2.2.3. Induction equation

The steady induction equation in ideal MHD,

$$\nabla \times (\boldsymbol{u} \times \boldsymbol{B}) = \boldsymbol{0},$$

implies

$$\boldsymbol{u} \times \boldsymbol{B} = -\boldsymbol{E} = \nabla \Phi_{\mathrm{e}},$$

where Φ_{e} is the electrostatic potential. Now

$$\boldsymbol{u} \times \boldsymbol{B} = (\boldsymbol{u}_{\mathrm{p}} + u_\phi\, \boldsymbol{e}_\phi) \times (\boldsymbol{B}_{\mathrm{p}} + B_\phi\, \boldsymbol{e}_\phi) = \Big[\boldsymbol{e}_\phi \times (u_\phi \boldsymbol{B}_{\mathrm{p}} - B_\phi \boldsymbol{u}_{\mathrm{p}})\Big] + \Big[\boldsymbol{u}_{\mathrm{p}} \times \boldsymbol{B}_{\mathrm{p}}\Big].$$

For an axisymmetric solution with $\partial\Phi_{\mathrm{e}}/\partial\phi = 0$, we have

$$\boldsymbol{u}_{\mathrm{p}} \times \boldsymbol{B}_{\mathrm{p}} = \boldsymbol{0},$$

i.e. the poloidal velocity is parallel to the poloidal magnetic field. Let

$$\rho \boldsymbol{u}_{\mathrm{p}} = k \boldsymbol{B}_{\mathrm{p}},$$

where k is the *mass loading*, i.e. the ratio of mass flux to magnetic flux.

The steady equation of mass conservation is

$$0 = \nabla \cdot (\rho \boldsymbol{u}) = \nabla \cdot (\rho \boldsymbol{u}_{\mathrm{p}}) = \nabla \cdot (k \boldsymbol{B}_{\mathrm{p}}) = \boldsymbol{B}_{\mathrm{p}} \cdot \nabla k.$$

Therefore

$$k = k(\psi),$$

i.e. k is a *surface function*, constant on each magnetic surface.

We now have

$$\boldsymbol{u} \times \boldsymbol{B} = \boldsymbol{e}_\phi \times (u_\phi \boldsymbol{B}_{\mathrm{p}} - B_\phi \boldsymbol{u}_{\mathrm{p}}) = \left(\frac{u_\phi}{r} - \frac{kB_\phi}{r\rho}\right) \nabla \psi.$$

Taking the curl of this, we find

$$\boldsymbol{0} = \nabla \left(\frac{u_\phi}{r} - \frac{kB_\phi}{r\rho}\right) \times \nabla \psi.$$

Therefore

$$\frac{u_\phi}{r} - \frac{kB_\phi}{r\rho} = \omega,$$

where $\omega(\psi)$ is another surface function, known as the *angular velocity of the magnetic surface.*

The complete velocity field may be written in the form

$$\boldsymbol{u} = \frac{k\boldsymbol{B}}{\rho} + r\omega\,\boldsymbol{e}_\phi,$$

i.e. the total velocity is parallel to the total magnetic field in a frame of reference rotating with angular velocity ω. It is useful to think of the fluid being constrained to move along the field line like a *bead on a rotating wire.*

2.2.4. Thermal energy equation

The steady thermal energy equation for an ideal fluid,

$$\boldsymbol{u} \cdot \nabla s = 0,$$

implies that

$$s = s(\psi)$$

is another surface function.

2.2.5. Equation of motion

The azimuthal component of the equation of motion is

$$\rho\left(\boldsymbol{u}_\mathrm{p} \cdot \nabla u_\phi + \frac{u_r u_\phi}{r}\right) = \frac{1}{\mu_0}\left(\boldsymbol{B}_\mathrm{p} \cdot \nabla B_\phi + \frac{B_r B_\phi}{r}\right)$$

$$\frac{1}{r}\rho\boldsymbol{u}_\mathrm{p} \cdot \nabla(r u_\phi) - \frac{1}{\mu_0 r}\boldsymbol{B}_\mathrm{p} \cdot \nabla(r B_\phi) = 0$$

$$\frac{1}{r}\boldsymbol{B}_\mathrm{p} \cdot \nabla\left(k r u_\phi - \frac{r B_\phi}{\mu_0}\right) = 0$$

and so

$$r u_\phi = \frac{r B_\phi}{\mu_0 k} + \ell,$$

where

$$\ell = \ell(\psi)$$

is another surface function, the *angular momentum invariant.* This is the angular momentum removed in the outflow per unit mass, but part of the torque is carried by the magnetic field.

Finally, consider the kinetic energy equation formed from the inner product of $\boldsymbol{u}$ with the equation of motion,

$$\rho\boldsymbol{u} \cdot \nabla(\tfrac{1}{2}u^2) = -\rho\boldsymbol{u} \cdot \nabla\Phi - \boldsymbol{u} \cdot \nabla p + \boldsymbol{u} \cdot (\boldsymbol{J} \times \boldsymbol{B}).$$

Now

$$\boldsymbol{u}\cdot\nabla p=\rho\boldsymbol{u}\cdot\nabla w,$$

since $\boldsymbol{u}\cdot\nabla s=0$. Therefore

$$\rho\boldsymbol{u}\cdot\nabla(\tfrac{1}{2}u^2+\Phi+w)=\boldsymbol{u}\cdot(\boldsymbol{J}\times\boldsymbol{B}).$$

Consider the divergence of the *Poynting flux*,

$$\begin{aligned}\nabla\cdot\left(\frac{1}{\mu_0}\boldsymbol{E}\times\boldsymbol{B}\right)&=\frac{1}{\mu_0}(\boldsymbol{B}\cdot\nabla\times\boldsymbol{E}-\boldsymbol{E}\cdot\nabla\times\boldsymbol{B})\\&=-\boldsymbol{E}\cdot\boldsymbol{J}, &&\text{(steady state)}\\&=(\boldsymbol{u}\times\boldsymbol{B})\cdot\boldsymbol{J} &&\text{(ideal MHD)}\\&=-\boldsymbol{u}\cdot(\boldsymbol{J}\times\boldsymbol{B}).\end{aligned}$$

On the other hand

$$-\boldsymbol{E}=\boldsymbol{u}\times\boldsymbol{B}=r\omega\,\boldsymbol{e}_\phi\times\boldsymbol{B}$$

and so

$$-\boldsymbol{E}\times\boldsymbol{B}=r\omega(B_\phi\boldsymbol{B}-B^2\,\boldsymbol{e}_\phi).$$

Thus

$$\boldsymbol{u}\cdot(\boldsymbol{J}\times\boldsymbol{B})=\nabla\cdot\left[\frac{r\omega}{\mu_0}(B_\phi\boldsymbol{B}-B^2\,\boldsymbol{e}_\phi)\right]=\boldsymbol{B}\cdot\nabla\left(\frac{r\omega B_\phi}{\mu_0}\right).$$

The steady kinetic energy equation is then

$$\rho\boldsymbol{u}_\mathrm{p}\cdot\nabla\left(\tfrac{1}{2}u^2+\Phi+w-\frac{r\omega B_\phi}{\mu_0 k}\right)=0.$$

Therefore

$$\tfrac{1}{2}u^2+\Phi+w-\frac{r\omega B_\phi}{\mu_0 k}=\varepsilon,$$

where

$$\varepsilon=\varepsilon(\psi)$$

is another surface function, the *energy invariant*.

An alternative invariant is

$$\begin{aligned}\varepsilon'&=\varepsilon-\ell\omega\\&=\tfrac{1}{2}u^2+\Phi+w-\frac{r\omega B_\phi}{\mu_0 k}-\left(ru_\phi-\frac{rB_\phi}{\mu_0 k}\right)\omega\\&=\tfrac{1}{2}u^2+\Phi+w-ru_\phi\omega\\&=\tfrac{1}{2}u_\mathrm{p}^2+\tfrac{1}{2}(u_\phi-r\omega)^2+\Phi^\mathrm{cg}+w,\end{aligned}$$

where

$$\Phi^{\rm cg} = \Phi - \tfrac{1}{2}\omega^2 r^2$$

is the centrifugal–gravitational potential associated with the magnetic surface. One can then see that ε' is the Bernoulli function of the flow in the frame rotating with angular velocity ω. In this frame the flow is strictly parallel to the field and the field therefore does no work because $\boldsymbol{J} \times \boldsymbol{B} \perp \boldsymbol{B}$ and so $\boldsymbol{J} \times \boldsymbol{B} \perp (\boldsymbol{u} - r\omega\, \boldsymbol{e}_\phi)$.

The component of the equation of motion perpendicular to the magnetic surfaces has not been used yet. This 'transfield' or 'Grad–Shafranov' equation ultimately determines the equilibrium shape of the magnetic surfaces. It is a very complicated nonlinear partial differential equation and cannot be reduced to simple terms. We do not consider it here.

2.2.6. The Alfvén surface

Define the *poloidal Alfvén number* (cf. the Mach number)

$$A = \frac{u_{\rm p}}{v_{\rm Ap}}.$$

Then

$$A^2 = \frac{\mu_0 \rho u_{\rm p}^2}{B_{\rm p}^2} = \frac{\mu_0 k^2}{\rho},$$

and so $A \propto \rho^{-1/2}$ on each magnetic surface.

Consider the two equations

$$\rho u_\phi = k B_\phi + \rho r \omega,$$

$$r u_\phi = \frac{r B_\phi}{\mu_0 k} + \ell.$$

Eliminate B_ϕ to obtain

$$\begin{aligned} u_\phi &= \frac{r^2\omega - A^2 \ell}{r(1 - A^2)} \\ &= \left(\frac{1}{1 - A^2}\right) r\omega + \left(\frac{A^2}{A^2 - 1}\right)\frac{\ell}{r}. \end{aligned}$$

For $A \ll 1$ we have

$$u_\phi \approx r\omega,$$

i.e. the fluid is in uniform rotation, corotating with the magnetic surface. For $A \gg 1$ we have

$$u_\phi \approx \frac{\ell}{r},$$

i.e. the fluid conserves its specific angular momentum. The point $r = r_{\rm A}(\psi)$ where $A = 1$ is the *Alfvén point*. To avoid a singularity we require

$$\ell = r_{\rm A}^2 \omega.$$