Order-of-magnitude treatment

Consider a simple order-of-magnitude treatment of the vertical structure in the case of a disc with Thomson opacity and negligible radiation pressure.

$$\frac{p}{H} \sim \rho\Omega^2 H,$$

$$\frac{F}{H} \sim \alpha p\Omega,$$

$$F \sim \frac{\sigma T^3}{\kappa\rho}\frac{T}{H},$$

$$p \sim \frac{k\rho T}{\mu_\mathrm{m} m_\mathrm{p}}.$$

The surface density,

$$\Sigma \sim \rho H,$$

should be regarded as a given quantity, analogous to the total mass in a stellar structure calculation.

These relations can be solved to give

$$H \sim \alpha^{1/6}\Sigma^{1/3}\Omega^{-5/6}\left(\frac{\mu_\mathrm{m} m_\mathrm{p}}{k}\right)^{-2/3}\left(\frac{\sigma}{\kappa}\right)^{-1/6}.$$

The other variables follow according to

$$\rho \sim \frac{\Sigma}{H},$$

$$p \sim \Sigma\Omega^2 H,$$

$$F \sim \alpha\Sigma\Omega^3 H^2,$$

$$T \sim \left(\frac{\mu_\mathrm{m} m_\mathrm{p}}{k}\right)\Omega^2 H^2.$$

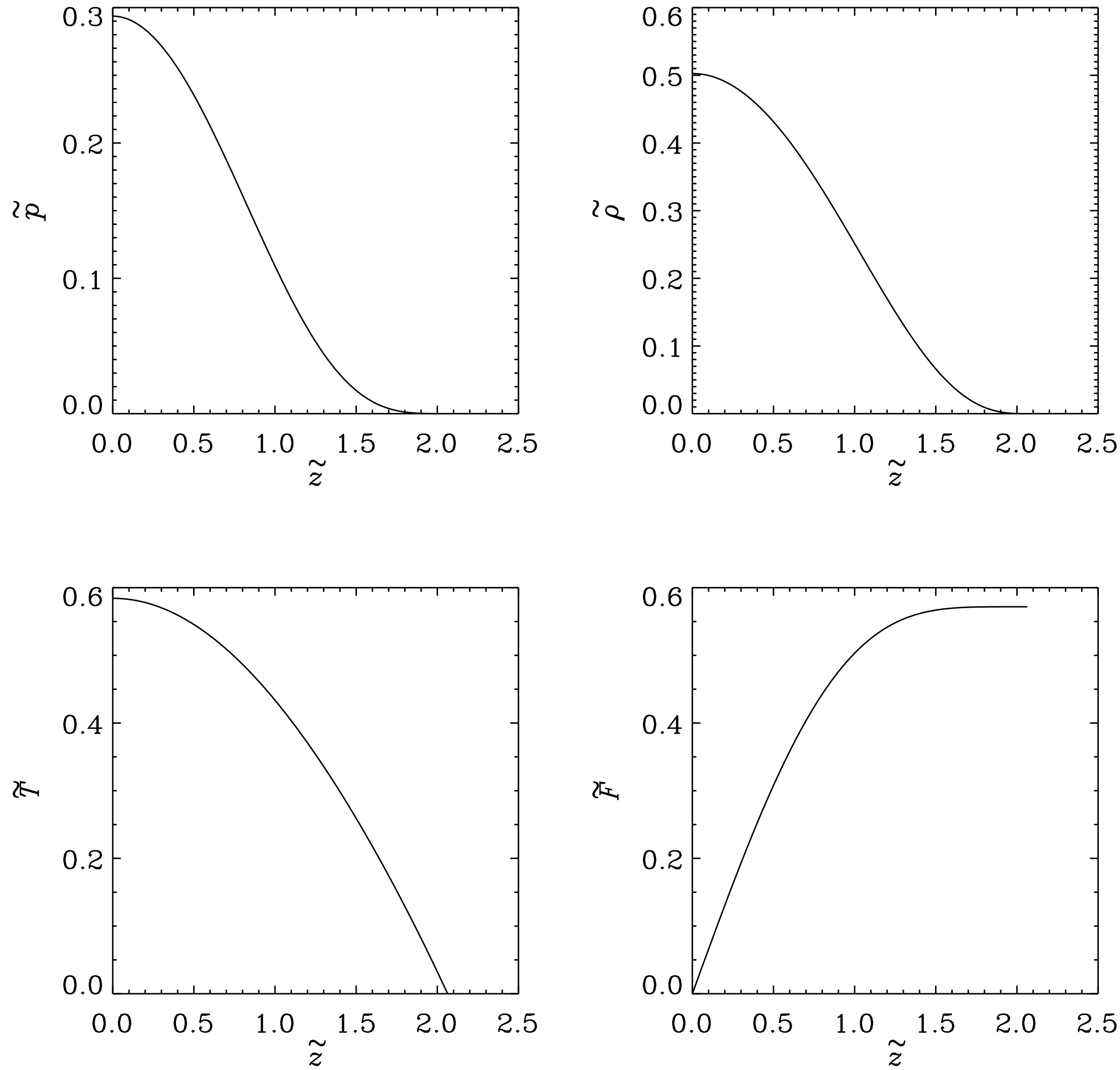
Also

$$\bar{\nu} \sim \alpha c_\mathrm{s} H \sim \alpha\Omega H^2 \sim \alpha^{4/3}\Sigma^{2/3}\Omega^{-2/3}\left(\frac{\mu_\mathrm{m} m_\mathrm{p}}{k}\right)^{-4/3}\left(\frac{\sigma}{\kappa}\right)^{-1/3},$$

which implies

$$\bar{\nu} \propto r\Sigma^{2/3}.$$

Full treatment



Vertical structure of a radiative model with gas pressure and Thomson opacity

Treating the above as a dimensional analysis, define characteristic physical units

$$U_H = \alpha^{1/6}\Sigma^{1/3}\Omega^{-5/6}\left(\frac{\mu_\mathrm{m}m_\mathrm{p}}{k}\right)^{-2/3}\left(\frac{\sigma}{\kappa}\right)^{-1/6},$$

$$U_\rho = \frac{\Sigma}{U_H},$$

$$U_p = \Sigma\Omega^2 U_H,$$

$$U_F = \alpha\Sigma\Omega^3 U_H^2,$$

$$U_T = \left(\frac{\mu_\mathrm{m}m_\mathrm{p}}{k}\right)\Omega^2 U_H^2.$$

Then introduce dimensionless variables according to

$$z = \tilde{z}\, U_H, \qquad H = \tilde{H}\, U_H, \qquad \rho = \tilde{\rho}(\tilde{z})\, U_\rho,$$

$$p = \tilde{p}(\tilde{z})\, U_p, \qquad F = \tilde{F}(\tilde{z})\, U_F, \qquad T = \tilde{T}(\tilde{z})\, U_T.$$

The dimensionless equations of vertical structure are then

$$\frac{\mathrm{d}\tilde{p}}{\mathrm{d}\tilde{z}} = -\tilde{\rho}\tilde{z},$$

$$\frac{\mathrm{d}\tilde{F}}{\mathrm{d}\tilde{z}} = \frac{9}{4}\tilde{p},$$

$$\tilde{F} = -\frac{16\tilde{T}^3}{3\tilde{\rho}}\frac{\mathrm{d}\tilde{T}}{\mathrm{d}\tilde{z}},$$

$$\tilde{p} = \tilde{\rho}\tilde{T}.$$

The boundary conditions are

$$\tilde{F}(0) = \tilde{p}(\tilde{H}) = \tilde{\rho}(\tilde{H}) = \tilde{T}(\tilde{H}) = 0.$$

Finally, the definition of the surface density requires

$$\int_{-\tilde{H}}^{\tilde{H}} \tilde{\rho}\,\mathrm{d}\tilde{z} = 1.$$

These equations have a unique solution that must be obtained numerically. The resulting profiles are almost indistinguishable from a polytropic model with index $n \approx 2.7$.

The vertically integrated viscosity is

$$\begin{aligned}\bar{\nu}\Sigma &= \int_{-H}^{H} \mu\,\mathrm{d}z \\ &= C_1\,\frac{\alpha U_p}{\Omega} U_H \\ &= C_1\,\alpha^{4/3}\Omega^{-2/3}\Sigma^{5/3}\left(\frac{\mu_\mathrm{m} m_\mathrm{p}}{k}\right)^{-4/3}\left(\frac{\sigma}{\kappa}\right)^{-1/3},\end{aligned}$$

where

$$C_1 = \int_{-\tilde{H}}^{\tilde{H}} \tilde{p}\,\mathrm{d}\tilde{z}$$

is a dimensionless constant of order unity. Again we find

$$\bar{\nu} \propto r\Sigma^{2/3}.$$

1.7. Thermal-viscous instability [non-examinable]

1.7.1. Viscous instability

We return to the diffusion equation for a Keplerian disc,

$$\frac{\partial\Sigma}{\partial t} = \frac{3}{r}\frac{\partial}{\partial r}\left[r^{1/2}\frac{\partial}{\partial r}(r^{1/2}\bar{\nu}\Sigma)\right],$$

with $\bar{\nu} = \bar{\nu}(r, \Sigma)$. Consider the stability of any given solution $\Sigma_0(r,t)$ of this equation with respect to small perturbations. Let the surface density be perturbed to

$$\Sigma(r,t) = \Sigma_0(r,t) + \Sigma'(r,t),$$

where $\Sigma'(r,t)$ is the *Eulerian perturbation* (i.e. at a fixed point in space) of surface density. For $|\Sigma'| \ll \Sigma$, the Eulerian perturbation of $\bar{\nu}\Sigma$ is given by a linear approximation,

$$(\bar{\nu}\Sigma)' = \frac{\partial(\bar{\nu}\Sigma)}{\partial\Sigma}\Sigma' = q\Sigma'.$$

The perturbed diffusion equation is then

$$\frac{\partial\Sigma'}{\partial t} = \frac{3}{r}\frac{\partial}{\partial r}\left[r^{1/2}\frac{\partial}{\partial r}(r^{1/2}q\Sigma')\right].$$

This is a *linear* diffusion equation for Σ', with a diffusion coefficient proportional to q. In particular, for wavelike perturbations with a wavelength short compared to r, we have

$$\frac{\partial\Sigma'}{\partial t} \approx 3q\frac{\partial^2\Sigma'}{\partial r^2}.$$

If $q > 0$, small perturbations will diffuse and dissipate. However, if $q < 0$, small perturbations will grow rapidly on short length-scales. The criterion for *viscous instability* is therefore

$$\frac{\partial(\bar{\nu}\Sigma)}{\partial\Sigma} < 0.$$

1.7.2. Thermal instability

So far we have assumed that a thermal balance between heating and cooling holds in the energy equation. Thus

$$\mathcal{H} = \mathcal{C},$$

where

$$\mathcal{H} = \frac{9}{4}\bar{\nu}\Sigma\Omega^2$$

is the vertically integrated viscous heating rate, and

$$\mathcal{C} = 2F^+$$

is the cooling rate (F^+ being the energy flux density leaving the upper surface of the disc).

Suppose that $\alpha \ll 1$ so that $t_{\rm dyn} \ll t_{\rm th} \ll t_{\rm visc}$. What happens if vertical thermal balance is not imposed, and the disc is allowed to vary on the thermal time-scale? Vertical hydrostatic equilibrium will still hold because $t_{\rm dyn} \ll t_{\rm th}$. Mass migrates radially on the time-scale $t_{\rm visc} \gg t_{\rm th}$, so each annulus retains its surface density Σ on the thermal time-scale.

Consider a single annulus at radius r. Solving the equations of vertical structure *other* than the equation of thermal balance, we can determine $\mathcal{H}$ and $\mathcal{C}$ as functions of the independent variables Σ and $\bar{\nu}\Sigma$. In fact $\mathcal{H}$ depends only on $\bar{\nu}\Sigma$. The equation of thermal balance, $\mathcal{H} = \mathcal{C}$, defines a curve in the $(\Sigma, \bar{\nu}\Sigma)$ plane, which determines the equilibrium relation between $\bar{\nu}\Sigma$ and Σ at fixed r. In general,

$$\mathrm{d}\mathcal{H} = \frac{\mathrm{d}\mathcal{H}}{\mathrm{d}(\bar{\nu}\Sigma)}\,\mathrm{d}(\bar{\nu}\Sigma),$$

$$\mathrm{d}\mathcal{C} = \frac{\partial\mathcal{C}}{\partial\Sigma}\,\mathrm{d}\Sigma + \frac{\partial\mathcal{C}}{\partial(\bar{\nu}\Sigma)}\,\mathrm{d}(\bar{\nu}\Sigma).$$

Along the equilibrium curve $\mathrm{d}\mathcal{H} = \mathrm{d}\mathcal{C}$ and $\mathrm{d}(\bar{\nu}\Sigma) = q\,\mathrm{d}\Sigma$. Thus

$$\frac{\mathrm{d}\mathcal{H}}{\mathrm{d}(\bar{\nu}\Sigma)} = \frac{1}{q}\frac{\partial\mathcal{C}}{\partial\Sigma} + \frac{\partial\mathcal{C}}{\partial(\bar{\nu}\Sigma)}. \qquad (1)$$

The thermal energy content of the disc per unit area is $\sim pH \sim (\Omega/\alpha)\bar{\nu}\Sigma$. If the equilibrium is perturbed by increasing the energy content slightly, $\bar{\nu}\Sigma$ increases, but recall that Σ is fixed on the thermal time-scale. The system will run away if the excess heating exceeds the excess cooling, i.e. if

$$\frac{\mathrm{d}\mathcal{H}}{\mathrm{d}(\bar{\nu}\Sigma)} > \frac{\partial\mathcal{C}}{\partial(\bar{\nu}\Sigma)}.$$

The criterion for *thermal instability* is therefore (applying equation 1)

$$\frac{1}{q}\frac{\partial\mathcal{C}}{\partial\Sigma} > 0.$$

Therefore both viscous and thermal instabilities depend on the sign of q. In practice $\partial\mathcal{C}/\partial\Sigma < 0$, so both instabilities occur when $q < 0$.

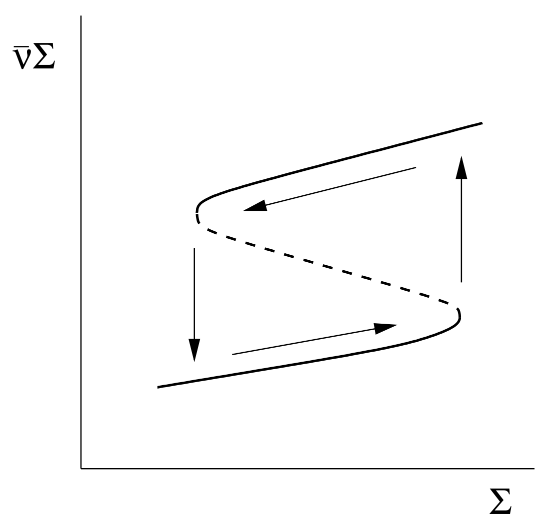
As we have seen, a radiative model with alpha viscosity, gas pressure and Thomson opacity has $\bar{\nu}\Sigma \propto r\Sigma^{5/3}$ and is thermally and viscously stable. The same is true when Kramers opacity dominates. However, for cooler discs undergoing hydrogen ionization, instability can occur.

When the equilibrium curve in the $(\Sigma, \bar{\nu}\Sigma)$ plane has an S-shape, limit-cycle behaviour can occur. There is then a range of $\bar{\nu}\Sigma$ for which no stable solution is available. For a steady state to exist in a binary star, we must have

$$\bar{\nu}\Sigma = \frac{\dot{M}}{3\pi}\left[1 - \left(\frac{r_{\mathrm{in}}}{r}\right)^{1/2}\right] \approx \frac{\dot{M}}{3\pi} \qquad \text{for} \quad r \gg r_{\mathrm{in}},$$

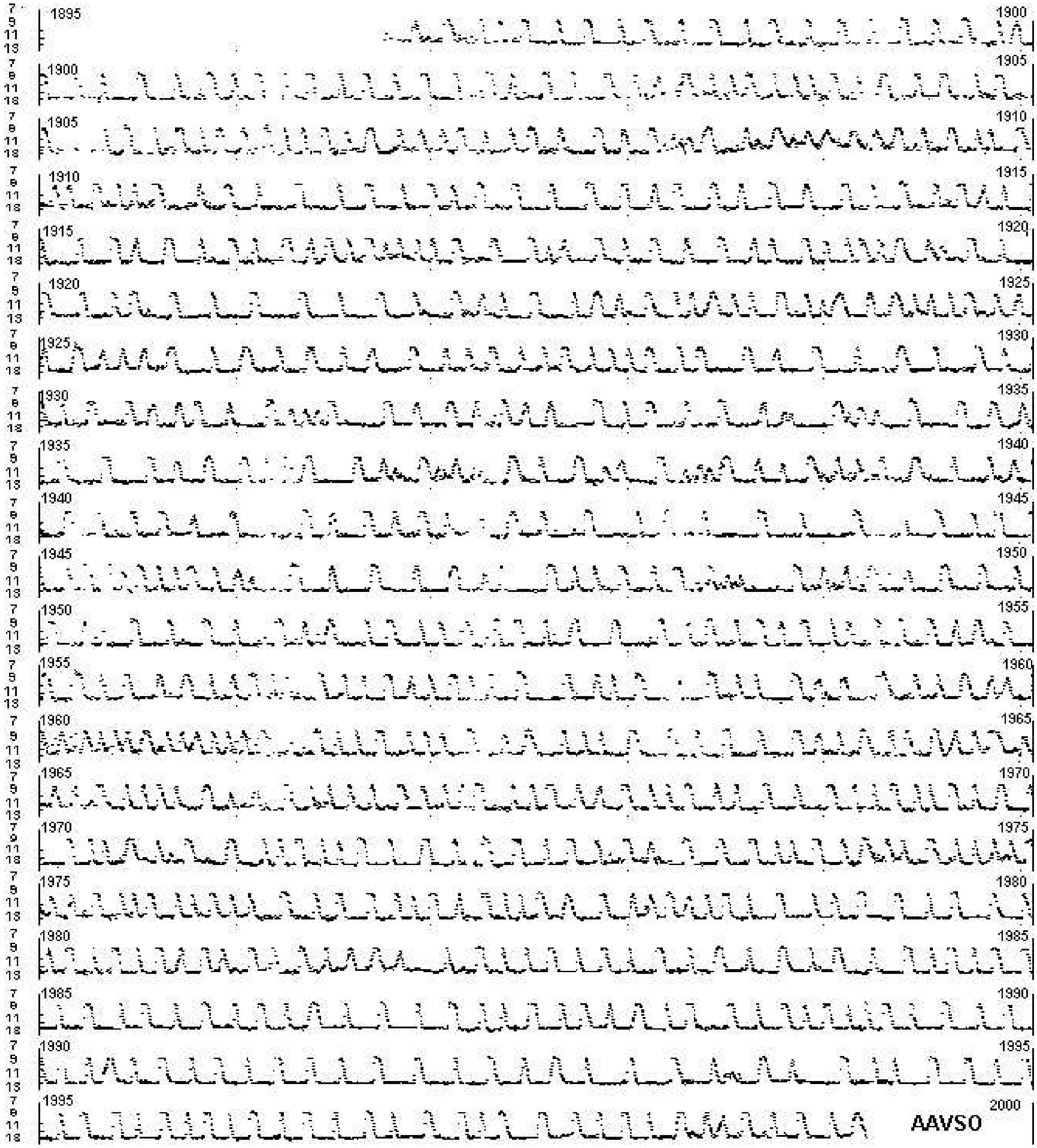
yet the value of $\dot{M}$ in a steady state is fixed externally, by the rate at which the companion star overflows its Roche lobe. For a certain range of mass-supply rates, no stable, steady state is possible.

On the upper stable branch, the viscous torque is too great and the surface density becomes depleted on the viscous time-scale. On the lower stable branch, the viscous torque is too small and the surface density accumulates. The result is a *limit cycle*.



Thermal-viscous instability: S-curve and limit cycle

This cyclical behaviour is the accepted explanation for the outbursts seen in many cataclysmic variables, and may also apply in some X-ray binaries. The disc cycles between a cool, poorly ionized state and a hot, highly ionized state.



Optical light-curve of the cataclysmic variable SS Cyg, from `http://www.aavso.org`

Thermal-viscous instability may also occur in the innermost parts of luminous discs around neutron stars and black holes, where radiation pressure and Thomson opacity dominate. Whether instability occurs here depends on exactly which form of the alpha viscosity prescription is adopted.