PART 1: CLASSICAL THEORY OF ACCRETION DISCS

1.1. Orbital dynamics

Reference: Lynden-Bell D. & Pringle J. E. (1974), *Mon. Not. R. Astron. Soc.*, **168**, 603

Consider the orbital motion, according to Newtonian dynamics, of a test particle in the gravitational potential Φ of a massive body (star, black hole, galaxy, etc.). Let (r,ϕ,z) be cylindrical polar coordinates and assume the potential is *axisymmetric*, *symmetric* and *'convex'*, i.e.

$$\Phi = \Phi(r,z), \qquad \Phi(r,-z) = \Phi(r,z), \qquad \Phi_{,zz}(r,0) > 0,$$

where the subscript comma denotes partial differentiation. Often we will assume the potential to be that of a point mass M (or the exterior potential of a spherical mass),

$$\Phi = -GM(r^2+z^2)^{-1/2}.$$

The orbits are then *Keplerian orbits.*

The equation of motion of the test particle is

$$\ddot{\boldsymbol{r}} = -\nabla\Phi.$$

The constants of the motion are the *specific energy*,

$$\tilde{\varepsilon} = \tfrac{1}{2}\dot{\boldsymbol{r}}^2 + \Phi,$$

and the *specific angular momentum*,

$$\tilde{h} = r^2\dot{\phi}.$$

The conservation of angular momentum may be used to reduce the equation of motion to the two-dimensional problem

$$\ddot{r} = -\Phi^{\text{eff}}_{,r}, \qquad \ddot{z} = -\Phi^{\text{eff}}_{,z},$$

where

$$\Phi^{\text{eff}} = \Phi + \frac{\tilde{h}^2}{2r^2}$$

is the *effective potential.* Then

$$\tilde{\varepsilon} = \tfrac{1}{2}(\dot{r}^2+\dot{z}^2) + \Phi^{\text{eff}}.$$

Suppose that the particle is able to dissipate energy (e.g. by radiation) but the angular momentum is conserved. The orbit of minimal energy for a given angular momentum $\tilde{h}$ is a circular orbit $\dot{r} = \dot{z} = 0$ in the mid-plane $z = 0$ at the radius r at which $\Phi^{\text{eff}}_{,r}(r, 0) = 0$. Then

$$0 = \Phi_{,r}(r, 0) - \frac{h^2}{r^3}$$

and

$$\varepsilon = \Phi(r, 0) + \frac{h^2}{2r^2},$$

where $\varepsilon(r)$ and $h(r)$ are the energy and angular momentum of the circular orbit at radius r. Note that

$$\frac{\mathrm{d}\varepsilon}{\mathrm{d}r} = \frac{h}{r^2}\frac{\mathrm{d}h}{\mathrm{d}r} \qquad \Rightarrow \qquad \frac{\mathrm{d}\varepsilon}{\mathrm{d}h} = \frac{h}{r^2} = \dot{\phi} = \Omega,$$

the *angular velocity*.

For Keplerian orbits in a point-mass potential we find

$$\Phi^{\text{eff}}(r, 0) = -\frac{GM}{r} + \frac{\tilde{h}^2}{2r^2},$$

and so

$$h = (GMr)^{1/2}, \qquad \varepsilon = -\frac{GM}{2r}, \qquad \Omega = \left(\frac{GM}{r^3}\right)^{1/2}.$$

[See example 1.1 for a more detailed revision of Keplerian orbits.]

1.2. Oscillations about minimal-energy orbits

For a minimal-energy orbit, $\nabla\Phi^{\text{eff}} = \mathbf{0}$. Consider small perturbations $(\delta r, \delta z)$ about such an orbit, at fixed angular momentum. The equation of motion becomes approximately

$$\ddot{\delta r} = -\kappa^2\, \delta r, \qquad \ddot{\delta z} = -\Omega_z^2\, \delta z,$$

where $\kappa(r)$ is the *epicyclic frequency* and $\Omega_z(r)$ the *vertical frequency*, defined by

$$\kappa^2 = \Phi^{\text{eff}}_{,rr}(r, 0), \qquad \Omega_z^2 = \Phi^{\text{eff}}_{,zz}(r, 0).$$

Thus

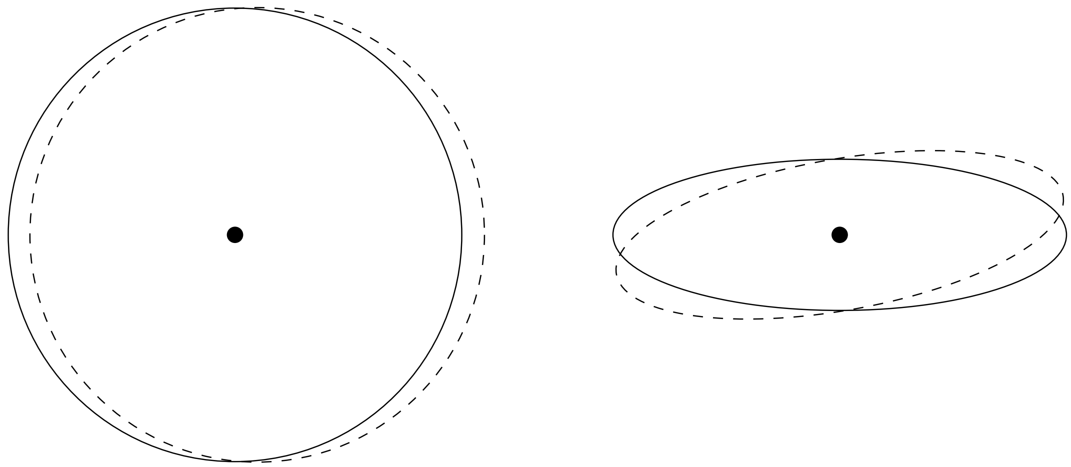
$$\kappa^2 = \Phi_{,rr}(r, 0) + \frac{3h^2}{r^4} = \frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{h^2}{r^3}\right) + \frac{3h^2}{r^4} = \frac{1}{r^3}\frac{\mathrm{d}h^2}{\mathrm{d}r}.$$

This is often written as

$$\kappa^2 = 4\Omega^2 + 2r\Omega\frac{\mathrm{d}\Omega}{\mathrm{d}r}.$$

Also

$$\Omega_z^2 = \Phi_{,zz}(r, 0).$$



Horizontal and vertical oscillations about a circular Keplerian orbit

For a point-mass potential we find

$$\kappa = \Omega_z = \Omega.$$

This commensurability means that either horizontal or vertical oscillations about a circular orbit result in closed figures. These are simply eccentric or inclined Keplerian orbits.

1.3. Angular momentum redistribution and energy dissipation

Consider two particles of masses m_1, m_2 in circular orbits of specific angular momenta h_1, h_2. Can the energy be further reduced by the exchange of angular momentum? The total energy and angular momentum are

$$E = E_1 + E_2 = m_1\varepsilon_1 + m_2\varepsilon_2,$$

$$H = H_1 + H_2 = m_1 h_1 + m_2 h_2,$$

and so

$$\mathrm{d}E = m_1\Omega_1\,\mathrm{d}h_1 + m_2\Omega_2\,\mathrm{d}h_2,$$

$$\mathrm{d}H = m_1\,\mathrm{d}h_1 + m_2\,\mathrm{d}h_2,$$

where we have used $\mathrm{d}\varepsilon/\mathrm{d}h = \Omega$. Subject to the constraint $\mathrm{d}H = 0$,

$$\mathrm{d}E = (\Omega_1 - \Omega_2)\,\mathrm{d}H_1.$$

Thus energy is released by transferring angular momentum to the particle with smaller Ω. Since $\mathrm{d}\Omega/\mathrm{d}r < 0$ in practice, this invariably means an *outward transfer of angular momentum.*

Now generalize the argument to allow mass to be exchanged as well. The total mass and angular momentum are fixed, so

$$\mathrm{d}M = \mathrm{d}m_1 + \mathrm{d}m_2 = 0,$$
$$\mathrm{d}H = \mathrm{d}H_1 + \mathrm{d}H_2 = 0,$$

where

$$\mathrm{d}H_1 = m_1\,\mathrm{d}h_1 + h_1\,\mathrm{d}m_1,$$

etc. Thus

$$\begin{aligned}\mathrm{d}E_1 &= m_1\Omega_1\,\mathrm{d}h_1 + \varepsilon_1\,\mathrm{d}m_1,\\ &= \Omega_1\,\mathrm{d}H_1 + (\varepsilon_1 - h_1\Omega_1)\,\mathrm{d}m_1,\end{aligned}$$

and so

$$\mathrm{d}E = (\Omega_1 - \Omega_2)\,\mathrm{d}H_1 + [(\varepsilon_1 - h_1\Omega_1) - (\varepsilon_2 - h_2\Omega_2)]\;\mathrm{d}m_1.$$

Now

$$\frac{\mathrm{d}}{\mathrm{d}r}(\varepsilon - h\Omega) = -h\frac{\mathrm{d}\Omega}{\mathrm{d}r}.$$

Since $\mathrm{d}\Omega/\mathrm{d}r < 0$ in practice, $(\mathrm{d}/\mathrm{d}r)(\varepsilon - h\Omega) > 0$. Thus energy is released by transferring *angular momentum outwards* and *mass inwards*. This is the basis of an accretion disc.