Linear case

In the linear case, the general solution may be found as a linear superposition of elementary solutions. One may look for elementary solutions in which the variables are separated,

$$\Sigma = r^{\beta}\sigma(r)\,\mathrm{e}^{-\lambda t}.$$

Here λ is a positive real number, representing the decay rate of the mode. β is a free parameter to be chosen at our convenience. Then

$$-\lambda r^{\beta}\sigma = \frac{3}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left[r^{1/2}\frac{\mathrm{d}}{\mathrm{d}r}(r^{\beta+1/2}\bar{\nu}\sigma)\right].$$

Suppose that $\bar{\nu} =$ constant. Then

$$r^2\frac{\mathrm{d}^2\sigma}{\mathrm{d}r^2} + \left(2\beta + \frac{3}{2}\right)r\frac{\mathrm{d}\sigma}{\mathrm{d}r} + \beta\left(\beta + \frac{1}{2}\right)\sigma + \frac{\lambda}{3\bar{\nu}}r^2\sigma = 0.$$

Choose $\beta = -1/4$ for convenience. Then

$$r^2\frac{\mathrm{d}^2\sigma}{\mathrm{d}r^2} + r\frac{\mathrm{d}\sigma}{\mathrm{d}r} + \left(k^2r^2 - \frac{1}{16}\right)\sigma = 0,$$

where $k^2 = \lambda/(3\bar{\nu})$. This is *Bessel's equation* of order 1/4. The general solution is

$$\sigma = AJ_{1/4}(kr) + BY_{1/4}(kr).$$

For small r, the Bessel functions behave as

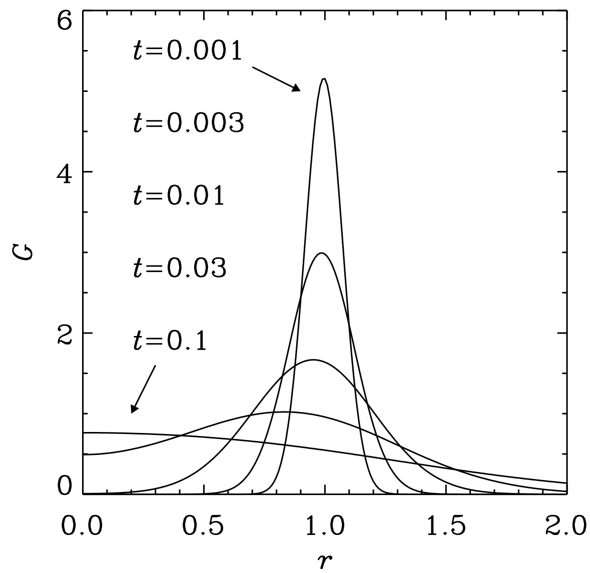
$$\begin{aligned} J_{1/4}(kr) &\propto r^{1/4}, \qquad \text{i.e.}\quad r^{1/2}\bar{\nu}\Sigma \to 0,\\ Y_{1/4}(kr) &\propto r^{-1/4}, \qquad \text{i.e.}\quad r^{1/2}\bar{\nu}\Sigma \to \text{constant}. \end{aligned}$$

The solution with vanishing torque at $r = 0$ is then

$$\Sigma \propto r^{-1/4}J_{1/4}(kr)\,\mathrm{e}^{-3\bar{\nu}k^2t}.$$

Now consider a general initial-value problem. Resolve the initial surface density into Bessel functions, i.e. let

$$\Sigma(r,0) = \int_0^{\infty} f(k)r^{-1/4}J_{1/4}(kr)\,\mathrm{d}k.$$



Green function of the linear diffusion equation, in units such that $s = 1$ and $\bar{\nu} = 1$.

The general solution is then found by evolving each mode as above, i.e.

$$\Sigma(r,t) = \int_0^\infty f(k) r^{-1/4} J_{1/4}(kr)\,\mathrm{e}^{-3\bar{\nu}k^2 t}\,\mathrm{d}k.$$

Recall the properties of *Hankel transforms*. The transform pair satisfy

$$A(r) = \int_0^\infty a(k) J_\nu(kr)(kr)^{1/2}\,\mathrm{d}k,$$
$$a(k) = \int_0^\infty A(r) J_\nu(kr)(kr)^{1/2}\,\mathrm{d}r,$$

where ν is the order of the Bessel functions used. We may write

$$r^{3/4}\Sigma(r,0) = \int_0^\infty k^{-1/2} f(k) J_{1/4}(kr)(kr)^{1/2}\,\mathrm{d}k,$$

and the inverse relation is then

$$k^{-1/2} f(k) = \int_0^\infty s^{3/4}\Sigma(s,0) J_{1/4}(ks)(ks)^{1/2}\,\mathrm{d}s.$$

The general solution may then be written in the form

$$\Sigma(r,t) = \int_0^\infty G(r,s,t)\Sigma(s,0)\,\mathrm{d}s,$$

where

$$G(r,s,t) = r^{-1/4} s^{5/4} \int_0^\infty J_{1/4}(kr) J_{1/4}(ks) k\,\mathrm{e}^{-3\bar{\nu}k^2 t}\,\mathrm{d}k$$

is the *Green function.* This may be evaluated explicitly as

$$G(r,s,t)=r^{-1/4}s^{5/4}\frac{1}{6\bar{\nu}t}\exp\left[-\frac{(r^2+s^2)}{12\bar{\nu}t}\right]I_{1/4}\left(\frac{rs}{6\bar{\nu}t}\right),$$

where I_ν is the modified Bessel function.

Note that the Green function is asymmetrical about $r=s$. Eventually all the mass ends up on the central object, while all the angular momentum is carried off to infinity by a negligible quantity of gas.

Nonlinear case

We will see later that, under certain conditions, viscosity laws of the form $\bar{\nu}=Ar^a\Sigma^b$ can be derived from plausible assumptions. Although we then cannot obtain the general solution of the nonlinear diffusion equation, there exist special *similarity solutions* which are algebraic in form and are thought to be attracting. These are valid if $r_{\text{in}}\to 0$, so that there is no physical length-scale present in the problem.

First, consider the following transformation of the diffusion equation. Let $x=r^{1/2}$, then

$$\frac{\partial\Sigma}{\partial t}=\frac{3}{4x^3}\frac{\partial^2}{\partial x^2}(x\bar{\nu}\Sigma).$$

The total mass of the disc is

$$\int_0^{r_{\text{out}}}\Sigma\,2\pi r\,\mathrm{d}r\propto\int_0^{x_{\text{out}}}\Sigma x^3\,\mathrm{d}x.$$

The total angular momentum of the disc is

$$\int_0^{r_{\text{out}}}\Sigma(GMr)^{1/2}\,2\pi r\,\mathrm{d}r\propto\int_0^{x_{\text{out}}}\Sigma x^4\,\mathrm{d}x.$$

Both mass and angular momentum are locally conserved. This is made evident by writing the diffusion equation in the forms

$$\frac{\partial}{\partial t}(\Sigma x^3)+\frac{\partial}{\partial x}\left[-\frac{\partial}{\partial x}\left(\frac{3}{4}x\bar{\nu}\Sigma\right)\right]=0$$

and

$$\frac{\partial}{\partial t}(\Sigma x^4)+\frac{\partial}{\partial x}\left[-\frac{3}{4}x^2\frac{\partial}{\partial x}(\bar{\nu}\Sigma)\right]=0. \quad (3)$$

Now our boundary condition requires $x\bar{\nu}\Sigma\to 0$ as $x\to 0$. If $x\bar{\nu}\Sigma\propto x$ as $x\to 0$, there is a non-vanishing mass flux at $x=0$, but zero angular momentum flux. We are looking for a solution in which mass is accreted at the origin, but no torque is exerted there. The only globally conserved quantity is then

$$\int_0^{x_{\text{out}}}\Sigma x^4\,\mathrm{d}x=C,$$

proportional to the total angular momentum of the disc.

Consider an initial-value problem for the diffusion equation with a viscosity law of the form $\bar{\nu} = Ar^a\Sigma^b$. The only quantity that is indelibly imprinted on the solution from the initial condition is the value of the conserved quantity C. Therefore A and C are the only dimensional constants in the problem. As there are three independent dimensions (mass, length and time), a unique product of powers of A, C and t will have the dimensions of $(\text{length})^{1/2}$. This characteristic scale $X(t)$ will usually increase as a power of t. A *similarity solution* is one that has a fixed shape that stretches along with the scale $X(t)$. It can expressed as a function of the similarity variable $\xi = x/X(t)$, e.g. $\bar{\nu}\Sigma \propto f(\xi)$.

Now consider a specific example. We will see later that the viscosity law

$$\bar{\nu} = Ax^{15/7}\Sigma^{3/7}$$

arises from the 'alpha models' in the case of 'Kramers opacity'. Start with a dimensional analysis of the problem. The two dimensional constants in the problem have dimensions

$$[A] = M^{-3/7}L^{25/14}T^{-1},$$
$$[C] = ML^{1/2}.$$

The variable

$$\xi = A^{-1/4}C^{-3/28}xt^{-1/4}$$

is then dimensionless, and is the similarity variable for this problem. We seek a solution in which

$$\bar{\nu}\Sigma = A^{-1/4}C^{25/28}t^{-5/4}f(\xi),$$

where f is a dimensionless function to be determined. Then

$$\Sigma = A^{-7/8}C^{5/8}x^{-3/2}t^{-7/8}f^{7/10}.$$

Substituting this into the diffusion equation (3), we obtain

$$\xi^{-3/2}\left(-\frac{7}{8}f^{7/10} - \frac{1}{4}\xi\frac{\mathrm{d}}{\mathrm{d}\xi}f^{7/10}\right) = \frac{3}{4}\xi^{-4}\frac{\mathrm{d}}{\mathrm{d}\xi}\left(\xi^2\frac{\mathrm{d}f}{\mathrm{d}\xi}\right).$$

Thus

$$\frac{\mathrm{d}}{\mathrm{d}\xi}\left(3\xi^2\frac{\mathrm{d}f}{\mathrm{d}\xi} + \xi^{7/2}f^{7/10}\right) = 0,$$

which can be integrated to give

$$3\xi^2\frac{\mathrm{d}f}{\mathrm{d}\xi} + \xi^{7/2}f^{7/10} = 0.$$

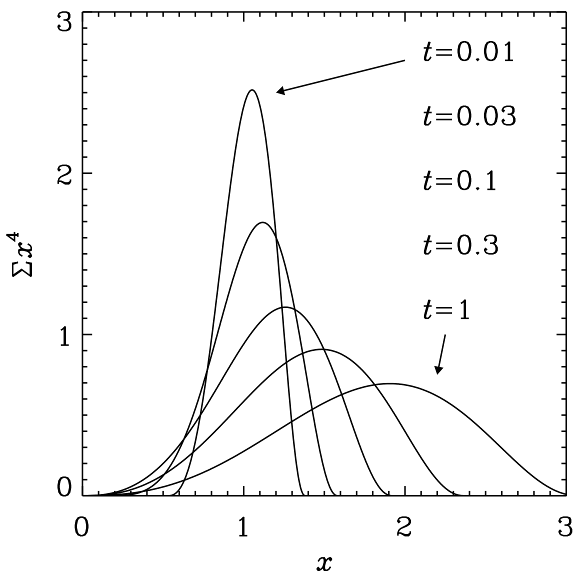
The constant vanishes because we want the solution with $f \to$ constant as $\xi \to 0$. Separate variables and integrate to obtain

$$f^{3/10} = \frac{1}{25}(\xi_{\text{out}}^{5/2} - \xi^{5/2}).$$

The integration constant ξ_{out} can be fixed by evaluating C, which implies

$$1 = \int_0^{\xi_{\text{out}}} f^{7/10} \xi^{5/2} \, \mathrm{d}\xi,$$

and one finds numerically $\xi_{\text{out}} \approx 3.018$.



Narrow-ring and similarity solutions of the nonlinear diffusion equation (see text).

The figure shows two solutions of the nonlinear diffusion equation, in units such that $A = 1$. The left-hand panel shows the evolution from a narrow ring of unit radius. The right-hand panel shows the attracting similarity solution for the same total angular momentum, $C = 1$. The plotted variables show the redistribution of angular momentum.