2.6. The magnetorotational instability: further developments

Reference: Balbus S. A. & Hawley J. F. (1998), *Rev. Mod. Phys.*, **70**, 1

2.6.1. Historical note

The magnetorotational instability (MRI) was first discovered by Velikhov (1959) in the context of vertically magnetized Couette flow between differentially rotating cylinders. His analysis was generalized by Chandrasekhar (1960) using a variational principle. The first astrophysical applications were made by Fricke (1969) and Acheson (1978), who considered differentially rotating stars containing magnetic fields. The instability has also appeared in geophysical studies. However, the more obvious application to accretion discs was neglected until the work of Balbus & Hawley (1991). Evidently the Velikhov–Chandrasekhar result had been widely overlooked or misunderstood; for example, Safronov (1969) explicitly dismissed it in connection with the evolution of the protosolar nebula. Since 1991, the MRI (or *Balbus–Hawley instability*) has acquired great importance in the theory of accretion discs. Many linear analyses and nonlinear simulations have been carried out, and have revealed several important properties.

2.6.2. Robustness

• *Field configuration:* The MRI exists whatever the direction of $\boldsymbol{B}$. In a local analysis, the important quantity is the Alfvén frequency $\omega_{\mathrm{A}} = (\mu_0\rho)^{-1/2}\boldsymbol{k}\cdot\boldsymbol{B}$, where $\boldsymbol{k}$ is the wavevector. The most effective perturbations have $\omega_{\mathrm{A}} \sim \Omega$. To have $\boldsymbol{k}\cdot\boldsymbol{B} \neq 0$, the perturbation must bend the field lines.

• *Field strength:* The MRI exists for intermediate magnetic field strengths. The lower limit is determined by dissipative processes, because the natural length-scale of the instability scales inversely with field strength. The upper limit is set by the finite size of the disc.

• *Other physics:* Including compressibility and stratification makes very little difference. The MRI also exists in weakly ionized gases. Some authors have estimated that an ionization fraction as low as 10^{-13} may be sufficient. Whether this can be achieved in the cooler parts of protoplanetary discs depends on factors such as cosmic-ray ionization and the abundance of radioactive species.

• *Global effects:* The MRI is essentially local and does not depend significantly on boundary conditions. In an extended disc, the fastest-growing normal modes are localized near the inner radius, where all the time-scales are shortest.

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2.6.3. *Nonlinear development*

The nonlinear development of the instability can be followed in a simple local model such as the incompressible shearing sheet. For computational purposes, the MHD equations must be solved in a finite box,

$$0 < x < L_x, \qquad 0 < y < L_y, \qquad 0 < z < L_z,$$

with some relatively neutral boundary conditions (e.g. periodic boundary conditions). The equations must also be discretized, either by representing the physical variables on a grid and applying finite differencing, or by projecting the equations on to a truncated set of functions such as Fourier modes.

In most implementations, any uniform vertical or azimuthal magnetic field imposed on the box is exactly conserved and cannot be altered by the internal dynamics. Such an imposed flux can act as a continuous source of instability, and therefore affects the nonlinear outcome.

The simplest case is that involving an *imposed vertical magnetic flux*. If the conditions for instability are met, the fastest-growing mode dominates the early evolution. This has the form of a 'channel flow' involving alternating layers of inward- and outward-moving fluid. The amplitude of this solution grows exponentially until it itself becomes unstable to three-dimensional 'parasitic modes' that feed off the gradients of velocity and magnetic field in the channel flow. The flow rapidly reaches a state of *magnetohydrodynamic turbulence*. A similar end-point is obtained in the case of an imposed azimuthal magnetic flux.

When there is *no imposed magnetic flux* there is no continuous source of linear instability. However, if a random magnetic field with zero mean is introduced initially into the box, it can act as a temporary source of instability, and the resulting motion can act to sustain or amplify the field. This is a *nonlinear dynamo process*, 'nonlinear' because the motion that sustains or amplifies the magnetic field is driven by the field itself through the MRI. The nonlinear dynamo process requires a significantly larger magnetic Reynolds number than the linear instability, but also leads to sustained MHD turbulence.

2.6.4. *Statistical properties of the turbulence*

The turbulent state has well-defined statistical properties and can be considered ideally to be *statistically steady*. Any fluctuating fluid property X can be regarded as the sum of its mean value plus a fluctuation of zero mean,

$$X = \bar{X} + X',$$

where $\bar{X} = \langle X \rangle$, $\langle X' \rangle = 0$ and the angle brackets denote a suitable averaging process. The mean value of a product is then

$$\langle XY \rangle = \bar{X}\bar{Y} + \langle X'Y' \rangle,$$

the second term being the correlation between the fluctuations.

Consider the equation of motion for an ideal, incompressible fluid in the shearing box,

$$\rho\left(\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u}\cdot\nabla\boldsymbol{u} + 2\boldsymbol{\Omega}\times\boldsymbol{u}\right) = -\nabla\Pi + \frac{1}{\mu_0}\boldsymbol{B}\cdot\nabla\boldsymbol{B}.$$

There are two nonlinear terms, $\boldsymbol{u}\cdot\nabla\boldsymbol{u}$ and $\boldsymbol{B}\cdot\nabla\boldsymbol{B}$. The averaged equation can be written in the form

$$\rho\left(\frac{\partial \bar{\boldsymbol{u}}}{\partial t} + \bar{\boldsymbol{u}}\cdot\nabla\bar{\boldsymbol{u}} + 2\boldsymbol{\Omega}\times\bar{\boldsymbol{u}}\right) = -\nabla\bar{\Pi} + \frac{1}{\mu_0}\bar{\boldsymbol{B}}\cdot\nabla\bar{\boldsymbol{B}} + \nabla\cdot\mathbf{T},$$

where

$$\mathbf{T} = \frac{\langle \boldsymbol{B}'\boldsymbol{B}' \rangle}{\mu_0} - \rho\langle \boldsymbol{u}'\boldsymbol{u}' \rangle$$

is the *mean turbulent stress*, and we have used $\nabla\cdot\boldsymbol{u}' = \nabla\cdot\boldsymbol{B}' = 0$. The first, magnetic contribution is the *turbulent Maxwell stress*, and the second, kinetic contribution is the *turbulent Reynolds stress*. These arise from correlations between fluctuating components of the magnetic and velocity fields.

As we have seen, the effective viscosity of accretion discs requires a stress $T_{r\phi} < 0$. In the shearing sheet this means $T_{xy} < 0$. In turbulence generated by the MRI, the mean turbulent stress always has this property, and the magnetic contribution always dominates.

In the turbulent state, the shear energy of the basic flow is continuously tapped by the instability. This energy goes into the kinetic and magnetic energies of the turbulent fluctuations. It is converted into heat by viscosity and resistivity at small length-scales.

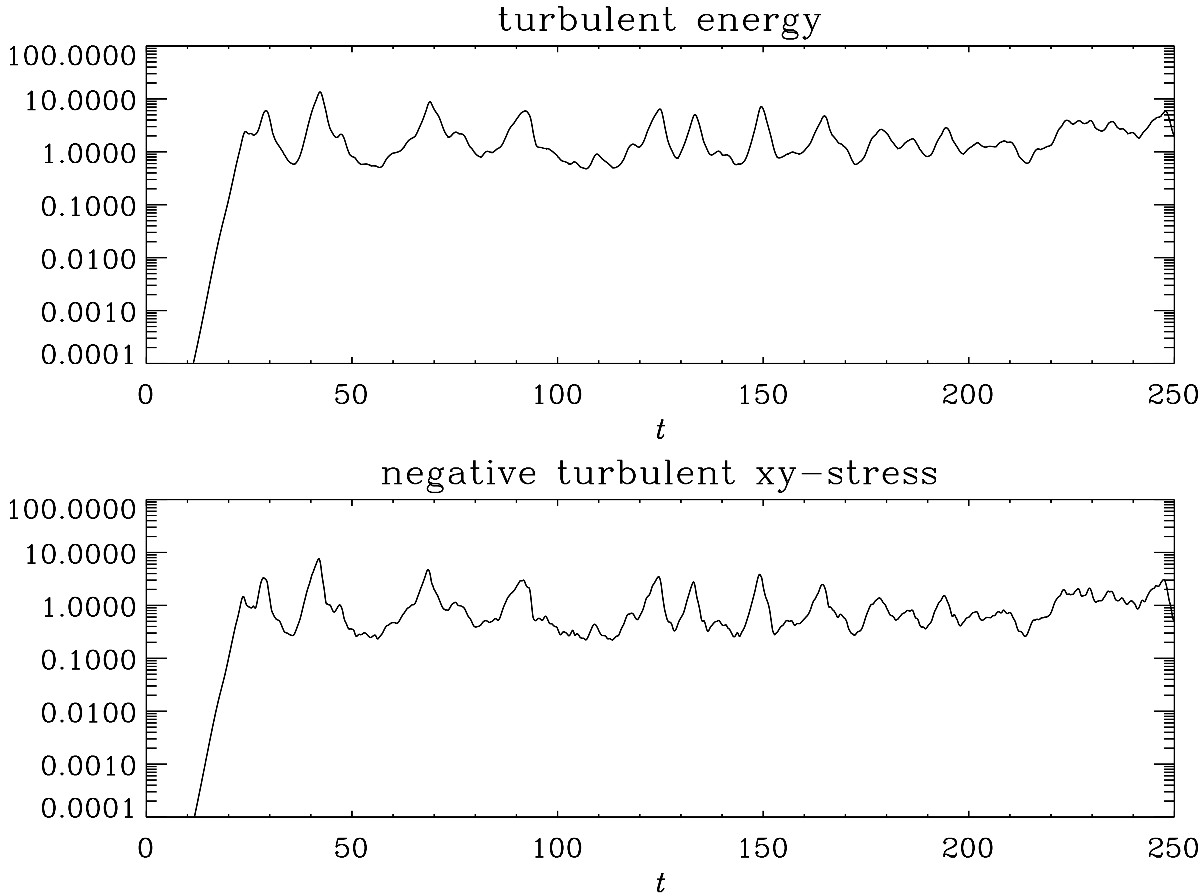
2.6.5. Alpha viscosity?

In summary, the MRI is widely applicable to gaseous accretion discs. Only a modest degree of ionization and a seed magnetic field are required. The instability saturates locally in a state of MHD turbulence, generating a mean turbulent stress that has the correct sign to explain the effective viscosity of accretion discs.

An important question: does the mean turbulent stress scale with the pressure, as assumed in the alpha viscosity prescription, and, if so, what is the value of alpha?

The simplest interpretation of the available facts is as follows. *This is a best guess, not to be taken as definitive.*

- When ν and η are made sufficiently small, their values do not affect the stress. The diffusivities merely provide an energy sink at small scales.



Development of statistically steady MHD turbulence in a incompressible shearing box with a weak imposed vertical magnetic field $B_z = (200)^{-1/2}$. Units are such that $L_x = L_y/4 = L_z = \Omega = \rho = \mu_0 = 1$. The shear is Keplerian ($A = 3/4$) and small diffusivities are included ($\nu = \eta = 0.001$). The instability initially grows exponentially from low-amplitude random perturbations but saturates in the nonlinear regime. The volume-averaged turbulent energy greatly exceeds that of the imposed magnetic flux (0.01). The *xy*-stress is dominated by the Maxwell component.

• When the horizontal dimensions of the box are increased, keeping L_z constant, the stress tends to a finite value. This limit represents the physical situation in a thin disc. The stress certainly depends on the value of L_z, which can be identified with the disc thickness $2H$.

• When there is no imposed magnetic flux, the typical velocity fluctuations scale as $|\boldsymbol{u}'| \sim \Omega H$, for purely dimensional reasons. The Reynolds stress then scales as

$$|T_{xy}| \sim \rho(\Omega H)^2 \sim p,$$

as assumed in the alpha viscosity prescription. The Maxwell stress is similar.

• The stress is larger in the presence of an imposed magnetic flux, which acts as a continuous source of instability, provided it is not so strong as to suppress the instability.

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In conclusion, the turbulent stress does resemble an alpha viscosity. Estimates of $\alpha \approx 0.01$ have been made when there is no imposed magnetic flux, increasing to $\alpha \approx 0.1 - 1$ when there is an imposed magnetic flux. These values are in the same range as those deduced from observations of time-dependent behaviour in accretion discs.