2.4.4. The case $k_y = 0$ (axisymmetric waves)

Here $\boldsymbol{k}$ is constant and the equations have constant coefficients. The solutions have an exponential form, with

$$(\hat{\boldsymbol{v}}, \hat{\psi}) \propto \exp\left[\pm\left|\frac{k_z}{k}\right| \mathrm{i}\kappa t\right]$$

when $\kappa^2 > 0$, or

$$(\hat{\boldsymbol{v}}, \hat{\psi}) \propto \exp\left[\pm\left|\frac{k_z}{k}\right| (-\kappa^2)^{1/2} t\right]$$

when $\kappa^2 < 0$. Recall that $\tilde{\boldsymbol{v}} = \hat{\boldsymbol{v}}\exp(-\nu k^2 t)$. Therefore all solutions decay ($\nu \neq 0$) or oscillate ($\nu = 0$) when $\kappa^2 > 0$. When $\kappa^2 < 0$, there exist growing solutions for wavevectors satisfying

$$\nu k^2 < \left|\frac{k_z}{k}\right| (-\kappa^2)^{1/2},$$

i.e. for sufficiently long wavelengths.

2.4.5. The case $k_y \neq 0$ (non-axisymmetric waves)

Here k^2 is not constant and the general solution of equation (1) involves Legendre functions. Consider the asymptotic behaviour as $t \to \infty$, where

$$k^2 \sim 4A^2 k_y^2 t^2.$$

(Here the symbol $\sim$ has its conventional meaning in asymptotic analysis.) Equation (1) has a regular singular point at $t = \infty$. The solutions have the asymptotic form

$$\hat{v}_x \propto t^{\sigma-(3/2)},$$

where σ is a complex number satisfying the indicial equation

$$(\sigma + \tfrac{1}{2})(\sigma - \tfrac{1}{2})4A^2 k_y^2 + \kappa^2 k_z^2 = 0,$$

or

$$\sigma^2 = \frac{1}{4} - \frac{\kappa^2 k_z^2}{4A^2 k_y^2}.$$

The other variables have the asymptotic forms

$$\hat{v}_y \propto t^{\sigma-(1/2)}, \qquad \hat{v}_z \propto t^{\sigma-(1/2)}, \qquad \hat{\psi} \propto t^{\sigma-(3/2)}.$$

There are three cases to consider.

(i) When $\kappa^2 > A^2(k_y^2/k_z^2)$, σ is imaginary and $|\hat{\boldsymbol{v}}|^2 \propto t^{-1}$ as $t \to \infty$.

(ii) When $0 < \kappa^2 < A^2(k_y^2/k_z^2)$, σ is real with $0 < \sigma^2 < \frac{1}{4}$. So $\sigma < \frac{1}{2}$ and $|\hat{\boldsymbol{v}}|^2 \propto t^{2\sigma-1} \to 0$ as $t \to \infty$.

(iii) When $\kappa^2 < 0$, σ is real with $\sigma^2 > \frac{1}{4}$. One solution has $\sigma > \frac{1}{2}$ and then $|\hat{\boldsymbol{v}}|^2 \propto t^{2\sigma-1} \to \infty$ as $t \to \infty$.

Therefore the inviscid solutions decay when $\kappa^2 > 0$ but grow (in the energy norm) when $\kappa^2 < 0$. When $\nu \neq 0$, however, the viscous decay factor E_ν kills off any algebraic growth as $t \to \infty$.

2.4.6. Rayleigh's criterion

When $\kappa^2 > 0$, the flow is hydrodynamically stable. When $\kappa^2 < 0$, the flow is hydrodynamically unstable. This is *Rayleigh's criterion*. It is the same criterion as for the stability of circular test-particle orbits.

The analysis of a cylindrical shear flow $\boldsymbol{u} = r\Omega(r)\,\boldsymbol{e}_\phi$ confirms this result, but only for infinitesimal, axisymmetric perturbations of an inviscid fluid.

The case $\kappa^2 = 0$ (either a non-rotating shear flow or one with uniform specific angular momentum) is marginally stable and requires special consideration [see example 2.2]. Axisymmetric inviscid solutions exhibit algebraic growth, $|\hat{\boldsymbol{v}}|^2 \propto t^2$ as $t \to \infty$. In the presence of a small viscosity this allows a large *transient amplification* of disturbances.

2.4.7. Comparison with laboratory shear flows

Non-rotating plane Couette flow between rigid walls is stable to all infinitesimal disturbances, for any finite Reynolds number.

However, nonlinear coupling of transiently amplified waves allows *nonlinear instability* and transition to turbulence at sufficiently high Reynolds number. This effect is a consequence of being close to marginal linear stability and (probably) does not apply to Keplerian discs.