2.5. Magnetohydrodynamic stability in the incompressible shearing sheet

2.5.1. Basic equations

For an incompressible shearing sheet of uniform density ρ, kinematic viscosity ν and magnetic diffusivity η, the equation of motion and induction equation may be written

$$\left(\frac{\partial}{\partial t} - 2Ax\frac{\partial}{\partial y}\right)\boldsymbol{v} - 2Av_x\,\boldsymbol{e}_y + 2\boldsymbol{\Omega}\times\boldsymbol{v} + \boldsymbol{v}\cdot\nabla\boldsymbol{v} = -\nabla\psi + \frac{1}{\mu_0\rho}\boldsymbol{B}\cdot\nabla\boldsymbol{B} + \nu\nabla^2\boldsymbol{v},$$

$$\left(\frac{\partial}{\partial t} - 2Ax\frac{\partial}{\partial y}\right)\boldsymbol{B} + \boldsymbol{v}\cdot\nabla\boldsymbol{B} = -2AB_x\,\boldsymbol{e}_y + \boldsymbol{B}\cdot\nabla\boldsymbol{v} + \eta\nabla^2\boldsymbol{B},$$

where now

$$\psi = \frac{\Pi}{\rho} + \Omega_z^2 z$$

and

$$\Pi = p + \frac{B^2}{2\mu_0}$$

is the *total pressure* (gas plus magnetic). The velocity perturbation and magnetic field are both solenoidal,

$$\nabla\cdot\boldsymbol{v} = \nabla\cdot\boldsymbol{B} = 0.$$

A simple solution with a uniform magnetic field is

$$\boldsymbol{v} = \boldsymbol{0}, \qquad \psi = \text{constant}, \qquad \boldsymbol{B} = \bar{\boldsymbol{B}}(t),$$

with

$$\bar{B}_x = \text{constant}, \qquad \bar{B}_y = \bar{B}_{y0} - 2A\bar{B}_x t, \qquad \bar{B}_z = \text{constant}.$$

Consider the stability of this solution to sheared plane-wave disturbances. Let

$$\boldsymbol{v} = \mathrm{Re}\left\{\tilde{\boldsymbol{v}}(t)\,\exp\left[\mathrm{i}\boldsymbol{k}(t)\cdot\boldsymbol{r}\right]\right\},$$
$$\psi = \mathrm{Re}\left\{\tilde{\psi}(t)\,\exp\left[\mathrm{i}\boldsymbol{k}(t)\cdot\boldsymbol{r}\right]\right\},$$
$$\boldsymbol{B} = \bar{\boldsymbol{B}}(t) + \mathrm{Re}\left\{\tilde{\boldsymbol{B}}(t)\,\exp\left[\mathrm{i}\boldsymbol{k}(t)\cdot\boldsymbol{r}\right]\right\},$$

with

$$\frac{\mathrm{d}\boldsymbol{k}}{\mathrm{d}t} = 2Ak_y\,\boldsymbol{e}_x.$$

The wave amplitudes satisfy

$$\frac{\mathrm{d}\tilde{\boldsymbol{v}}}{\mathrm{d}t} - 2A\tilde{v}_x\,\boldsymbol{e}_y + 2\boldsymbol{\Omega}\times\tilde{\boldsymbol{v}} = -\mathrm{i}\boldsymbol{k}\tilde{\psi} + \frac{1}{\mu_0\rho}(\mathrm{i}\boldsymbol{k}\cdot\bar{\boldsymbol{B}})\tilde{\boldsymbol{B}} - \nu k^2\tilde{\boldsymbol{v}},$$

$$\frac{\mathrm{d}\tilde{\boldsymbol{B}}}{\mathrm{d}t} = -2A\tilde{B}_x\,\boldsymbol{e}_y + (\mathrm{i}\boldsymbol{k}\cdot\bar{\boldsymbol{B}})\tilde{\boldsymbol{v}} - \eta k^2\tilde{\boldsymbol{B}},$$

$$\mathrm{i}\boldsymbol{k}\cdot\tilde{\boldsymbol{v}} = \mathrm{i}\boldsymbol{k}\cdot\tilde{\boldsymbol{B}} = 0.$$

Again, the nonlinear wave-wave self-interaction terms vanish.

The background field $\bar{\boldsymbol{B}}$ affects the dynamics only through the *Alfvén frequency*

$$\omega_\mathrm{A} = \boldsymbol{k}\cdot\boldsymbol{v}_\mathrm{A} = (\mu_0\rho)^{-1/2}\boldsymbol{k}\cdot\bar{\boldsymbol{B}},$$

which is constant:

$$\frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{k}\cdot\bar{\boldsymbol{B}}) = 2Ak_y\bar{B}_x - k_y 2A\bar{B}_x = 0.$$

ω_A measures the restoring effect of magnetic tension on the disturbance.

The analysis of general disturbances is difficult. Consider purely horizontal disturbances with a purely vertical wavevector:

$$k_x = k_y = 0, \qquad \tilde{v}_z = \tilde{B}_z = \tilde{\psi} = 0.$$

The equations have constant coefficients and admit solutions of an exponential form $\propto \exp(-\mathrm{i}\omega t)$, where ω is the frequency eigenvalue. Instability occurs if

$$\mathrm{Im}(\omega) > 0.$$

We obtain

$$\begin{aligned}
-\mathrm{i}\omega\tilde{v}_x \qquad\quad - 2\Omega\tilde{v}_y &= \mathrm{i}\omega_\mathrm{A}\tilde{b}_x - \nu k^2\tilde{v}_x,\\
-\mathrm{i}\omega\tilde{v}_y + 2(\Omega - A)\tilde{v}_x &= \mathrm{i}\omega_\mathrm{A}\tilde{b}_y - \nu k^2\tilde{v}_y,\\
-\mathrm{i}\omega\tilde{b}_x \qquad\qquad\qquad\quad &= \mathrm{i}\omega_\mathrm{A}\tilde{v}_x - \eta k^2\tilde{b}_x,\\
-\mathrm{i}\omega\tilde{b}_y \qquad\quad + 2A\tilde{b}_x &= \mathrm{i}\omega_\mathrm{A}\tilde{v}_y - \eta k^2\tilde{b}_y,
\end{aligned}$$

where $\tilde{\boldsymbol{b}} = (\mu_0\rho)^{-1/2}\tilde{\boldsymbol{B}}$. For a non-trivial solution, ω must satisfy the *magnetorotational dispersion relation*

$$\det\begin{bmatrix} -\mathrm{i}\omega+\nu k^2 & -2\Omega & -\mathrm{i}\omega_\mathrm{A} & 0 \\ 2(\Omega-A) & -\mathrm{i}\omega+\nu k^2 & 0 & -\mathrm{i}\omega_\mathrm{A} \\ -\mathrm{i}\omega_\mathrm{A} & 0 & -\mathrm{i}\omega+\eta k^2 & 0 \\ 0 & -\mathrm{i}\omega_\mathrm{A} & 2A & -\mathrm{i}\omega+\eta k^2 \end{bmatrix} = 0,$$

which simplifies to

$$\left[(\omega+\mathrm{i}\nu k^2)(\omega+\mathrm{i}\eta k^2)-\omega_\mathrm{A}^2\right]^2 - 4\Omega(\Omega-A)(\omega+\mathrm{i}\eta k^2)^2 - 4\Omega A\omega_\mathrm{A}^2 = 0.$$

Recall that $4\Omega(\Omega-A)=\kappa^2$.

In the absence of a magnetic field,

$$\omega = \pm\kappa - \mathrm{i}\nu k^2,$$

the combination of an inertial oscillation (if $\kappa^2>0$) with viscous damping.

2.5.3. Ideal case

In the ideal case $\nu=\eta=0$, the dispersion relation is a quadratic equation for ω^2,

$$\omega^4 - (2\omega_\mathrm{A}^2+\kappa^2)\omega^2 + \omega_\mathrm{A}^2(\omega_\mathrm{A}^2-4\Omega A) = 0,$$

with solutions

$$\omega^2 = \omega_\mathrm{A}^2 + \frac{1}{2}\kappa^2\left[1\pm\left(1+\frac{16\omega_\mathrm{A}^2\Omega^2}{\kappa^4}\right)^{1/2}\right].$$

Assume that $\kappa^2>0$, since otherwise the flow is already hydrodynamically unstable according to Rayleigh's criterion. Both roots are then real, and the sum of roots is positive. Therefore at least one root is positive. Instability occurs if and only if the product of roots is negative, i.e. if

$$-2r\Omega\frac{\mathrm{d}\Omega}{\mathrm{d}r} > \omega_\mathrm{A}^2.$$

This is the *magnetorotational instability* (MRI). The unstable mode corresponds to the root

$$\omega^2 = \omega_\mathrm{A}^2 + \tfrac{1}{2}\kappa^2\left[1-\left(1+\frac{16\omega_\mathrm{A}^2\Omega^2}{\kappa^4}\right)^{1/2}\right].$$

The maximal possible growth rate (as a function of k) is obtained by minimizing this expression with respect to ω_A^2:

$$0 = \frac{\partial\omega^2}{\partial\omega_\mathrm{A}^2} = 1 - \frac{4\Omega^2}{\kappa^2}\left(1+\frac{16\omega_\mathrm{A}^2\Omega^2}{\kappa^4}\right)^{-1/2},$$

which implies

$$\omega_{\mathrm{A}}^2 = \Omega^2 - \frac{\kappa^4}{16\Omega^2}$$

and

$$(\omega^2)_{\mathrm{min}} = -A^2.$$

Therefore the maximal possible growth rate is the Oort parameter (shear rate) A. For a Keplerian disc ($\kappa = \Omega$) the maximal growth rate is $\frac{3}{4}\Omega$ and the optimal vertical wavenumber is

$$k = \sqrt{\frac{15}{16}}\frac{\Omega}{v_{\mathrm{A}}}.$$

In the limit $B_z \to 0$ this corresponds to a smaller and smaller length scale. Ultimately diffusion will become important.