1.6. Vertical disc structure

1.6.1. Vertical hydrostatic equilibrium

We return to the equations governing the structure of a disc. The dominant balance in the vertical component of the equation of motion is

$$0 = -\rho\frac{\partial\Phi}{\partial z} - \frac{\partial p}{\partial z}.$$

The potential may be expanded about the mid-plane $z = 0$:

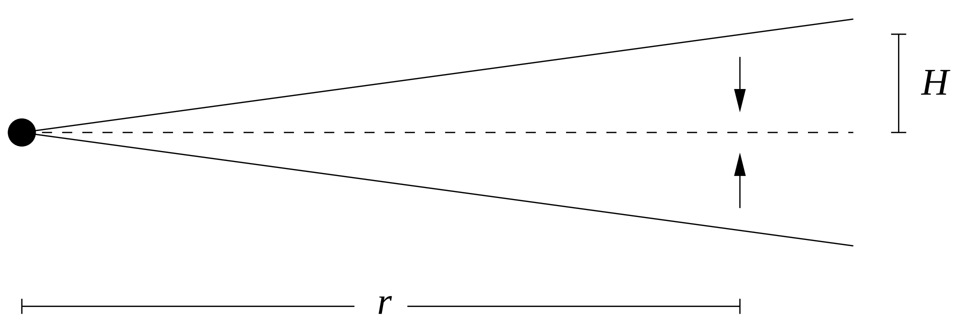
$$\Phi(r, z) = \Phi(r, 0) + \frac{1}{2}\Phi_{,zz}(r, 0)z^2 + \cdots$$

For a thin disc, therefore,

$$\frac{\partial\Phi}{\partial z} \approx \Omega_z^2 z,$$

where $\Omega_z(r)$ is the vertical frequency. (For a Keplerian disc, $\Omega_z = \Omega$.) The *equation of vertical hydrostatic equilibrium* is then

$$\frac{\partial p}{\partial z} = -\rho\Omega_z^2 z.$$



Vertical gravitational acceleration in a thin disc

As in a star, pressure supports the disc against gravity in the vertical direction. But note that the disc is centrifugally supported in the radial direction.

1.6.2. *Order-of-magnitude estimates and time-scales*

Simple scaling relations, omitting numerical factors of order unity, are commonly used to make order-of-magnitude estimates where exact calculations would be cumbersome or impossible.

An important dimensionless parameter is the angular semi-thickness of the disc, H/r. For a thin disc,

$$\frac{H}{r} \ll 1.$$

Here H represents either the true semi-thickness of the disc, or the height of the photosphere above the mid-plane, or a measure of the density scale-height.

From vertical hydrostatic equilibrium,

$$\frac{p}{H} \sim \rho\Omega^2 H,$$

or

$$c_\mathrm{s} \sim \Omega H,$$

where

$$c_\mathrm{s} = \left(\frac{p}{\rho}\right)^{1/2}$$

is the *isothermal sound speed.*

The dimensions of the viscosity μ, i.e.

$$[\mu] = ML^{-1}T^{-1},$$

are the same as those of p/Ω. We may therefore write

$$\mu = \frac{\alpha p}{\Omega},$$

where α is the *dimensionless viscosity parameter.* This relation is called the *alpha viscosity prescription.* It follows that

$$\bar{\nu} \sim \frac{\mu}{\rho} \sim \frac{\alpha c_\mathrm{s}^2}{\Omega} \sim \alpha c_\mathrm{s} H.$$

In kinetic theory, the kinematic viscosity is found to be

$$\nu \sim v\ell,$$

where v is the mean speed of the molecules and ℓ the mean free path. This quantity is entirely negligible for astrophysical discs. A similar kind of estimate, however, can be made for the effective 'eddy viscosity' of turbulence, if v represents the mean turbulent speed and

ℓ the correlation length of the turbulence. For subsonic turbulence, $v \lesssim c_\mathrm{s}$. For eddies no larger than the disc thickness, $\ell \lesssim H$. This suggests that $\alpha \lesssim 1$.

We then find for the stress in a Keplerian disc

$$T_{r\phi} = \mu r \frac{\mathrm{d}\Omega}{\mathrm{d}r} = -\frac{3}{2}\alpha p$$

The equation $T_{r\phi} = -\alpha p$ is an alternative version of the alpha prescription, used in the original paper by Shakura & Sunyaev [Shakura N. I & Sunyaev R. A. (1973), *Astron. Astrophys.* **24**, 337]. It is motivated by the dimensional concept that whatever physical process gives rise to the stress, the result should scale with the pressure in different situations. For local processes such as small-scale turbulent motions, this assumption is probably correct.

Three important characteristic time-scales can be defined. The *dynamical time-scale* is the time-scale of the orbital motion,

$$t_\mathrm{dyn} \sim \frac{1}{\Omega}.$$

It is also the time-scale for establishing the vertical hydrostatic equilibrium,

$$t_\mathrm{dyn} \sim \frac{H}{c_\mathrm{s}}.$$

The *viscous time-scale* may be defined as

$$t_\mathrm{visc} \sim \frac{r^2}{\bar{\nu}} \sim \alpha^{-1}\left(\frac{H}{r}\right)^{-2} t_\mathrm{dyn}.$$

This is the time-scale of the radial motion in the disc. The *thermal time-scale* is the time-scale for establishing the vertical thermal balance. This can be estimated as the thermal energy content per unit area divided by the rate of viscous heating per unit area,

$$t_\mathrm{th} \sim \frac{pH}{\bar{\nu}\Sigma\Omega^2} \sim \frac{c_\mathrm{s}^2}{\bar{\nu}\Omega^2} \sim \frac{H^2}{\bar{\nu}} \sim \alpha^{-1} t_\mathrm{dyn}.$$

For a thin disc with $\alpha < 1$, therefore,

$$t_\mathrm{dyn} < t_\mathrm{th} \ll t_\mathrm{visc}.$$

All three time-scales usually increase with increasing radius. The global time-scales therefore correspond to the time-scales at the outer edge of the disc.

The Mach number of the orbital motion is

$$\mathrm{Ma} \sim \frac{r\Omega}{c_\mathrm{s}} \sim \left(\frac{H}{r}\right)^{-1}.$$

The characteristic radial velocity due to viscosity is

$$|\bar{u}_r| \sim \frac{\bar{\nu}}{r} \sim \alpha \left(\frac{H}{r}\right) c_\mathrm{s}.$$

For a thin disc, therefore,

$$|\bar{u}_r| \ll c_\mathrm{s} \ll r\Omega,$$

i.e. the orbital motion is highly supersonic while the accretion flow is highly subsonic.

Finally, consider the relative contribution of the radial pressure gradient to the radial component of the equation of motion.

$$\frac{\partial p}{\partial r}\bigg/\rho r\Omega^2 \sim \frac{c_\mathrm{s}^2}{r^2\Omega^2} \sim \left(\frac{H}{r}\right)^2.$$

Vertical variations of the radial gravitational acceleration are also of this order. Other terms in the radial equation of motion, such as inertial terms associated with the radial motion, are smaller still. Therefore

$$u_\phi = r\Omega\left[1 + O\left(\frac{H}{r}\right)^2\right],$$

and treating the azimuthal fluid velocity as equal to the orbital velocity of a test particle is an excellent approximation for a thin disc. In general, the thin-disc approximations involve fractional errors of $O(H/r)^2$. A formal asymptotic treatment of thin discs is possible, using as small parameter a characteristic value of $(H/r)^2$.