1.5.2. Steady solutions

If the disc is in a steady state ($\partial/\partial t = 0$), the equation of mass conservation implies

$$\mathcal{F} = -\dot{M} = \text{constant},$$

where $\dot{M}$ is the *mass accretion rate*. [Although the central mass does indeed increase with time, we always neglect the slow variation of the potential with time.]

The angular momentum equation is

$$\mathcal{F}\frac{\mathrm{d}h}{\mathrm{d}r} = -\frac{\mathrm{d}\mathcal{G}}{\mathrm{d}r},$$

and this can be integrated to give

$$-\dot{M}h + \mathcal{G} = \text{constant}.$$

The constant can be determined from the inner boundary condition, giving

$$\mathcal{G} = \dot{M}(h - h_{\mathrm{in}}),$$

where $h_{\mathrm{in}} = h(r_{\mathrm{in}})$. Even if the inner boundary condition is different, a steady solution can always be written in this form, for some (possibly negative) constant h_{in}.

For a Keplerian disc, we then find

$$\bar{\nu}\Sigma = \frac{\dot{M}}{3\pi}\left[1 - \left(\frac{r_{\mathrm{in}}}{r}\right)^{1/2}\right].$$

If we know the function $\bar{\nu}(r, \Sigma)$, this provides a complete solution for the disc.

1.5.3. Spectrum of emitted radiation

The rate of energy dissipation per unit volume is

$$T_{r\phi} r\frac{\mathrm{d}\Omega}{\mathrm{d}r} = \mu\left(r\frac{\mathrm{d}\Omega}{\mathrm{d}r}\right)^2.$$

Assume that this is radiated away locally in radius, rather than being advected through the disc or retained in the gas. Then the rate at which energy is emitted per unit area of the disc (averaged, if necessary, over azimuth) is

$$\frac{1}{2}\bar{\nu}\Sigma\left(r\frac{\mathrm{d}\Omega}{\mathrm{d}r}\right)^2,$$

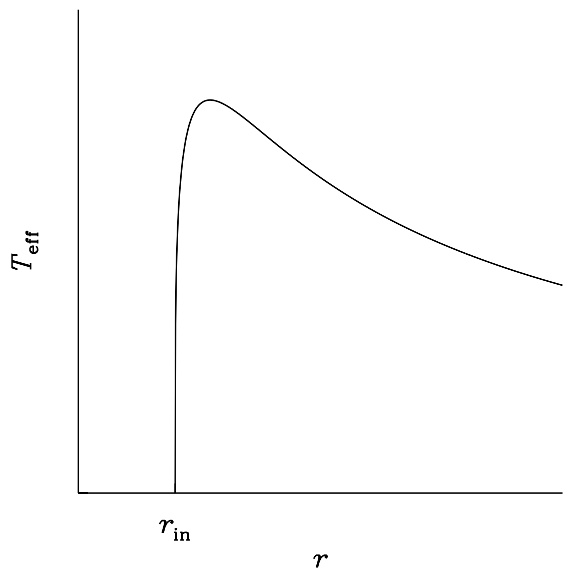
on each face of the disc. If this is carried by black-body radiation, the surface temperature $T_{\rm eff}(r,t)$ is given by

$$\sigma T_{\rm eff}^4 = \frac{9}{8}\bar{\nu}\Sigma\Omega^2,$$

for a Keplerian disc, where σ is the Stefan-Boltzmann constant. Even if the radiation is not black-body, this equation defines the *effective temperature* of the disc.

For a steady disc, we then have

$$\sigma T_{\rm eff}^4 = \frac{3GM\dot{M}}{8\pi r^3}\left[1-\left(\frac{r_{\rm in}}{r}\right)^{1/2}\right].$$



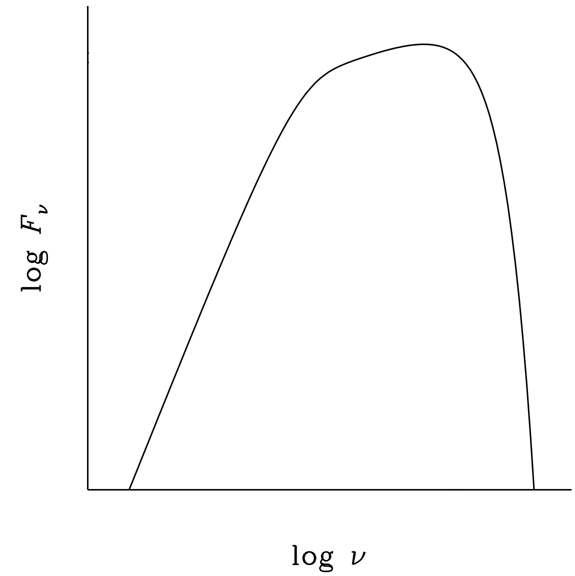
Effective temperature profile for a steady accretion disc

If the radiation has a black-body spectrum, each annulus has a specific emissivity proportional to the Planck function,

$$B_\nu(T_{\rm eff}) \propto \frac{\nu^3}{\exp(h\nu/kT_{\rm eff})-1}.$$

[Here h is Planck's constant, ν is a frequency in the electromagnetic spectrum (not the kinematic viscosity) and k is Boltzmann's constant.] The total emitted spectrum is obtained by integrating over all annuli, i.e.

$$F_\nu \propto \int_{r_{\rm in}}^{r_{\rm out}} \frac{\nu^3\, 2\pi r\,{\rm d}r}{\exp(h\nu/kT_{\rm eff}(r))-1}.$$



Spectrum of a black-body accretion disc with $r_{\text{out}}/r_{\text{in}} = 1000$

This gives a stretched black-body spectrum. The low-frequency emission comes mainly from the outer part of the disc, and the high-frequency emission from the inner part.

The total luminosity of a disc extending to $r_{\text{out}} = \infty$ is

$$L_{\text{disc}} = \int_{r_{\text{in}}}^{\infty} \frac{3GM\dot{M}}{4\pi r^3} \left[1 - \left(\frac{r_{\text{in}}}{r}\right)^{1/2}\right] 2\pi r\,\mathrm{d}r = \frac{1}{2}\frac{GM\dot{M}}{r_{\text{in}}}.$$

This is exactly equal to the rate at which orbital binding energy is transferred to the gas as it passes from ∞ to r_{in}. But it is only half the potential energy released. In the case of accretion on to a stellar surface, the remaining energy is released in the boundary layer.

1.5.4. Time-dependent solutions

For a Keplerian disc, the diffusion equation is

$$\frac{\partial \Sigma}{\partial t} = \frac{3}{r}\frac{\partial}{\partial r}\left[r^{1/2}\frac{\partial}{\partial r}(r^{1/2}\bar{\nu}\Sigma)\right],$$

subject to

$$r^{1/2}\bar{\nu}\Sigma = 0 \qquad \text{at} \quad r = r_{\text{in}}.$$

We suppose that $\bar{\nu}(r, \Sigma)$ is a given function. There are two possibilities.
(i) In the special case $\bar{\nu} = \bar{\nu}(r)$, we have a *linear diffusion equation.*
(ii) If $\bar{\nu} = \bar{\nu}(r, \Sigma)$, we have a *nonlinear diffusion equation.*

For simplicity, we assume further that $r_{\text{in}} \to 0$. The inner boundary condition is then

$$r^{1/2}\bar{\nu}\Sigma \to 0 \qquad \text{as} \quad r \to 0.$$