2.5.4. Non-ideal case

In the simplest non-ideal case, $\nu = \eta$, the frequency eigenvalues are simply shifted in the complex plane:

$$\omega = \omega_{\text{ideal}} - \mathrm{i}\eta k^2.$$

This either damps oscillatory modes ($\omega_{\text{ideal}}^2 > 0$) or reduces the growth rates of unstable modes ($\omega_{\text{ideal}}^2 < 0$) by an amount ηk^2. If k can take any real value, instability always persists for sufficiently small k, although the growth rate may then be very small.

2.5.5. Effect of vertical boundaries

In reality the finite thickness of the disc restricts the values of k. Suppose that k is limited to the discrete values

$$k = \frac{n\pi}{2H}, \qquad n \in \mathbf{Z},$$

i.e. the eigenfunctions are proportional to

$$\sin\left(\frac{n\pi z}{2H}\right) \qquad \text{or} \qquad \cos\left(\frac{n\pi z}{2H}\right).$$

For $n = 0$ there is no instability, so consider the lowest possible non-trivial wavenumber $n = 1$. In the ideal case there is instability when

$$-2r\Omega\frac{\mathrm{d}\Omega}{\mathrm{d}r} > \omega_{\mathrm{A}}^2.$$

For a Keplerian disc this implies

$$v_{\mathrm{A}} < \frac{2\sqrt{3}}{\pi} H\Omega,$$

or

$$v_{\mathrm{A}} \lesssim c_{\mathrm{s}}.$$

The diffusive damping rate of the $n = 1$ mode is $\eta(\pi/2H)^2$, while the ideal growth rate is $\sim kv_{\mathrm{A}} = v_{\mathrm{A}}(\pi/2H)$. The instability is therefore found for an intermediate range of field strengths, approximately

$$\frac{\eta}{H} \lesssim v_{\mathrm{A}} \lesssim c_{\mathrm{s}}.$$

The MRI requires sufficient ionization of the disc, and may fail in the cooler parts of protoplanetary discs.

2.5.6. Summary

In the absence of a magnetic field, the ideal, incompressible shearing sheet is linearly unstable when

$$4\Omega^2 + 2r\Omega\frac{\mathrm{d}\Omega}{\mathrm{d}r} < 0.$$

In the presence of an arbitrarily weak magnetic field with some vertical component, the ideal, incompressible shearing sheet is linearly unstable when

$$2r\Omega\frac{\mathrm{d}\Omega}{\mathrm{d}r} < 0.$$

This may be regarded as *Chandrasekhar's criterion.*

Alternatively, the system is *hydrodynamically unstable* if the *specific angular momentum* $|r^2\Omega|$ decreases outwards, but is *magnetohydrodynamically unstable* if the *angular velocity* $|\Omega|$ decreases outwards.

The angular velocity in a Keplerian accretion disc is

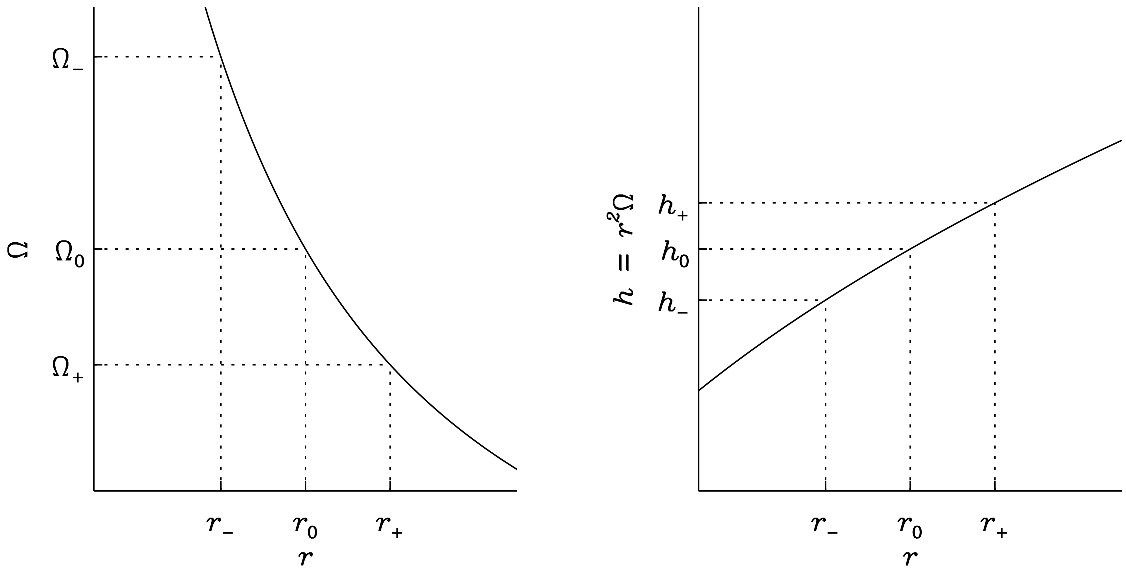
$$\Omega = \left(\frac{GM}{r^3}\right)^{1/2}.$$

This is hydrodynamically stable according to Rayleigh's criterion, but magnetohydrodynamically unstable in the presence of a weak magnetic field, according to Chandrasekhar's criterion.

The apparent paradox between these criteria is resolved when allowance is made for the finite height of the disc and for magnetic diffusivity, because then the field strength must then exceed a certain value to initiate instability.

2.5.7. *Physical interpretation*

To interpret these stability analyses in physical terms, consider the dynamics of a rotating ring of fluid embedded in a flow in which $|\Omega|$ decreases outwards while $|r^2\Omega|$ increases outwards, as in a Keplerian disc.



Angular velocity and specific angular momentum profiles

Unmagnetized fluid

Initially the ring has radius r_0, angular velocity Ω_0 and specific angular momentum h_0. Perturb it by spinning it up, increasing its angular momentum to $h_+ > h_0$. The excess centrifugal force causes it to expand. It can find an equilibrium at a radius $r_+ > r_0$ where $h(r_+) = h_+$. In fact the ring will overshoot and oscillate around r_+. These are stable epicyclic oscillations.

Now consider two rings connected by magnetic field lines. Perturb the rings by spinning up the upper ring, and spinning down the lower ring. If the magnetic field were inert, the upper ring would expand and oscillate around the radius r_+. In doing so it would slow down to an average angular velocity $\Omega_+ < \Omega_0$. The lower ring would contract and oscillate around the radius r_-, speeding up to an average angular velocity $\Omega_- > \Omega_0$.

Since $\Omega_+ < \Omega_-$, the upper ring lags behind the lower ring and the magnetic field lines are wound up. The tension in the field lines provides a torque, transferring angular momentum from the lower ring to the upper ring. (The magnetic field tries to enforce corotation.) However, the upper ring was given a positive angular momentum perturbation initially. The perturbation is therefore enhanced, causing an instability.

If the field is very weak, the growth rate is very small unless the rings are close together, corresponding to a mode with a short vertical wavelength (large k).