2.3.4. Equation of motion in the sheet

Write the total velocity as

$$\begin{aligned}\boldsymbol{u} &= \boldsymbol{u}_0 + \boldsymbol{v} \\ &= -2A_0 x\,\boldsymbol{e}_y + \boldsymbol{v},\end{aligned}$$

where $\boldsymbol{v}$ is the velocity relative to the background shear flow. The equation of motion in the rotating frame is

$$\rho\left(\frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} + 2\boldsymbol{\Omega}_0 \times \boldsymbol{u}\right) = -\rho\nabla\Phi^{\mathrm{cg}} - \nabla p + \nabla\cdot\mathbf{T},$$

where $\Phi^{\mathrm{cg}} = \Phi - \frac{1}{2}\Omega_0^2 r^2$ is the centrifugal–gravitational potential. Thus

$$\left(\frac{\partial}{\partial t} - 2A_0 x\frac{\partial}{\partial y} + \boldsymbol{v}\cdot\nabla\right)(-2A_0 x\,\boldsymbol{e}_y + \boldsymbol{v}) + 2\boldsymbol{\Omega}_0 \times (-2A_0 x\,\boldsymbol{e}_y + \boldsymbol{v}) = -\nabla\Phi^{\mathrm{cg}} - \frac{1}{\rho}\nabla p + \frac{1}{\rho}\nabla\cdot\mathbf{T}.$$

In a centrifugally supported disc, the horizontal inertial and gravitational forces balance exactly, i.e.

$$-\rho r\Omega^2 = -\rho\frac{\partial\Phi}{\partial r},$$

and so

$$\begin{aligned}\frac{\partial\Phi^{\mathrm{cg}}}{\partial r} &= r(\Omega^2 - \Omega_0^2) \\ &= 2r_0\Omega_0\left(\frac{\mathrm{d}\Omega}{\mathrm{d}r}\right)_0 (r - r_0) + O(r - r_0)^2.\end{aligned}$$

In the shearing sheet, the shear flow is represented in a linear approximation, and this relation becomes

$$\frac{\partial\Phi^{\mathrm{cg}}}{\partial x} = -4\Omega_0 A_0 x.$$

These two terms cancel out of the equation of motion, leaving the equation of motion in the sheet,

$$\left(\frac{\partial}{\partial t} - 2Ax\frac{\partial}{\partial y}\right)\boldsymbol{v} - 2Av_x\,\boldsymbol{e}_y + 2\boldsymbol{\Omega}\times\boldsymbol{v} + \boldsymbol{v}\cdot\nabla\boldsymbol{v} = -\Omega_z^2 z\,\boldsymbol{e}_z - \frac{1}{\rho}\nabla p + \frac{1}{\rho}\nabla\cdot\mathbf{T}.$$

Here and below, we drop the subscript '0' on Ω and A. The basic state of the sheet is $\boldsymbol{v} = \boldsymbol{0}$, with vertical hydrostatic equilibrium.

2.4. Hydrodynamic stability in the incompressible shearing sheet

2.4.1. Introduction

An accretion disc has a systematic *differential rotation*, or shear. This is potentially a source of energy and therefore of instability.

Exercise: Suppose that the density $\rho(\boldsymbol{r})$ and the total angular momentum of a fluid body,

$$H = \int \rho r u_\phi \,\mathrm{d}V,$$

are fixed. Show that the total kinetic energy,

$$K = \int \tfrac{1}{2}\rho u^2 \,\mathrm{d}V,$$

is minimized when the fluid is in uniform rotation.

The hydrodynamic stability of circular Keplerian motion is of fundamental interest in studies of discs. If the motion is unstable it may develop turbulence, which may transport angular momentum. The shearing sheet provides a convenient framework within which to study the local stability of a differentially rotating disc.

2.4.2. Basic equations

For an incompressible shearing sheet with a uniform density ρ and uniform kinematic viscosity ν, but no magnetic fields, the equation of motion may be written

$$\left(\frac{\partial}{\partial t} - 2Ax\frac{\partial}{\partial y}\right)\boldsymbol{v} - 2Av_x\,\boldsymbol{e}_y + 2\boldsymbol{\Omega}\times\boldsymbol{v} + \boldsymbol{v}\cdot\nabla\boldsymbol{v} = -\nabla\psi + \nu\nabla^2\boldsymbol{v},$$

where

$$\psi = \frac{p}{\rho} + \frac{1}{2}\Omega_z^2 z^2$$

is a modified pressure. In fact ψ measures the deviation from vertical hydrostatic equilibrium. The equation of mass conservation is

$$\nabla\cdot\boldsymbol{v} = 0.$$

It is conventional to include a small viscosity ν in the stability analysis, partly to regularize the problem, and partly to clarify the role of the Reynolds number.

2.4.3. Plane-wave solutions

Although the basic equations contain x explicitly, a modified Fourier analysis can still be applied. Solutions exist in the form of *sheared plane waves*, a technique due to Kelvin.

Consider the action of the operator $\partial_t - 2Ax\partial_y$ on a plane wave of the form

$$f(\boldsymbol{r},t) = \mathrm{Re}\left\{\tilde{f}(t)\,\exp\left[\mathrm{i}\boldsymbol{k}(t)\cdot\boldsymbol{r}\right]\right\},$$

where $\boldsymbol{k}(t)$ is a time-dependent wavevector:

$$\left(\frac{\partial}{\partial t} - 2Ax\frac{\partial}{\partial y}\right) f = \mathrm{Re}\left\{\left[\frac{\mathrm{d}\tilde{f}}{\mathrm{d}t} + \left(\mathrm{i}\frac{\mathrm{d}\boldsymbol{k}}{\mathrm{d}t}\cdot\boldsymbol{r} - 2\mathrm{i}Axk_y\right)\tilde{f}\right]\exp\left[\mathrm{i}\boldsymbol{k}(t)\cdot\boldsymbol{r}\right]\right\}.$$

Let $\boldsymbol{k}$ evolve according to

$$\frac{\mathrm{d}\boldsymbol{k}}{\mathrm{d}t} = 2Ak_y\,\boldsymbol{e}_x,$$

then

$$\left(\frac{\partial}{\partial t} - 2Ax\frac{\partial}{\partial y}\right) f = \mathrm{Re}\left\{\frac{\mathrm{d}\tilde{f}}{\mathrm{d}t}\,\exp\left[\mathrm{i}\boldsymbol{k}(t)\cdot\boldsymbol{r}\right]\right\}.$$

This evolution of $\boldsymbol{k}$ represents the action of the background shear flow on the wavefronts:

$$k_x = k_{x0} + 2Ak_y t, \qquad k_y = \text{constant}, \qquad k_z = \text{constant}.$$

Consider plane-wave solutions of this form, i.e.

$$\boldsymbol{v} = \mathrm{Re}\left\{\tilde{\boldsymbol{v}}(t)\,\exp\left[\mathrm{i}\boldsymbol{k}(t)\cdot\boldsymbol{r}\right]\right\},$$
$$\psi = \mathrm{Re}\left\{\tilde{\psi}(t)\,\exp\left[\mathrm{i}\boldsymbol{k}(t)\cdot\boldsymbol{r}\right]\right\}.$$

Then we obtain

$$\frac{\mathrm{d}\tilde{\boldsymbol{v}}}{\mathrm{d}t} - 2A\tilde{v}_x\,\boldsymbol{e}_y + 2\boldsymbol{\Omega}\times\tilde{\boldsymbol{v}} = -\mathrm{i}\boldsymbol{k}\tilde{\psi} - \nu k^2\tilde{\boldsymbol{v}},$$

$$\mathrm{i}\boldsymbol{k}\cdot\tilde{\boldsymbol{v}} = 0.$$

The nonlinear term $\boldsymbol{v}\cdot\nabla\boldsymbol{v}$ vanishes owing to the incompressibility condition:

$$\tilde{\boldsymbol{v}}\cdot\nabla\,\mathrm{e}^{\pm\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}} = \pm(\mathrm{i}\boldsymbol{k}\cdot\tilde{\boldsymbol{v}})\,\mathrm{e}^{\pm\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}} = 0.$$

The viscous term is taken care of by a *viscous decay factor*

$$\begin{aligned} E_\nu(t) &= \exp\left(-\int \nu k^2\,\mathrm{d}t\right) \\ &= \exp\left\{-\nu\left[(k_{x0}^2 + k_y^2 + k_z^2)t + 2Ak_{x0}k_y t^2 + \frac{4}{3}A^2k_y^2t^3\right]\right\}. \end{aligned}$$

When $k_y = 0$ (unsheared, 'axisymmetric' waves), E_ν decays exponentially. When $k_y \neq 0$ (sheared, 'non-axisymmetric' waves), E_ν decays superexponentially.

Let

$$\tilde{\boldsymbol{v}} = E_\nu(t)\hat{\boldsymbol{v}}(t), \qquad \tilde{\psi} = E_\nu(t)\hat{\psi}(t).$$

We then obtain the inviscid problem

$$\begin{aligned} \frac{\mathrm{d}\hat{v}_x}{\mathrm{d}t} \qquad\quad - 2\Omega\hat{v}_y &= -\mathrm{i}k_x\hat{\psi}, \\ \frac{\mathrm{d}\hat{v}_y}{\mathrm{d}t} + 2(\Omega - A)\hat{v}_x &= -\mathrm{i}k_y\hat{\psi}, \\ \frac{\mathrm{d}\hat{v}_z}{\mathrm{d}t} \qquad\qquad\qquad &= -\mathrm{i}k_z\hat{\psi}, \\ \mathrm{i}\boldsymbol{k}\cdot\hat{\boldsymbol{v}} &= 0. \end{aligned}$$

Eliminate variables in favour of $\hat{v}_x$ to obtain

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}(k^2\hat{v}_x) + \kappa^2k_z^2\hat{v}_x = 0, \tag{1}$$

where

$$\begin{aligned} \kappa^2 &= 4\Omega(\Omega - A) \\ &= 4\Omega^2 + 2r\Omega\frac{\mathrm{d}\Omega}{\mathrm{d}r} \end{aligned}$$

is the square of the epicyclic frequency in the shearing sheet.

Exercise: Derive equation (1). It is helpful to consider $(\mathrm{d}/\mathrm{d}t)(\mathrm{i}\boldsymbol{k}\cdot\hat{\boldsymbol{v}}) = 0$.

In the following, we assume that $\Omega \neq 0$ and $\kappa^2 \neq 0$. [See example 2.2 for the case of a non-rotating shearing sheet.]