

Lagrangian formulation of ideal MHD

(non-examinable)

0.1 The Lagrangian viewpoint

The flow of a fluid can be considered as a time-dependent map,

$$\mathbf{a} \mapsto \mathbf{r}(\mathbf{a}, t),$$

where \mathbf{a} is the position vector of a fluid element at some initial time t_0 , and \mathbf{r} is its position vector at time t . The Cartesian components of \mathbf{a} are examples of *Lagrangian variables*, labelling the fluid element. The components of \mathbf{r} are *Eulerian variables*, labelling a fixed point in space. Any fluid property X (scalar, vector or tensor) can be regarded as a function of either Lagrangian or Eulerian variables:

$$X = X^L(\mathbf{a}, t) = X^E(\mathbf{r}, t).$$

The velocity of the fluid is simply

$$\mathbf{u} = \left(\frac{\partial \mathbf{r}}{\partial t} \right)_{\mathbf{a}},$$

and the Lagrangian time-derivative is

$$\frac{D}{Dt} = \left(\frac{\partial}{\partial t} \right)_{\mathbf{a}}.$$

The aim of a Lagrangian formulation of ideal MHD is to derive a non-linear evolutionary equation for the function $\mathbf{r}(\mathbf{a}, t)$. The dynamics is Hamiltonian in character and can be derived from a Lagrangian function or action principle. There are many similarities with classical field theories.

0.2 Geometrical conservation laws

The equations of ideal MHD comprise the equation of motion and three ‘geometrical’ conservation laws. These are the conservation of specific entropy (thermal energy equation),

$$\frac{Ds}{Dt} = 0, \tag{1}$$

the conservation of mass,

$$\frac{D\rho}{Dt} = -\rho \nabla \cdot \mathbf{u}, \tag{2}$$

and the conservation of magnetic flux (induction equation),

$$\frac{D}{Dt} \left(\frac{\mathbf{B}}{\rho} \right) = \left(\frac{\mathbf{B}}{\rho} \right) \cdot \nabla \mathbf{u}. \tag{3}$$

These equations describe the pure advection of fluid properties in a manner equivalent to the advection of various geometrical objects. The specific entropy is advected as a simple scalar, so that its numerical value is conserved by material points. The specific volume $v = 1/\rho$ is advected in the same way as an infinitesimal volume element dV . The quantity \mathbf{B}/ρ is advected in the same way as an infinitesimal line element $d\mathbf{x}$. All three conservation laws can be integrated exactly in Lagrangian variables.

Introduce the *deformation tensor* of the flow,

$$F_{ij} = \frac{\partial r_i}{\partial a_j},$$

and its determinant

$$F = \det(F_{ij})$$

and inverse

$$G_{ij} = \frac{\partial a_i}{\partial r_j}.$$

We note the following properties. First, the derivative

$$\frac{\partial F}{\partial F_{ij}} = C_{ij} = FG_{ji} = \frac{1}{2}\epsilon_{ikl}\epsilon_{jmn}F_{km}F_{ln} \quad (4)$$

is equal to the cofactor C_{ij} of the matrix element F_{ij} . Second, the matrix of cofactors has zero divergence on its second index:

$$\frac{\partial C_{ij}}{\partial a_j} = \frac{\partial}{\partial a_j} \left(\frac{1}{2}\epsilon_{ikl}\epsilon_{jmn} \frac{\partial r_k}{\partial a_m} \frac{\partial r_\ell}{\partial a_n} \right) = 0. \quad (5)$$

Now

$$\frac{DF_{ij}}{Dt} = \frac{\partial u_i}{\partial a_j}$$

and, according to equation (4),

$$\frac{D \ln F}{Dt} = G_{ji} \frac{DF_{ij}}{Dt} = \frac{\partial a_j}{\partial r_i} \frac{\partial u_i}{\partial a_j} = \frac{\partial u_i}{\partial r_i} = \nabla \cdot \mathbf{u}.$$

The exact solutions of equations (1), (2) and (3) are then

$$\begin{aligned} s^L(\mathbf{a}, t) &= s_0(\mathbf{a}), \\ \rho^L(\mathbf{a}, t) &= F^{-1}\rho_0(\mathbf{a}), \\ B_i^L(\mathbf{a}, t) &= F^{-1}F_{ij}B_{j0}(\mathbf{a}), \end{aligned}$$

where s_0 , ρ_0 , and \mathbf{B}_0 are the initial values at time t_0 . The verification of equation (3) is

$$\frac{D}{Dt} \left(\frac{B_i}{\rho} \right) = \frac{\partial u_i}{\partial a_j} \frac{B_{j0}}{\rho_0} = F_{kj} \frac{\partial u_i}{\partial r_k} \frac{B_{j0}}{\rho_0} = \left(\frac{B_k}{\rho} \right) \frac{\partial u_i}{\partial r_k}.$$

Note that the advected quantities at time t depend only on the initial values and on the instantaneous mapping $\mathbf{a} \mapsto \mathbf{r}$, not on the intermediate history of the flow. The ‘memory’ of an ideal fluid is perfect.

0.3 The Lagrangian of ideal MHD

Newtonian dynamics can be formulated using Hamilton’s principle of stationary action,

$$\delta \int L dt = 0,$$

where the Lagrangian L is the difference between the kinetic energy and the potential energy of the system. By analogy, we may expect the Lagrangian of ideal MHD to take the form

$$L = \int \mathcal{L} dV$$

where (for a non-self-gravitating fluid)

$$\mathcal{L} = \rho \left(\frac{1}{2}u^2 - \Phi - e - \frac{B^2}{2\mu_0\rho} \right)$$

is the *Lagrangian density*.

To verify this, we assume that the equation of state can be written in the form $e = e(v, s)$, where $v = \rho^{-1}$ is the specific volume. Since $de = T ds - p dv$, we have

$$\left(\frac{\partial e}{\partial v} \right)_s = -p, \quad \left(\frac{\partial^2 e}{\partial v^2} \right)_s = \frac{\gamma p}{v}$$

(strictly, γ should be Γ_1 here).

We then write the action using Lagrangian variables,

$$S[\mathbf{r}] = \int \int \tilde{\mathcal{L}}(\mathbf{r}, \mathbf{u}, \mathbf{F}) d^3\mathbf{a} dt,$$

with

$$\tilde{\mathcal{L}} = \rho_0 \left[\frac{1}{2}u^2 - \Phi(\mathbf{r}) - e(F\rho_0^{-1}, s_0) - \frac{F^{-1}F_{ij}B_{j0}F_{ik}B_{k0}}{2\mu_0\rho_0} \right].$$

This uses the fact that F is the Jacobian determinant of the transformation $\mathbf{a} \mapsto \mathbf{r}$, or, equivalently, that $\rho d^3\mathbf{r} = \rho_0 d^3\mathbf{a} = dm$ is an invariant mass measure. $\tilde{\mathcal{L}}$ is now expressed in terms of the function $\mathbf{r}(\mathbf{a}, t)$ and its derivatives with respect to time (\mathbf{u}) and space (\mathbf{F}). The Euler–Lagrange equation for the variational principle $\delta S = 0$ is

$$\frac{D}{D} \frac{\partial \tilde{\mathcal{L}}}{\partial u_i} + \frac{\partial}{\partial a_j} \frac{\partial \tilde{\mathcal{L}}}{\partial F_{ij}} - \frac{\partial \tilde{\mathcal{L}}}{\partial r_i} = 0.$$

The straightforward terms are

$$\frac{\partial \tilde{\mathcal{L}}}{\partial u_i} = \rho_0 u_i, \quad \frac{\partial \tilde{\mathcal{L}}}{\partial r_i} = -\rho_0 \frac{\partial \Phi}{\partial r_i}.$$

Now

$$\begin{aligned} \frac{\partial \tilde{\mathcal{L}}}{\partial F_{ij}} &= \left(p + \frac{B^2}{2\mu_0} \right) \frac{\partial F}{\partial F_{ij}} - \frac{F^{-1} B_{j0} F_{ik} B_{k0}}{\mu_0} \\ &= C_{ij} \left(p + \frac{B^2}{2\mu_0} \right) - \frac{1}{\mu_0} C_{kj} B_i B_k \\ &= -C_{kj} V_{ik}, \end{aligned}$$

where

$$V_{ik} = - \left(p + \frac{B^2}{2\mu_0} \right) \delta_{ik} + \frac{B_i B_k}{\mu_0}$$

is the stress tensor including gas pressure and magnetic fields.

The Euler–Lagrange equation is therefore

$$\rho_0 \frac{D u_i}{D t} = -\rho_0 \frac{\partial \Phi}{\partial r_i} + \frac{\partial}{\partial a_j} (C_{kj} V_{ik}).$$

Using equation (5) we note that

$$\frac{\partial}{\partial a_j} (C_{kj} V_{ik}) = C_{kj} \frac{\partial V_{ik}}{\partial a_j} = F G_{jk} \frac{\partial V_{ik}}{\partial a_j} = F \frac{\partial V_{ik}}{\partial r_k}.$$

On dividing through by F , the Euler–Lagrange equation becomes the desired equation of motion,

$$\rho \frac{D \mathbf{u}}{D t} = -\rho \nabla \Phi + \nabla \cdot \mathbf{V}.$$

In this construction, the fluid flow is viewed as a field $\mathbf{r}(\mathbf{a}, t)$ on the initial state space. Ideal MHD is seen as a nonlinear field theory derived from an action principle.

When considering stability problems, it is useful to generalize this concept and to view a perturbed flow as a field on an unperturbed flow.

0.4 The Lagrangian displacement

Now consider two different flows, $\mathbf{r}(\mathbf{a}, t)$ and $\hat{\mathbf{r}}(\mathbf{a}, t)$, for which the initial values of the advected quantities, s_0 , ρ_0 and \mathbf{B}_0 , are the same. The two deformation tensors are related by the chain rule,

$$\hat{F}_{ij} = J_{ik} F_{kj},$$

where

$$J_{ik} = \frac{\partial \hat{r}_i}{\partial r_k}$$

is the Jacobian matrix of the map $\mathbf{r} \mapsto \hat{\mathbf{r}}$. Similarly,

$$\hat{F} = JF,$$

where

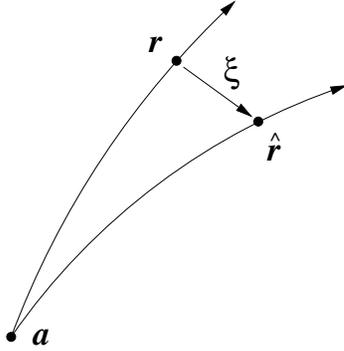
$$J = \det(J_{ij})$$

is the Jacobian determinant. The advected quantities in the two flows are therefore related by the composition of maps,

$$\begin{aligned} \hat{s}^L(\mathbf{a}, t) &= s^L(\mathbf{a}, t), \\ \hat{\rho}^L(\mathbf{a}, t) &= J^{-1} \rho^L(\mathbf{a}, t), \\ \hat{B}_i^L(\mathbf{a}, t) &= J^{-1} J_{ij} B_j^L(\mathbf{a}, t). \end{aligned}$$

The *Lagrangian displacement* is the relative displacement of the fluid element in the two flows,

$$\boldsymbol{\xi} = \hat{\mathbf{r}} - \mathbf{r}.$$



Thus

$$J_{ij} = \delta_{ij} + \frac{\partial \xi_i}{\partial r_j},$$

$$J = 1 + \frac{\partial \xi_i}{\partial r_i} + \frac{1}{2} \left(\frac{\partial \xi_i}{\partial r_i} \frac{\partial \xi_j}{\partial r_j} - \frac{\partial \xi_i}{\partial r_j} \frac{\partial \xi_j}{\partial r_i} \right) + O(\xi^3),$$

$$J^{-1} = 1 - \frac{\partial \xi_i}{\partial r_i} + \frac{1}{2} \left(\frac{\partial \xi_i}{\partial r_i} \frac{\partial \xi_j}{\partial r_j} + \frac{\partial \xi_i}{\partial r_j} \frac{\partial \xi_j}{\partial r_i} \right) + O(\xi^3).$$

0.5 The Lagrangian for a perturbed flow

We now use the action principle to construct a theory for the displacement as a field on the unperturbed flow: $\boldsymbol{\xi} = \boldsymbol{\xi}(\mathbf{r}, t)$. The action for the perturbed flow is

$$\hat{S}[\boldsymbol{\xi}] = \iiint \hat{\mathcal{L}} \left(\boldsymbol{\xi}, \frac{\partial \boldsymbol{\xi}}{\partial t}, \nabla \boldsymbol{\xi} \right) d^3 \mathbf{r} dt,$$

where

$$\begin{aligned} \hat{\mathcal{L}} &= \rho \left(\frac{1}{2} \hat{u}^2 - \hat{\Phi} - \hat{e} - \frac{\hat{B}^2}{2\mu_0 \hat{\rho}} \right) \\ &= \rho \left[\frac{1}{2} \hat{u}^2 - \Phi(\mathbf{r} + \boldsymbol{\xi}) - e(J\rho^{-1}, s) - \frac{J^{-1} J_{ij} B_j J_{ik} B_k}{2\mu_0 \rho} \right], \end{aligned}$$

with

$$\hat{u} = \frac{D\hat{\mathbf{r}}}{Dt} = \mathbf{u} + \frac{\partial \boldsymbol{\xi}}{\partial t} + \mathbf{u} \cdot \nabla \boldsymbol{\xi}.$$

The Euler–Lagrange equation for the variational principle $\delta \hat{S} = 0$ is

$$\frac{\partial}{\partial t} \frac{\partial \hat{\mathcal{L}}}{\partial (\partial \xi_i / \partial t)} + \frac{\partial}{\partial r_j} \frac{\partial \hat{\mathcal{L}}}{\partial (\partial \xi_i / \partial r_j)} - \frac{\partial \hat{\mathcal{L}}}{\partial \xi_i} = 0.$$

We expand the various terms of $\hat{\mathcal{L}}$ in powers of $\boldsymbol{\xi}$:

$$\begin{aligned} \frac{1}{2} \rho \hat{u}^2 &= \frac{1}{2} \rho u^2 + \rho u_i \left(\frac{\partial \xi_i}{\partial t} + u_j \frac{\partial \xi_i}{\partial r_j} \right) \\ &\quad + \frac{1}{2} \rho \left(\frac{\partial \xi_i}{\partial t} + u_j \frac{\partial \xi_i}{\partial r_j} \right) \left(\frac{\partial \xi_i}{\partial t} + u_k \frac{\partial \xi_i}{\partial r_k} \right), \\ -\rho \Phi(\mathbf{r} + \boldsymbol{\xi}) &= -\rho \left[\Phi(\mathbf{r}) + \xi_i \frac{\partial \Phi}{\partial r_i} + \frac{1}{2} \xi_i \xi_j \frac{\partial^2 \Phi}{\partial r_i \partial r_j} + O(\xi^3) \right], \\ -\rho \left(\hat{e} - \frac{\hat{B}^2}{2\mu_0 \hat{\rho}} \right) &= -\left(\rho e + \frac{B^2}{2\mu_0} \right) - V_{ij} \frac{\partial \xi_i}{\partial r_j} \\ &\quad - \frac{1}{2} V_{ijkl} \frac{\partial \xi_i}{\partial r_j} \frac{\partial \xi_k}{\partial r_l} + O(\xi^3). \end{aligned}$$

This last expression uses the fact that \hat{e} , $\hat{\mathbf{B}}$ and $\hat{\rho}$ depend only on $\nabla \boldsymbol{\xi}$ (through J and J_{ij}) and can therefore be expanded in a Taylor series in this quantity. A short calculation of this expansion gives

$$V_{ij} = -\left(p + \frac{B^2}{2\mu_0} \right) \delta_{ij} + \frac{B_i B_j}{\mu_0},$$

which is the stress tensor used above, and

$$V_{ijkl} = \left[(\gamma - 1)p + \frac{B^2}{2\mu_0} \right] \delta_{ij}\delta_{kl} + \left(p + \frac{B^2}{2\mu_0} \right) \delta_{il}\delta_{jk} \\ + \frac{1}{\mu_0} B_j B_\ell \delta_{ik} - \frac{1}{\mu_0} (B_i B_j \delta_{kl} + B_k B_\ell \delta_{ij}),$$

which has the symmetry

$$V_{ijkl} = V_{klij}$$

necessitated by its function in the Taylor series. We now have

$$\frac{\partial \hat{\mathcal{L}}}{\partial(\partial\xi_i/\partial t)} = \rho u_i + \rho \frac{\partial \xi_i}{\partial t}, \\ \frac{\partial \hat{\mathcal{L}}}{\partial(\partial\xi_i/\partial r_j)} = \rho u_i u_j + \rho u_j \frac{D\xi_i}{Dt} - V_{ij} - V_{ijkl} \frac{\partial \xi_k}{\partial r_\ell} + O(\xi^2), \\ \frac{\partial \hat{\mathcal{L}}}{\partial \xi_i} = -\rho \frac{\partial \Phi}{\partial r_i} - \rho \xi_j \frac{\partial^2 \Phi}{\partial r_i \partial r_j} + O(\xi^2).$$

Since $\boldsymbol{\xi} = \mathbf{0}$ must be a solution of the Euler–Lagrange equation, it is no surprise that the terms independent of $\boldsymbol{\xi}$ cancel by virtue of the equation of motion of the unperturbed flow,

$$\frac{\partial}{\partial t}(\rho u_i) + \frac{\partial}{\partial r_j}(\rho u_i u_j - V_{ij}) + \rho \frac{\partial \Phi}{\partial r_i} = 0.$$

The remaining terms are

$$\frac{\partial}{\partial t} \left(\rho \frac{\partial \xi_i}{\partial t} \right) + \frac{\partial}{\partial r_j} \left(\rho u_j \frac{D\xi_i}{Dt} - V_{ijkl} \frac{\partial \xi_k}{\partial r_\ell} \right) + \rho \xi_j \frac{\partial^2 \Phi}{\partial r_i \partial r_j} + O(\xi^2) = 0,$$

or

$$\rho \frac{D^2 \xi_i}{Dt^2} = \frac{\partial}{\partial r_j} \left(V_{ijkl} \frac{\partial \xi_k}{\partial r_\ell} \right) - \rho \xi_j \frac{\partial^2 \Phi}{\partial r_i \partial r_j} + O(\xi^2).$$

This equation, which can be extended to any order in $\boldsymbol{\xi}$, provides the basis for a nonlinear perturbation theory for any flow in ideal MHD. In a linear theory we would neglect terms of $O(\xi^2)$. Note that, in the case of perturbations of a magnetostatic equilibrium ($\mathbf{u} = \mathbf{0}$), the linearized equation reduces to

$$\rho \frac{\partial^2 \boldsymbol{\xi}}{\partial t^2} = \rho \mathcal{F} \boldsymbol{\xi},$$

where the operator \mathcal{F} is given by

$$\rho(\mathcal{F}\boldsymbol{\xi})_i = \frac{\partial}{\partial r_j} \left(V_{ijkl} \frac{\partial \xi_k}{\partial r_\ell} \right) - \rho \xi_j \frac{\partial^2 \Phi}{\partial r_i \partial r_j}.$$

The self-adjointness of \mathcal{F} follows from the symmetry of the expression

$$\int \boldsymbol{\xi}_1^* \cdot (\mathcal{F}\boldsymbol{\xi}_2) \rho \, dV = - \int \left(V_{ijkl} \frac{\partial \xi_{1i}^*}{\partial r_j} \frac{\partial \xi_{2k}}{\partial r_\ell} + \rho \xi_{1i}^* \xi_{2j} \frac{\partial^2 \Phi}{\partial r_i \partial r_j} \right) dV,$$

which follows from an integration by parts, assuming suitable boundary conditions.

For perturbations of steady flows, the linearized operator contains additional contributions from the $\mathbf{u} \cdot \nabla$ terms in $D^2 \boldsymbol{\xi}/Dt^2$; it is not self-adjoint and its eigenvalues are not real in general.

0.6 Further notes on linear perturbations

The *Lagrangian perturbation* ΔX of a quantity X is the difference in the values of the quantity in the two flows for the same fluid element,

$$\Delta X = \hat{X}^L(\mathbf{a}, t) - X^L(\mathbf{a}, t).$$

It follows that

$$\Delta s = 0,$$

$$\Delta \rho = -\rho \frac{\partial \xi_i}{\partial r_i} + O(\xi^2),$$

$$\Delta B_i = B_j \frac{\partial \xi_i}{\partial r_j} - B_i \frac{\partial \xi_j}{\partial r_j} + O(\xi^2).$$

Also

$$\Delta u_i = \frac{D\xi_i}{Dt}.$$

In linear theory, $\nabla \xi$ is small and terms higher than the first order are neglected. Thus

$$\Delta s = 0,$$

$$\Delta \rho = -\rho \nabla \cdot \xi,$$

$$\Delta \mathbf{B} = \mathbf{B} \cdot \nabla \xi - (\nabla \cdot \xi) \mathbf{B}.$$

In linear theory, $\Delta s = 0$ implies

$$\Delta p = \frac{\gamma p}{\rho} \Delta \rho = -\gamma p \nabla \cdot \xi.$$

The *Eulerian perturbation* δX of a quantity X is the difference in the values of the quantity in the two flows *at the same point in space*,

$$\delta X = \hat{X}^E(\mathbf{r}, t) - X^E(\mathbf{r}, t).$$

By Taylor's theorem,

$$\Delta X = \delta X + \xi \cdot \nabla X + O(\xi^2),$$

and so, in linear theory,

$$\delta X = \Delta X - \xi \cdot \nabla X.$$

Thus

$$\delta \rho = -\rho \nabla \cdot \xi - \xi \cdot \nabla \rho,$$

$$\delta p = -\gamma p \nabla \cdot \xi - \xi \cdot \nabla p,$$

$$\delta \mathbf{B} = \mathbf{B} \cdot \nabla \xi - \xi \cdot \nabla \mathbf{B} - (\nabla \cdot \xi) \mathbf{B}.$$

exactly as was obtained in the lectures for perturbations of magneto-static equilibria.

The relation

$$\delta \mathbf{u} = \frac{D\xi}{Dt} - \xi \cdot \nabla \mathbf{u}$$

can be used to introduce the Lagrangian displacement into a linear theory based on Eulerian perturbations. Only in the case of a static basic state, $\mathbf{u} = \mathbf{0}$, does this reduce to the simple relation $\delta \mathbf{u} = \partial \xi / \partial t$.