1.4. Viscous evolution of an accretion disc

1.4.1. Introduction

The evolution of an accretion disc is regulated by two conservation laws: conservation of mass, and conservation of angular momentum. These laws are embodied in the three-dimensional equations of fluid dynamics, but we must first reduce them to *one-dimensional* versions. The *equation of mass conservation* is

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \boldsymbol{u}) = 0,$$

where ρ is the density and $\boldsymbol{u}$ the velocity of the fluid. The *equation of motion* is

$$\rho \frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} = -\rho \nabla \Phi - \nabla p + \nabla \cdot \mathbf{T},$$

where

$$\frac{\mathrm{D}}{\mathrm{D}t} = \frac{\partial}{\partial t} + \boldsymbol{u} \cdot \nabla$$

is the *Lagrangian time-derivative* following the fluid flow, Φ is the gravitational potential, p is the pressure and $\mathbf{T}$ is the stress tensor. We adopt cylindrical polar coordinates (r, ϕ, z) for all calculations, such that the central mass is at $r = z = 0$ and the mid-plane of the disc is $z = 0$.

The origin of the stress $\mathbf{T}$ is a matter of central importance in accretion disc theory. It may be a turbulent stress, most likely involving tangled magnetic fields. There may also be contributions from large-scale magnetic fields, or from waves in the disc. These possibilities will be explored later in the course. In the classical theory, the stress is parametrized as a viscous stress.

1.4.2. Conservation of mass

Define the *surface density* $\Sigma(r, t)$ according to

$$\Sigma = \frac{1}{2\pi} \int_0^{2\pi} \int_{-\infty}^{\infty} \rho \, \mathrm{d}z \, \mathrm{d}\phi.$$

The mass contained between r_1 and r_2 is then

$$\int_{r_1}^{r_2} \Sigma \, 2\pi r \, \mathrm{d}r.$$

The equation of mass conservation is

$$\frac{\partial\rho}{\partial t}+\frac{1}{r}\frac{\partial}{\partial r}(r\rho u_r)+\frac{1}{r}\frac{\partial}{\partial\phi}(\rho u_\phi)+\frac{\partial}{\partial z}(\rho u_z)=0.$$

Integrate with respect to ϕ and z, over the full extent of the disc, to obtain

$$2\pi\frac{\partial\Sigma}{\partial t}+\frac{1}{r}\frac{\partial\mathcal{F}}{\partial r}=0,$$

where $\mathcal{F}(r,t)$ is the *radial mass flux*, defined by

$$\mathcal{F}=\int_0^{2\pi}\int_{-\infty}^{\infty}r\rho u_r\,\mathrm{d}z\,\mathrm{d}\phi,$$

and we assume that there is no vertical loss of mass to $z=\pm\infty$. One can also define a (density-weighted) *mean radial velocity* $\bar{u}_r(r,t)$ according to

$$\mathcal{F}=2\pi r\Sigma\bar{u}_r.$$

We then have

$$\frac{\partial\Sigma}{\partial t}+\frac{1}{r}\frac{\partial}{\partial r}(r\Sigma\bar{u}_r)=0, \tag{1}$$

which expresses the conservation of mass in one dimension.

1.4.3. Conservation of angular momentum

The azimuthal component of the equation of motion is

$$\rho\left(\frac{\mathrm{D}u_\phi}{\mathrm{D}t}+\frac{u_r u_\phi}{r}\right)=-\frac{\rho}{r}\frac{\partial\Phi}{\partial\phi}-\frac{1}{r}\frac{\partial p}{\partial\phi}+\frac{1}{r^2}\frac{\partial}{\partial r}(r^2T_{r\phi})+\frac{1}{r}\frac{\partial T_{\phi\phi}}{\partial\phi}+\frac{\partial T_{\phi z}}{\partial z}.$$

In the case of an axisymmetric potential, it follows that

$$\rho\frac{\mathrm{D}}{\mathrm{D}t}(ru_\phi)=\frac{1}{r}\frac{\partial}{\partial r}(r^2T_{r\phi})+\frac{1}{r}\frac{\partial}{\partial\phi}(-rp+rT_{\phi\phi})+\frac{\partial}{\partial z}(rT_{\phi z}).$$

Therefore angular momentum is conserved, but is transported radially outwards by a negative shear stress $T_{r\phi}$. Such a stress is necessary for accretion.

Now assume that the azimuthal velocity in the disc is given by $u_\phi=r\Omega$, where $\Omega(r)$ is the angular velocity of circular orbits in the potential Φ. We return to examine this approximation later. Then

$$\rho u_r\frac{\mathrm{d}h}{\mathrm{d}r}=\frac{1}{r}\frac{\partial}{\partial r}(r^2T_{r\phi})+\frac{1}{r}\frac{\partial}{\partial\phi}(-rp+rT_{\phi\phi})+\frac{\partial}{\partial z}(rT_{\phi z}),$$

where $h=r^2\Omega$ is the specific orbital angular momentum. Multiply by r and integrate with respect to ϕ and z to obtain

$$\mathcal{F}\frac{\mathrm{d}h}{\mathrm{d}r}=-\frac{\partial\mathcal{G}}{\partial r},$$

where $\mathcal{G}(r,t)$ is the *viscous torque*, defined by

$$\mathcal{G} = -\int_0^{2\pi}\int_{-\infty}^{\infty} r^2 T_{r\phi}\,\mathrm{d}z\,\mathrm{d}\phi,$$

and we assume that there is no vertical loss of angular momentum to $z = \pm\infty$.

In astrophysical discs the molecular viscosity is much too small to account for the torque. Nevertheless, it is conventional to *parametrize* the torque in terms of an *effective viscosity*. According to the Navier–Stokes equation, we have a viscous stress

$$\mathbf{T} = \mu\left[\nabla\boldsymbol{u} + (\nabla\boldsymbol{u})^{\mathrm{T}}\right] + (\mu_{\mathrm{b}} - \tfrac{2}{3}\mu)(\nabla\cdot\boldsymbol{u})\mathbf{1},$$

where μ is the viscosity and μ_{b} the bulk viscosity. (Recall that the *dynamic viscosity* μ and the *kinematic viscosity* ν are related by $\mu = \rho\nu$.) In the case of circular orbital motion, the only stress component is

$$T_{r\phi} = T_{\phi r} = \mu r\frac{\mathrm{d}\Omega}{\mathrm{d}r},$$

i.e. the viscosity multiplied by the shear rate. Define the (density-weighted) *mean kinematic viscosity* $\bar{\nu}(r,t)$ according to

$$\bar{\nu}\Sigma = \frac{1}{2\pi}\int_0^{2\pi}\int_{-\infty}^{\infty}\mu\,\mathrm{d}z\,\mathrm{d}\phi.$$

Then we find

$$\mathcal{G} = -2\pi\bar{\nu}\Sigma r^3\frac{\mathrm{d}\Omega}{\mathrm{d}r}.$$

Although this form is derived from a Navier–Stokes viscosity, the torque can always be parametrized in this form for some suitable function $\bar{\nu}(r,t)$.

We then have

$$\Sigma\bar{u}_r\frac{\mathrm{d}h}{\mathrm{d}r} = \frac{1}{r}\frac{\partial}{\partial r}\left(\bar{\nu}\Sigma r^3\frac{\mathrm{d}\Omega}{\mathrm{d}r}\right), \tag{2}$$

which expresses the conservation of angular momentum in one dimension.

1.4.4. Diffusion equation for surface density

Equations (1) and (2) may be combined to eliminate $\bar{u}_r$, leading to

$$\frac{\partial\Sigma}{\partial t} + \frac{1}{r}\frac{\partial}{\partial r}\left[\left(\frac{\mathrm{d}h}{\mathrm{d}r}\right)^{-1}\frac{\partial}{\partial r}\left(\bar{\nu}\Sigma r^3\frac{\mathrm{d}\Omega}{\mathrm{d}r}\right)\right] = 0.$$

For a point-mass potential (Keplerian disc), we have

$$\Omega = \left(\frac{GM}{r^3}\right)^{1/2}, \qquad h = (GMr)^{1/2},$$

and so

$$\frac{\partial\Sigma}{\partial t} = \frac{3}{r}\frac{\partial}{\partial r}\left[r^{1/2}\frac{\partial}{\partial r}\left(r^{1/2}\bar{\nu}\Sigma\right)\right].$$

This has the character of a *diffusion equation for the surface density.*

This may be interpreted as follows. Viscous torques cause the redistribution of angular momentum, and therefore there is a viscous 'spreading' or diffusion. Most of the mass goes to smaller radii to be accreted by the central object, but some goes to larger radii in order to take up the angular momentum that is transported there.

An alternative form, more obviously related to the classical diffusion equation, is

$$\frac{\partial\mathcal{G}}{\partial t} = \bar{\nu}r^2\frac{\mathrm{d}h}{\mathrm{d}r}\left(-\frac{\mathrm{d}\Omega}{\mathrm{d}r}\right)\frac{\partial^2\mathcal{G}}{\partial h^2},$$

which describes the torque diffusing in the space of specific angular momentum. This form is valid only if $\bar{\nu}$ is independent of t.

1.5. Analysis of the diffusion equation

1.5.1. Inner boundary condition

The inner boundary condition depends on the nature of the central object. There are three important possibilities.

Weakly magnetized star

If the central object is a non-magnetic (or weakly magnetized) star, the disc may extend to the stellar surface. Usually the star rotates at only a fraction of the Keplerian angular velocity at its surface, i.e.

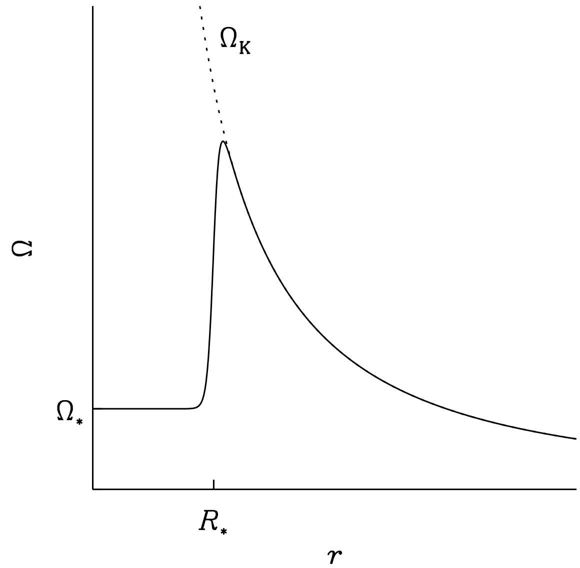
$$\Omega_* < \left(\frac{GM}{R_*^3}\right)^{1/2}.$$

This is because the star is mainly supported by pressure, not by the centrifugal force.

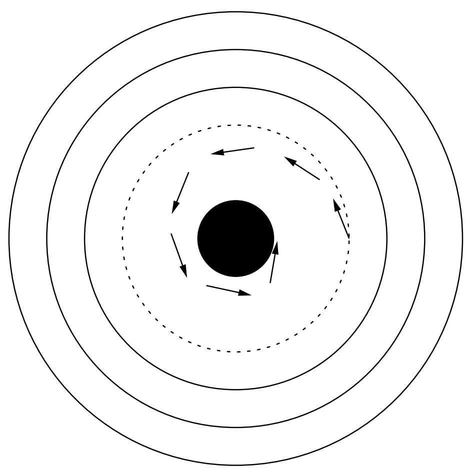
The angular velocity of the fluid makes a rapid transition from the Keplerian value to the stellar value, in a *viscous boundary layer.* Somewhere in the boundary layer is a radius r_{in} at which the shear rate vanishes, and therefore the viscous torque $\mathcal{G} = 0$. This may be regarded as the inner radius of the accretion disc. To a good approximation, $r_{\mathrm{in}} \approx R_*$.

Black hole

If the central object is a black hole, the disc does not extend to the event horizon. This is because of the existence of a *marginally stable circular orbit* at $r = r_{\mathrm{ms}}$. For $r < r_{\mathrm{ms}}$, circular orbits are unstable and the gas spirals rapidly into the black hole without need for a viscous torque.



Angular velocity profile for accretion from a disc on to a star



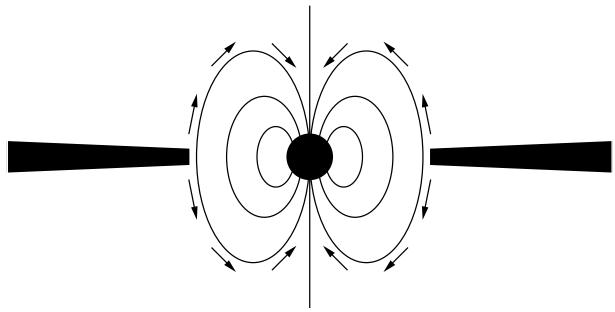
Accretion from a disc on to a black hole

For a non-rotating (Schwarzschild) black hole, the marginally stable orbit is at $r_{\rm ms} = 6GM/c^2$, while the event horizon is at $r_{\rm S} = 2GM/c^2$. To obtain this result properly, of course, requires a relativistic treatment of orbital motion.

In order to conserve mass, the surface density Σ decreases very rapidly just inside $r_{\rm ms}$ as the gas accelerates into the hole. The viscous stress is then essentially zero at $r_{\rm ms}$.

Strongly magnetized star

If the central object is a strongly magnetized star, the inner part of the disc may be disrupted by the strong magnetic field. The disc terminates at a *magnetospheric radius* which depends on complicated (and controversial) physics. The accretion flow is then channelled along the magnetic field lines on to the magnetic poles of the star.



Accretion from a disc on to a strongly magnetized star

In summary, the inner boundary condition can usually be considered to be

$$\mathcal{G} = 0 \qquad \text{at} \quad r = r_{\text{in}}.$$

If the angular velocity is treated as Keplerian throughout the disc, this implies

$$r^{1/2}\bar{\nu}\Sigma = 0 \qquad \text{at} \quad r = r_{\text{in}}.$$