1.6.3. Isothermal and polytropic models

We return to the equation of vertical hydrostatic equilibrium,

$$\frac{\partial p}{\partial z} = -\rho \Omega_z^2 z.$$

Simple solutions can be obtained if the pressure is a known function of the density. These are known as *barotropic models*.

In the case of a *vertically isothermal model*,

$$p = c_{\mathrm{s}}^2 \rho,$$

where c_{s} is independent of z. Then

$$\frac{1}{\rho}\frac{\partial \rho}{\partial z} = -\frac{z}{H^2},$$

where

$$H = \frac{c_{\mathrm{s}}}{\Omega_z}$$

is the *isothermal scale-height*. The solution is

$$\rho(r,z) = \rho(r,0)\,\mathrm{e}^{-z^2/2H^2},$$

where $\rho(r,0)$ is the mid-plane density. Both density and pressure have Gaussian distributions centred on the mid-plane, and H represents the 'standard deviation'. Formally, the disc extends to $|z| \to \infty$, and the thin-disc approximations break down once z/r is no longer small, but there is essentially no mass at such heights.

In the case of a *vertically polytropic model*,

$$p = K\rho^{1+1/n},$$

where K is independent of z, and n is the *polytropic index* (positive but not necessarily an integer). Define the *pseudo-enthalpy*

$$w = (n+1)K\rho^{1/n},$$

so that

$$\frac{\mathrm{d}p}{\rho} = \mathrm{d}w.$$

Then

$$\frac{\partial w}{\partial z} = -\Omega_z^2 z,$$

with the solution

$$w = \frac{1}{2}\Omega_z^2(H^2 - z^2),$$

where H is the true semi-thickness of the disc. Therefore

$$\rho(r,z) = \rho(r,0)\left(1 - \frac{z^2}{H^2}\right)^n,$$

$$p(r,z) = p(r,0)\left(1 - \frac{z^2}{H^2}\right)^{n+1},$$

for $|z| \leq H$. Note that there is a true surface at $z = H$, at which the pressure and density go to zero. Above that is a vacuum.

Isothermal models apply in the case of an ideal gas if the temperature is independent of z. This might apply if the disc were very optically thin (transparent). Polytropic models resemble optically thick (opaque) discs, but would also apply in the case of an ideal gas if the specific entropy were independent of z. In that case $1 + 1/n = \gamma$ is the adiabatic exponent, or ratio of specific heats, and w is the true specific enthalpy.

1.6.4. Radiative models

A *radiative model* is one in which the energy dissipated by viscosity is carried away by radiation (rather than by convection). If the disc is optically thick the radiation is locally close to a black-body distribution and may be treated in the *diffusion approximation.* The radiative energy flux density is then directed down the temperature gradient,

$$\boldsymbol{F} = -\frac{16\sigma T^3}{3\kappa\rho}\nabla T,$$

where κ is the *Rosseland mean opacity.*

In general, the *thermal energy equation* for an ideal gas is

$$\left(\frac{1}{\gamma - 1}\right)\left(\frac{\mathrm{D}p}{\mathrm{D}t} - \frac{\gamma p}{\rho}\frac{\mathrm{D}\rho}{\mathrm{D}t}\right) = \mathbf{T} : \boldsymbol{\nabla}\boldsymbol{u} - \nabla \cdot \boldsymbol{F},$$

where

$$\gamma = \left(\frac{\partial \ln p}{\partial \ln \rho}\right)_s$$

is the *adiabatic exponent* of the gas. The left-hand side of this equation is proportional to the rate of change of entropy of the fluid. The right-hand side represents heating by the 'viscous' stress and cooling by radiation.

Provided that the disc is steady on the thermal time-scale, viscous heating and radiative cooling must balance in the thermal energy equation:

$$0 = \mu \left(r \frac{\mathrm{d}\Omega}{\mathrm{d}r} \right)^2 - \frac{\partial F_z}{\partial z}.$$

The other components of $\nabla \cdot \boldsymbol{F}$ are much smaller because, in a thin disc, vertical derivatives of most quantities are much larger than horizontal derivatives. Thus

$$\frac{F_r}{F_z} \sim \frac{H}{r} \qquad \text{and} \qquad \frac{1}{r}\frac{\partial}{\partial r}(rF_r) \bigg/ \frac{\partial F_z}{\partial z} \sim \left(\frac{H}{r}\right)^2.$$

Hereafter we write simply F for F_z. Other terms in the thermal energy equation are also much smaller, e.g.

$$\left(\frac{1}{\gamma - 1}\right) u_r \frac{\partial p}{\partial r} \bigg/ \mu \left(r \frac{\mathrm{d}\Omega}{\mathrm{d}r} \right)^2 \sim \frac{(\bar{\nu}/r)(p/r)}{\rho\bar{\nu}\Omega^2} \sim \frac{c_\mathrm{s}^2}{r^2\Omega^2} \sim \left(\frac{H}{r}\right)^2.$$

The full set of equations governing the vertical structure of a Keplerian disc is then

$$\frac{\partial p}{\partial z} = -\rho\Omega^2 z,$$

$$\frac{\partial F}{\partial z} = \frac{9}{4}\mu\Omega^2,$$

$$F = -\frac{16\sigma T^3}{3\kappa\rho}\frac{\partial T}{\partial z},$$

together with an *equation of state*, $p = p(\rho, T)$, and a specification of the effective viscosity μ and opacity κ. These equations are closely analogous to those of stellar structure. Here, however, the potential is due to the central mass and not to the self-gravitation of the gas. There are no geometrical factors associated with spherical geometry. Also, the energy is generated here by viscous dissipation rather than nuclear reactions.

The two potentially important contributors to the pressure are the *ideal gas pressure* and the *radiation pressure*:

$$p = \frac{k\rho T}{\mu_\mathrm{m} m_\mathrm{p}} + \frac{4\sigma T^4}{3c},$$

where k is Boltzmann's constant, m_p the mass of the proton, and μ_m the mean molecular weight (2 for molecular hydrogen, 1 for atomic hydrogen, 0.5 for fully ionized hydrogen and about 0.6 for ionized matter of cosmic abundances). Radiation pressure is usually negligible except in the innermost part of luminous discs around neutron stars and black holes, where it can dominate.

The viscosity is parametrized using the alpha prescription,

$$\mu = \frac{\alpha p}{\Omega},$$

and we usually assume that α is independent of z.

The solution should be symmetrical about the mid-plane, such that

$$F = 0 \qquad \text{at} \quad z = 0.$$

At the upper surface, the solution should properly be matched to an atmospheric model at the photosphere $z = H$ where the gas becomes optically thin. However, a good approximation for an optically thick disc is obtained by assuming that

$$p = \rho = T = 0 \qquad \text{at} \quad z = H.$$

These *zero boundary conditions* are also used in stellar structure. Like a polytropic model, a radiative model with zero boundary conditions has a definite surface, with vacuum outside.

Opacity

The opacity κ is a complicated function of the density and temperature of the gas, and also depends on the chemical composition. It requires extensive numerical calculations based on atomic physics, and is available only as a tabulated function. However, there are certain regimes in which one opacity process is dominant and a simple power-law relation holds. *Thomson opacity* holds when electron scattering is the dominant process:

$$\kappa = \text{constant} \approx 0.33\,\text{cm}^2\,\text{g}^{-1}.$$

Kramers opacity holds when free-free and bound-free transitions are dominant:

$$\kappa = C_\kappa \rho T^{-7/2}, \qquad C_\kappa \approx 4.5 \times 10^{24}\,\text{cm}^5\,\text{g}^{-2}\,\text{K}^{7/2}.$$

These opacity laws hold fairly well in ionized accretion discs, Kramers opacity in the outer parts and Thomson opacity close to the central object if it is a compact star or black hole. The opacity in cooler discs is entirely different, and may be due to dust, molecules, etc.