2.2.7. Acceleration from the surface of the disc

If the flow starts essentially from rest in high-density material ($A \ll 1$), we have

$$\omega \approx \Omega,$$

i.e. the angular velocity of the magnetic surface is the angular velocity of the disc at the foot-point of the field line. In the sub-Alfvénic region we have

$$\varepsilon' \approx \tfrac{1}{2}u_\mathrm{p}^2 + \Phi^\mathrm{cg} + w.$$

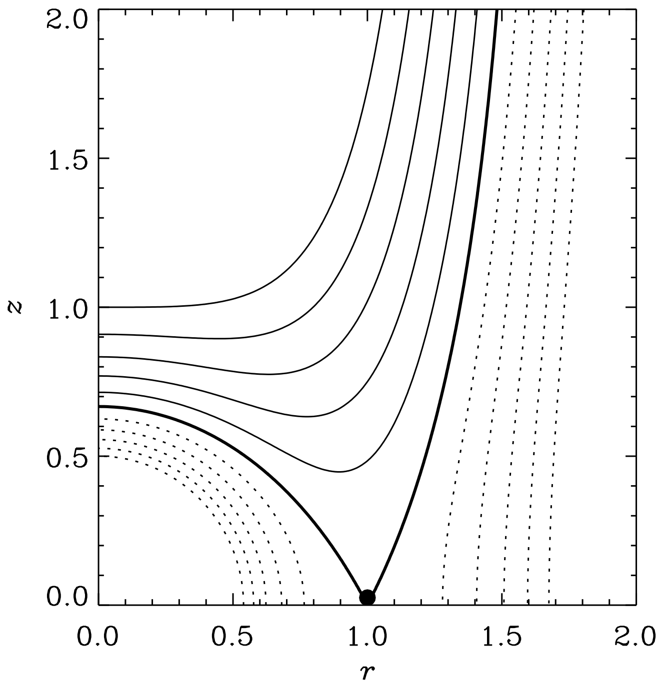
As in the case of stellar winds, if the gas is hot (comparable to the escape temperature) an outflow can be driven by thermal pressure. Of more interest here is the possibility of a dynamically driven outflow. For a 'cold' wind the enthalpy makes a negligible contribution in this equation. Whether the flow accelerates or not above the disc then depends on the variation of the centrifugal–gravitational potential along the field line.

Consider a Keplerian disc in a point-mass potential. Let the foot-point of the field line be at $r = r_0$, and let the angular velocity of the field line be

$$\omega = \Omega_0 = \left(\frac{GM}{r_0^3}\right)^{1/2},$$

as argued above. Then

$$\Phi^{\rm cg} = -GM(r^2+z^2)^{-1/2} - \frac{1}{2}\frac{GM}{r_0^3}r^2.$$



Contours of $\Phi^{\rm cg}$, in units such that $r_0 = 1$. The downhill directions are indicated by dashed contours.

In units such that $r_0 = 1$, the equation of the equipotential passing through the foot-point (r_0, z) is

$$(r^2+z^2)^{-1/2} + \frac{r^2}{2} = \frac{3}{2}.$$

This can be rearranged into the form

$$z^2 = \frac{(2-r)(r-1)^2(r+1)^2(r+2)}{(3-r^2)^2}.$$

Close to the foot-point $(1, 0)$ we have

$$z^2 \approx 3(r-1)^2$$

and so

$$z \approx \pm\sqrt{3}(r-1).$$

The foot-point lies at a saddle point of Φ^{cg}. If the inclination of the field line to the vertical, i, at the surface of the disc exceeds 30°, the flow is accelerated without thermal assistance. This is *magnetocentrifugal acceleration*.

The critical equipotential has an asymptote at $r = r_0\sqrt{3}$. The field line must continue to expand sufficiently in the radial direction in order to sustain the magnetocentrifugal acceleration.

2.2.8. Magnetically driven accretion

To allow a quantity of mass ΔM_{acc} to be accreted from radius r_0, the angular momentum that must be removed is $r_0^2\Omega_0\,\Delta M_{\text{acc}}$. The angular momentum removed by a quantity of mass ΔM_{jet} flowing out in a magnetized jet from radius r_0 is $\ell\,\Delta M_{\text{jet}} = r_{\text{A}}^2\Omega_0\,\Delta M_{\text{jet}}$. Therefore accretion can in principle be driven by an outflow, with

$$\frac{\dot{M}_{\text{acc}}}{\dot{M}_{\text{jet}}} \sim \frac{r_{\text{A}}^2}{r_0^2}.$$

For the efficient removal of angular momentum, the Alfvén radius should be large compared to the foot-point radius. This effect is the *magnetic lever arm*.

2.3. The shearing sheet

2.3.1. Basic concept

Reference: Goldreich P. & Lynden-Bell D. (1965), *Mon. Not. R. Astron. Soc.*, **130**, 125

The *shearing sheet* is a very useful local model of a differentially rotating disc.

Consider a reference point on the mid-plane of the disc, orbiting the central mass. It has cylindrical coordinates $(r_0, \phi_0 + \Omega_0 t, 0)$. Use this point as the origin of a local Cartesian coordinate system (x, y, z), with unit vectors $(\boldsymbol{e}_x, \boldsymbol{e}_y, \boldsymbol{e}_z)$ pointing in the radial, azimuthal and vertical directions, respectively.

The flow is represented locally as a uniform rotation $\boldsymbol{\Omega}_0 = \Omega_0\, \boldsymbol{e}_z$ plus a linear shear flow

$$\boldsymbol{u}_0 = -2A_0 x\, \boldsymbol{e}_y,$$

where

$$A = -\frac{r}{2}\frac{\mathrm{d}\Omega}{\mathrm{d}r}$$

is *Oort's first parameter* (proportional to the shear rate). This is the simplest local representation of differential rotation, and originated in studies of galactic dynamics. The flow is considered to be unbounded in the xy-plane.

2.3.2. Possible treatments of the vertical direction

- Ignore the vertical direction completely (2D shearing sheet).
- Treat the fluid as incompressible, so that vertical gravity and stratification can be ignored.
- Include the full vertical structure (e.g. isothermal, polytropic or radiative).

2.3.3. Symmetries of the sheet

- Every point in the xy-plane is equivalent, modulo a Galilean boost. The system is therefore spatially homogeneous, and can develop statistically homogeneous (but anisotropic) turbulence if it is unstable.
- The inside and outside of the sheet cannot be distinguished.