Useful Mathematical Results

1. Vector calculus in cylindrical polar coordinates

Right-handed orthogonal coordinate system (r, ϕ, z), referred to as *radial*, *azimuthal* and *vertical*.
Line element:

$$\mathrm{d}s^2 = \mathrm{d}r^2 + r^2\,\mathrm{d}\phi^2 + \mathrm{d}z^2$$

Volume element:

$$\mathrm{d}V = r\,\mathrm{d}r\,\mathrm{d}\phi\,\mathrm{d}z$$

Derivatives of unit vectors:

$$\frac{\partial \boldsymbol{e}_r}{\partial \phi} = \boldsymbol{e}_\phi, \qquad \frac{\partial \boldsymbol{e}_\phi}{\partial \phi} = -\boldsymbol{e}_r$$

Gradient of a scalar field:

$$\nabla\Phi = \boldsymbol{e}_r\,\frac{\partial\Phi}{\partial r} + \boldsymbol{e}_\phi\,\frac{1}{r}\frac{\partial\Phi}{\partial\phi} + \boldsymbol{e}_z\,\frac{\partial\Phi}{\partial z}$$

Divergence of a vector field:

$$\nabla\cdot\boldsymbol{B} = \frac{1}{r}\frac{\partial}{\partial r}(rB_r) + \frac{1}{r}\frac{\partial B_\phi}{\partial\phi} + \frac{\partial B_z}{\partial z}$$

Curl of a vector field:

$$\nabla\times\boldsymbol{B} = \frac{1}{r}\det\begin{bmatrix} \boldsymbol{e}_r & r\,\boldsymbol{e}_\phi & \boldsymbol{e}_z \\ \partial/\partial r & \partial/\partial\phi & \partial/\partial z \\ B_r & rB_\phi & B_z \end{bmatrix}$$

Divergence of a second-rank symmetric tensor field:

$$\begin{aligned}\nabla\cdot\mathbf{T} = {} & \boldsymbol{e}_r\left[\frac{1}{r}\frac{\partial}{\partial r}(rT_{rr}) + \frac{1}{r}\frac{\partial T_{r\phi}}{\partial\phi} + \frac{\partial T_{rz}}{\partial z} - \frac{T_{\phi\phi}}{r}\right] \\ & + \boldsymbol{e}_\phi\left[\frac{1}{r^2}\frac{\partial}{\partial r}(r^2T_{r\phi}) + \frac{1}{r}\frac{\partial T_{\phi\phi}}{\partial\phi} + \frac{\partial T_{\phi z}}{\partial z}\right] \\ & + \boldsymbol{e}_z\left[\frac{1}{r}\frac{\partial}{\partial r}(rT_{rz}) + \frac{1}{r}\frac{\partial T_{\phi z}}{\partial\phi} + \frac{\partial T_{zz}}{\partial z}\right]\end{aligned}$$

Gradient of a vector field:

$$\begin{aligned}\nabla\boldsymbol{u} = {} & \boldsymbol{e}_r\boldsymbol{e}_r\,\frac{\partial u_r}{\partial r} + \boldsymbol{e}_r\boldsymbol{e}_\phi\,\frac{\partial u_\phi}{\partial r} + \boldsymbol{e}_r\boldsymbol{e}_z\,\frac{\partial u_z}{\partial r} \\ & + \boldsymbol{e}_\phi\boldsymbol{e}_r\left(\frac{1}{r}\frac{\partial u_r}{\partial\phi} - \frac{u_\phi}{r}\right) + \boldsymbol{e}_\phi\boldsymbol{e}_\phi\left(\frac{1}{r}\frac{\partial u_\phi}{\partial\phi} + \frac{u_r}{r}\right) + \boldsymbol{e}_\phi\boldsymbol{e}_z\,\frac{1}{r}\frac{\partial u_z}{\partial\phi} \\ & + \boldsymbol{e}_z\boldsymbol{e}_r\,\frac{\partial u_r}{\partial z} + \boldsymbol{e}_z\boldsymbol{e}_\phi\,\frac{\partial u_\phi}{\partial z} + \boldsymbol{e}_z\boldsymbol{e}_z\,\frac{\partial u_z}{\partial z}\end{aligned}$$

Gradient of a vector field along another vector:

$$\boldsymbol{B}\cdot\nabla\boldsymbol{u} = \boldsymbol{e}_r\left(\boldsymbol{B}\cdot\nabla u_r - \frac{B_\phi u_\phi}{r}\right) + \boldsymbol{e}_\phi\left(\boldsymbol{B}\cdot\nabla u_\phi + \frac{B_\phi u_r}{r}\right) + \boldsymbol{e}_z\left(\boldsymbol{B}\cdot\nabla u_z\right)$$

2. Special functions used in the course

2.1 Bessel functions

Bessel's equation of order ν:

$$x^2\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + x\frac{\mathrm{d}y}{\mathrm{d}x} + (x^2-\nu^2)y = 0$$

Solutions $y = J_\nu(x)$, $y = Y_\nu(x)$. Assume $\mathrm{Re}(\nu)\geq 0$ WLOG.
Limiting forms when $x \to 0$:

$$J_\nu(x) \sim \frac{1}{\Gamma(\nu+1)}\left(\frac{x}{2}\right)^\nu, \qquad Y_\nu(x) \sim -\frac{\Gamma(\nu)}{\pi}\left(\frac{x}{2}\right)^{-\nu}, \qquad \text{except} \quad Y_0(x) \sim \frac{2}{\pi}\ln x$$

2.2 Modified Bessel functions

Modified Bessel equation of order ν:

$$x^2\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + x\frac{\mathrm{d}y}{\mathrm{d}x} - (x^2+\nu^2)y = 0$$

Solutions $y = I_\nu(x)$, $y = K_\nu(x)$. Assume $\mathrm{Re}(\nu)\geq 0$ WLOG.
Limiting forms when $x \to 0$:

$$I_\nu(x) \sim \frac{1}{\Gamma(\nu+1)}\left(\frac{x}{2}\right)^\nu, \qquad K_\nu(x) \sim \frac{\Gamma(\nu)}{2}\left(\frac{x}{2}\right)^{-\nu}, \qquad \text{except} \quad K_0(x) \sim -\ln x$$

2.3 Hankel transforms

Symmetrical transform pair:

$$A(r) = \int_0^\infty a(k)J_\nu(kr)(kr)^{1/2}\,\mathrm{d}k,$$

$$a(k) = \int_0^\infty A(r)J_\nu(kr)(kr)^{1/2}\,\mathrm{d}r$$