PART 2: MAGNETIC FIELDS IN ACCRETION DISCS

2.1. Magnetohydrodynamics

Reference: Roberts P. H. (1967), An Introduction to Magnetohydrodynamics

Magnetohydrodynamics (or MHD) is the dynamics of an electrically conducting fluid containing a magnetic field. It is a fusion of fluid dynamics and electromagnetism.

2.1.1. The induction equation

We restrict ourselves to a non-relativistic theory in which fluid motions are slow compared to the speed of light. The electromagnetic fields are governed by Maxwell's equations without the displacement current,

$$\begin{aligned}
\nabla\cdot\boldsymbol{E} &= \frac{\rho_\mathrm{e}}{\epsilon_0},\\
\nabla\cdot\boldsymbol{B} &= 0,\\
\nabla\times\boldsymbol{E} &= -\frac{\partial\boldsymbol{B}}{\partial t},\\
\nabla\times\boldsymbol{B} &= \mu_0\boldsymbol{J},
\end{aligned}$$

where ρ_e is the charge density and $\boldsymbol{J}$ the current density.

Exercise: Show that these equations are invariant under the *Galilean transformation* to a frame of reference moving with uniform relative velocity $\boldsymbol{v}$,

$$\begin{aligned}
\boldsymbol{r}' &= \boldsymbol{r}-\boldsymbol{v}t,\\
t' &= t,\\
\boldsymbol{E}' &= \boldsymbol{E}+\boldsymbol{v}\times\boldsymbol{B},\\
\boldsymbol{B}' &= \boldsymbol{B},\\
\rho_\mathrm{e}' &= \rho_\mathrm{e}-\mu_0\epsilon_0\boldsymbol{v}\cdot\boldsymbol{J},\\
\boldsymbol{J}' &= \boldsymbol{J},
\end{aligned}$$

as required by a 'non-relativistic' theory. (In fact, this is simply 'Galilean relativity'.)

Ohm's law for a static medium with electrical conductivity σ is

$$\boldsymbol{J}=\sigma\boldsymbol{E}.$$

For an electrically conducting fluid moving with velocity $\boldsymbol{u}$ this becomes

$$\boldsymbol{J}=\sigma(\boldsymbol{E}+\boldsymbol{u}\times\boldsymbol{B}),$$

or

$$\boldsymbol{E} = -\boldsymbol{u} \times \boldsymbol{B} + \frac{\boldsymbol{J}}{\sigma}.$$

From Maxwell's equations we then obtain the *induction equation*

$$\frac{\partial \boldsymbol{B}}{\partial t} = \nabla \times (\boldsymbol{u} \times \boldsymbol{B}) - \nabla \times (\eta \nabla \times \boldsymbol{B}),$$

where

$$\eta = \frac{1}{\mu_0 \sigma}$$

is the *magnetic diffusivity*. Note that this is an evolutionary equation for $\boldsymbol{B}$ alone, and $\boldsymbol{E}$ and $\boldsymbol{J}$ have been eliminated. The divergence of the induction equation is

$$\frac{\partial}{\partial t}(\nabla \cdot \boldsymbol{B}) = 0,$$

so the solenoidal character of $\boldsymbol{B}$ is preserved.

2.1.2. Lorentz force

A fluid carrying a current density $\boldsymbol{J}$ in a magnetic field $\boldsymbol{B}$ experiences a *Lorentz force*

$$\boldsymbol{F}_\mathrm{m} = \boldsymbol{J} \times \boldsymbol{B} = \frac{1}{\mu_0}(\nabla \times \boldsymbol{B}) \times \boldsymbol{B}$$

per unit volume. (The electrostatic force is negligible in the non-relativistic theory.) The equation of motion of the fluid is therefore

$$\rho \frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} = -\rho \nabla \Phi - \nabla p + \nabla \cdot \mathbf{T} + \frac{1}{\mu_0}(\nabla \times \boldsymbol{B}) \times \boldsymbol{B}.$$

In Cartesian components,

$$(\mu_0 \boldsymbol{F}_\mathrm{m})_i = -\epsilon_{ijk} B_j \epsilon_{klm} \partial_l B_m = -B_j \partial_i B_j + B_j \partial_j B_i.$$

Thus

$$\boldsymbol{F}_\mathrm{m} = -\nabla \left(\frac{B^2}{2\mu_0} \right) + \frac{1}{\mu_0} \boldsymbol{B} \cdot \nabla \boldsymbol{B}.$$

The first term is the gradient of an isotropic *magnetic pressure*

$$p_\mathrm{m} = \frac{B^2}{2\mu_0}.$$

The second term is a *curvature force* due to a tension in the field lines.

Alternatively, one can write

$$\boldsymbol{F}_\mathrm{m} = \nabla \cdot \mathbf{M},$$

where

$$M_{ij} = \frac{1}{\mu_0}(B_i B_j - \tfrac{1}{2}B^2\delta_{ij})$$

is the *Maxwell stress tensor*. If the magnetic field is locally aligned with the z-axis, then

$$\mathbf{M} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & +\frac{B^2}{\mu_0} \end{bmatrix} + \begin{bmatrix} -\frac{B^2}{2\mu_0} & 0 & 0 \\ 0 & -\frac{B^2}{2\mu_0} & 0 \\ 0 & 0 & -\frac{B^2}{2\mu_0} \end{bmatrix}.$$

The first term represents a *magnetic tension* $T_{\mathrm{m}} = B^2/\mu_0$ per unit area in the field lines. This gives rise to *Alfvén waves*, which travel parallel to the field with characteristic speed

$$v_{\mathrm{A}} = \left(\frac{T_{\mathrm{m}}}{\rho}\right)^{1/2} = \frac{B}{(\mu_0\rho)^{1/2}},$$

the *Alfvén speed*. The magnetic pressure also affects the propagation of sound waves, which become *magnetosonic waves*.

2.1.3. Ideal MHD

For a perfect electrical conductor, $\sigma \to \infty$ and so $\eta \to 0$. This limit is known as *ideal MHD*. The induction equation becomes

$$\frac{\partial \boldsymbol{B}}{\partial t} = \nabla \times (\boldsymbol{u} \times \boldsymbol{B}).$$

This equation has a beautiful geometrical interpretation: the magnetic field lines are 'frozen in' to the fluid and can be identified with material lines. To show this, write the induction equation in the form

$$\frac{\mathrm{D}\boldsymbol{B}}{\mathrm{D}t} = \boldsymbol{B} \cdot \nabla \boldsymbol{u} - (\nabla \cdot \boldsymbol{u})\boldsymbol{B},$$

and use the equation of mass conservation,

$$\frac{\mathrm{D}\rho}{\mathrm{D}t} = -\rho \nabla \cdot \boldsymbol{u},$$

to obtain

$$\frac{\mathrm{D}}{\mathrm{D}t}\left(\frac{\boldsymbol{B}}{\rho}\right) = \left(\frac{\boldsymbol{B}}{\rho}\right) \cdot \nabla \boldsymbol{u}.$$

This is exactly the same equation satisfied by an infinitesimal *material line element* $\delta \boldsymbol{r}$ as it is stretched by the velocity gradient:

$$\frac{\mathrm{D}}{\mathrm{D}t}\delta \boldsymbol{r} = \delta \boldsymbol{u} = \delta \boldsymbol{r} \cdot \nabla \boldsymbol{u}.$$

2.1.4. Non-ideal MHD

When $\eta > 0$ the resistivity of the fluid causes *diffusion* of the magnetic field and *dissipation* of magnetic energy. In the case of a uniform magnetic diffusivity, the induction equation becomes

$$\frac{\partial \boldsymbol{B}}{\partial t} = \nabla \times (\boldsymbol{u} \times \boldsymbol{B}) + \eta \nabla^2 \boldsymbol{B}.$$

Magnetic field lines are no longer frozen in to the fluid. If L and U represent characteristic scales of length and velocity for the flow, the characteristic time-scales for advection and diffusion of the field are

$$T_{\text{advection}} = \frac{L}{U},$$

$$T_{\text{diffusion}} = \frac{L^2}{\eta}.$$

The relative importance of advection is measured by the *magnetic Reynolds number*

$$\text{Rm} = \frac{T_{\text{diffusion}}}{T_{\text{advection}}} = \frac{LU}{\eta}.$$

When $\text{Rm} \gg 1$, as is typical in astrophysics, ideal MHD is a good approximation. However, a highly conducting fluid can violate the constraints of ideal MHD by developing very small-scale structures for which Rm is not large. This happens in reconnection and in turbulence.

The magnetic diffusivity has the same dimensions as the kinematic viscosity ν. Their ratio is (one definition of) the *magnetic Prandtl number*

$$\text{Pm} = \frac{\nu}{\eta}.$$

2.2. Jets and large-scale magnetic fields

2.2.1. Motivation

Collimated jets and other outflows are commonly observed in connection with discs around young stars, in interacting binary stars and in active galactic nuclei. Some observations clearly show twin jets emitted from the central regions of discs, in directions perpendicular to the plane of the disc.

Theoretical models involving large-scale magnetic fields have been the most successful in explaining aspects of this phenomenon. Outflows can also remove angular momentum vertically and thereby allow accretion. This can be done much more efficiently using magnetic fields, and could be in addition to the radial transport of angular momentum by small-scale, turbulent magnetic fields.

We examine *steady, axisymmetric* models based on the equations of *compressible ideal MHD*,

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \boldsymbol{u}) = 0,$$

$$\rho \left(\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} \right) = -\rho \nabla \Phi - \nabla p + \boldsymbol{J} \times \boldsymbol{B},$$

$$\boldsymbol{J} = \frac{1}{\mu_0} \nabla \times \boldsymbol{B},$$

$$\frac{\partial \boldsymbol{B}}{\partial t} = \nabla \times (\boldsymbol{u} \times \boldsymbol{B}),$$

$$\nabla \cdot \boldsymbol{B} = 0,$$

$$\frac{\mathrm{D}s}{\mathrm{D}t} = 0.$$

Here s is the specific entropy, and we will also need the thermodynamic relation

$$\mathrm{d}w = T\,\mathrm{d}s + \frac{\mathrm{d}p}{\rho},$$

where w is the specific enthalpy.

2.2.2. Representation of an axisymmetric magnetic field

The solenoidal condition for an axisymmetric magnetic field is

$$\frac{1}{r}\frac{\partial}{\partial r}(rB_r) + \frac{\partial B_z}{\partial z} = 0.$$

One may write

$$B_r = -\frac{1}{r}\frac{\partial \psi}{\partial z}, \qquad B_z = \frac{1}{r}\frac{\partial \psi}{\partial r}$$

where $\psi(r, z)$ is the *magnetic flux function.* This is related to the magnetic vector potential by $\psi = rA_\phi$. The magnetic flux contained inside the circle $r =$ constant, $z =$ constant is

$$\int_0^r B_z(r', z)\, 2\pi r' \,\mathrm{d}r' = 2\pi\psi(r, z) \qquad (+\text{constant}).$$

Since $\boldsymbol{B} \cdot \nabla\psi = 0$, ψ labels magnetic field lines or their surfaces of revolution, *magnetic surfaces.* The magnetic field may be written in the form

$$\boldsymbol{B} = \nabla\psi \times \nabla\phi + B_\phi\, \boldsymbol{e}_\phi = \left[-\frac{1}{r}\boldsymbol{e}_\phi \times \nabla\psi\right] + \left[B_\phi\, \boldsymbol{e}_\phi\right].$$

The two square brackets represent the *poloidal* (meridional) and *toroidal* (azimuthal) parts of the magnetic field:

$$\boldsymbol{B} = \boldsymbol{B}_\mathrm{p} + B_\phi\, \boldsymbol{e}_\phi.$$

Note that

$$\nabla \cdot \boldsymbol{B} = \nabla \cdot \boldsymbol{B}_\mathrm{p} = 0.$$

Similarly, one can write the velocity in the form

$$\boldsymbol{u} = \boldsymbol{u}_\mathrm{p} + u_\phi\, \boldsymbol{e}_\phi.$$