathrm{d}r} > 0$$

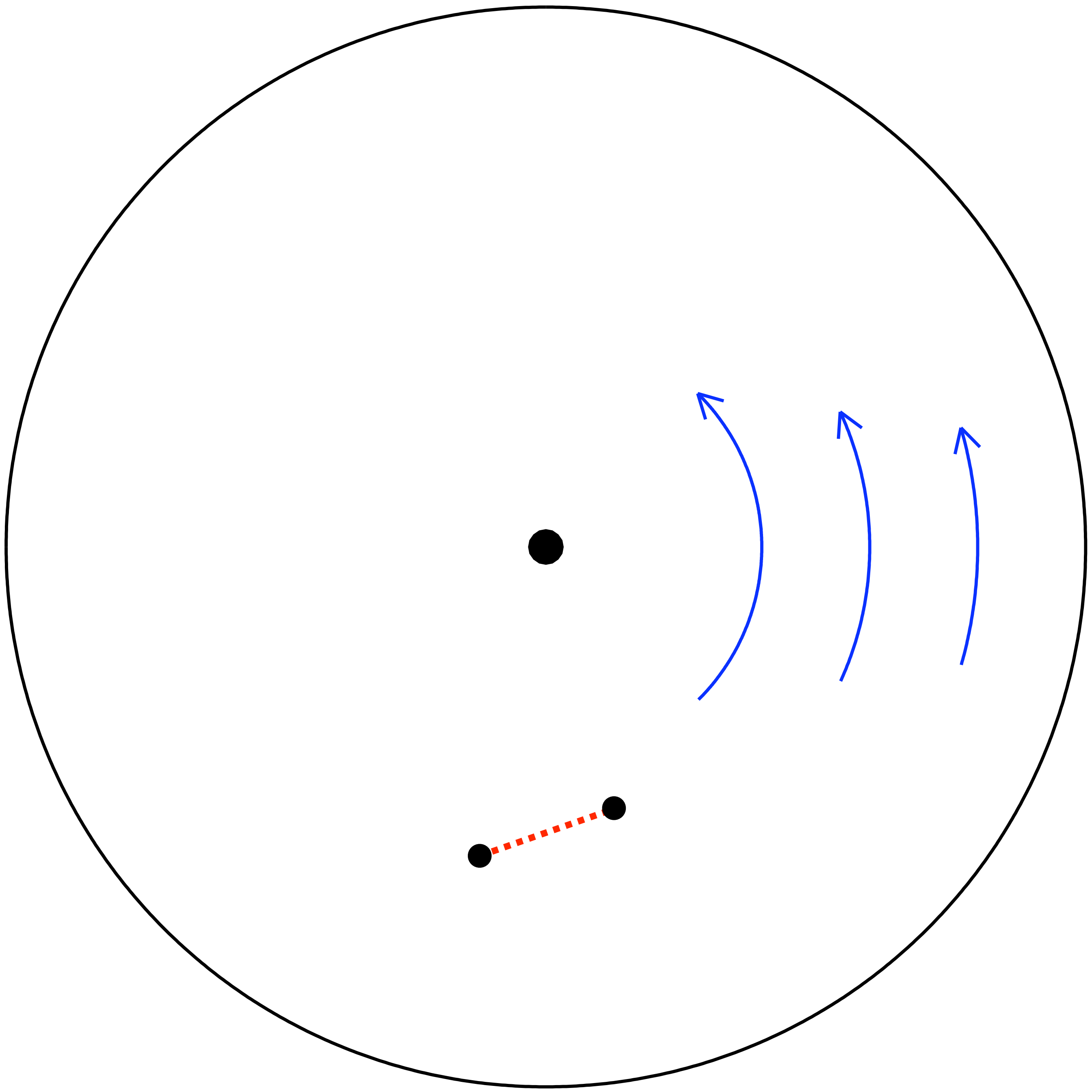
Magnetorotational instability



$$\frac{\mathrm{d}\Omega}{\mathrm{d}r} < 0$$

$$\frac{\mathrm{d}(r^2\Omega)}{\mathrm{d}r} > 0$$

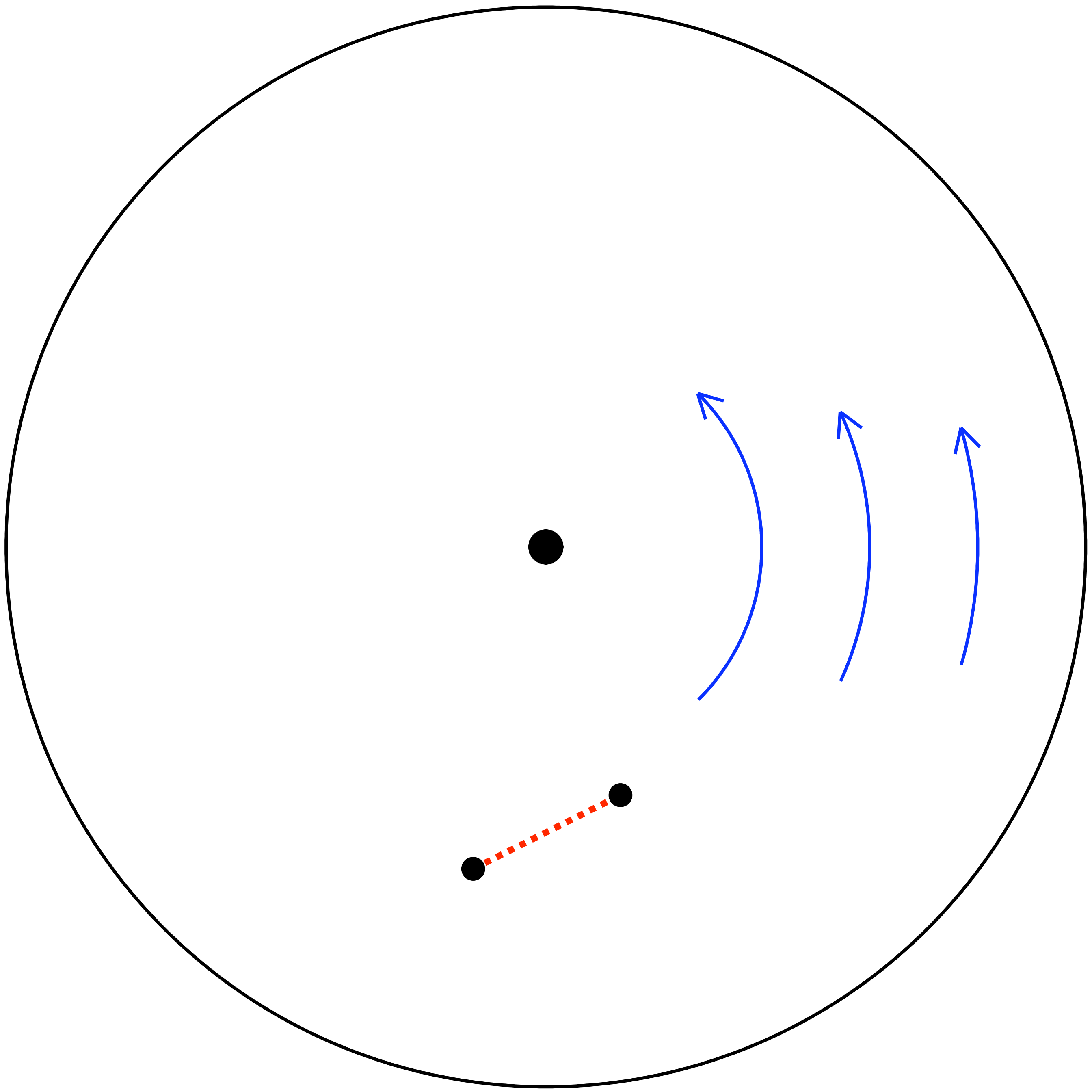
Magnetorotational instability



$$\frac{\mathrm{d}\Omega}{\mathrm{d}r} < 0$$

$$\frac{\mathrm{d}(r^2\Omega)}{\mathrm{d}r} > 0$$

Magnetorotational instability



$$\frac{\mathrm{d}\Omega}{\mathrm{d}r} < 0$$

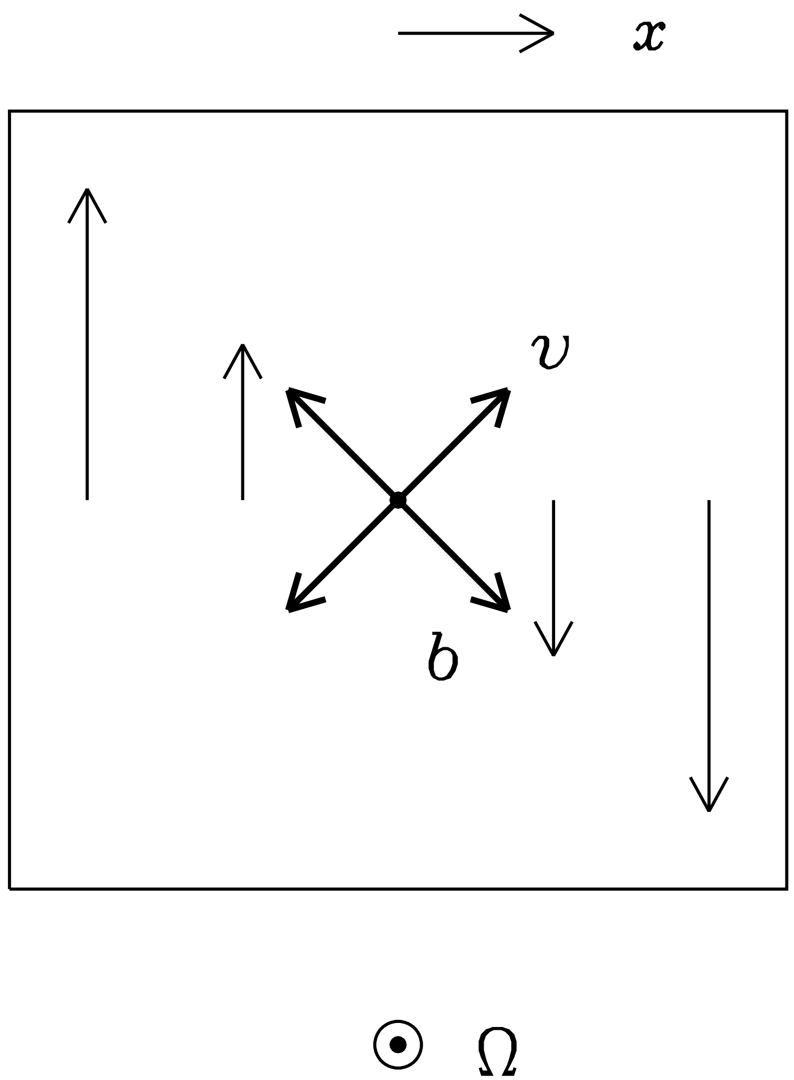
$$\frac{\mathrm{d}(r^2\Omega)}{\mathrm{d}r} > 0$$

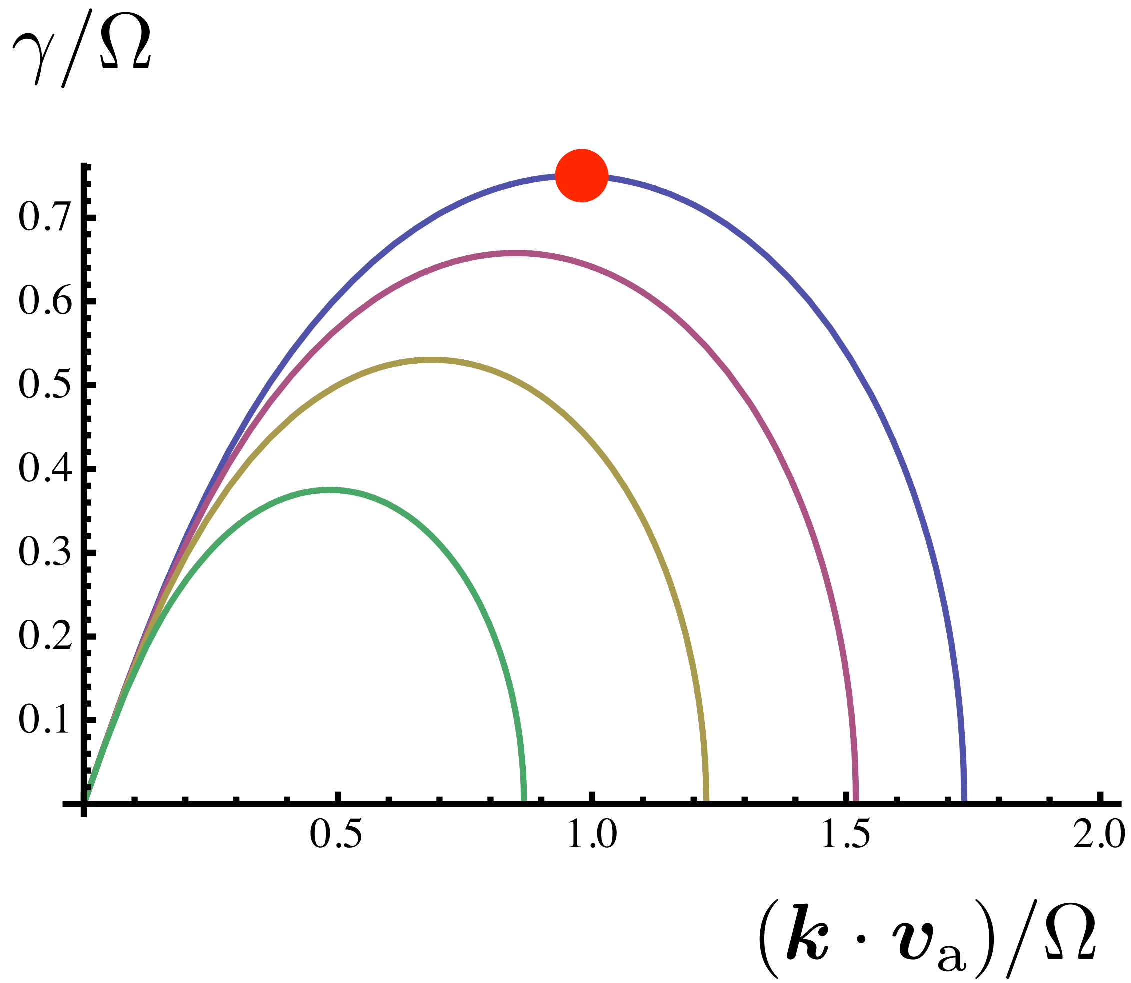
Magnetorotational instability

- Effective potential $\Phi = -\Omega S x^2$ has a maximum at $x = 0$
- Gyroscopic stabilization is defeated by the tension force, allowing instability

Magnetorotational instability

Optimal "channel mode":





Magnetorotational instability

- Nonlinear outcome:
  - With imposed magnetic field: sustained MHD turbulence (intensity depends on imposed magnetic field)
  - Without imposed magnetic field: nonlinear dynamo?

$$\boldsymbol{B} \rightleftarrows \boldsymbol{u}$$

Summary

- Mechanisms of activity and angular momentum transport in astrophysical discs:
  - Viscous transport
  - Hydrodynamic instability
  - Vortex dynamics
  - Gravitational instability
  - Satellite–disc interaction
  - Magnetorotational instability

Summary

- Viscous transport
  - Relevant for planetary rings (macroscopic particles)
- Hydrodynamic instability
  - Mostly thought to be absent or ineffective in standard discs (but controversial)
  - Can be present in non-circular or warped discs

Summary

- Vortex dynamics
  - Can be effective if vortices can be produced and maintained
  - Vortices excite density waves that transport angular momentum
  - Production:
    - “Baroclinic instability”
    - “Rossby vortex instability”, etc.
  - Destruction:
    - Elliptical instability, etc.
    - Inward migration
  - May be relevant in protoplanetary discs
    (also for planet formation)

Summary

- Gravitational instability
  - Occurs in sufficiently massive and cool discs
  - May produce turbulence or fragmentation depending on cooling
  - Relevant for outer parts of protoplanetary discs and discs around black holes in active galactic nuclei
  - Also relevant for planetary rings
- Satellite–disc interaction
  - Embedded or external satellites excite waves and induce induce angular momentum transport
  - Applications are quite specific and localized

Summary

- Magnetorotational instability
  - Occurs in sufficiently ionized discs
  - Relevant for high-energy (plasma) accretion discs and for sufficiently ionized layers of protoplanetary discs
  - Questions remain over efficiency of dynamo and transport