J. Fluid Mech. (1976), *vol.* 75, *part* 4, *pp.* 705–714
Printed in Great Britain

Slender-body theory for slow viscous flow

By JOSEPH B. KELLER
Courant Institute of Mathematical Sciences, New York University, New York 10012
AND SOL I. RUBINOW
Graduate School of Medical Sciences, Cornell University, New York 10021

(Received 15 October 1975 and in revised form 1 April 1976)

Slow flow of a viscous incompressible fluid past a slender body of circular cross-section is treated by the method of matched asymptotic expansions. The main result is an integral equation for the force per unit length exerted on the body by the fluid. The novelty is that the body is permitted to twist and dilate in addition to undergoing the translating, bending and stretching, which have been considered by others. The method of derivation is relatively simple, and the resulting integral equation does not involve the limiting processes which occur in the previous work.

1. Introduction

We consider slow flow of a viscous incompressible fluid past a slender body of circular cross section. The body can translate, bend, twist, stretch and dilate. Our main result is an integral equation for the distribution along the length of the body of the force exerted on it by the fluid. When twisting and dilating are absent, the equation is equivalent to that of Hancock (1953), and then the first approximation to its solution by iterations is the result of Cox (1970). However our results do not involve the limiting process which occurs in their results. Furthermore the derivation, by the method of matched asymptotic expansions, seems to be simpler than theirs. For the motion of slender axially symmetric rigid bodies, related integral equations have been formulated and solved asymptotically by Tuck (1964), Tillett (1970), and more completely by Geer (1976). Non-axially symmetric straight bodies have been treated by Batchelor (1970).

As an example, we have applied the integral equation to the transverse and the longitudinal motion of a rigid circular cylinder of finite length, when it agrees with those of the authors just mentioned. In both cases we have calculated the second approximation to the force distribution by iterations, and from it we have found the total force. The results include and extend the previous ones for this case.

It follows from the work of Tillett (1970) and Geer (1976) that for axially symmetric bodies the solution of the integral equation contains *all* terms in the force distribution which are powers of $(\log a/L)^{-1}$. Here a is the radius and L the length of the cylinder. Therefore an infinite number of such terms would have to be calculated to yield the same result as could be obtained by solving the integral equation. This is in contrast to Prandtl's 'lifting line' integral equation for the force distribution along a slender airfoil. In that case, as Van Dyke (1964, p. 175) has shown, two or three terms in the asymptotic expansion of the solution are more accurate than the solution of the integral equation.

An extensive study of flows around axially symmetric bodies which are not necessarily slender has been made by Chwang & Wu (1974, 1975) and Chwang (1975). They have analysed in detail the fundamental solutions (Stokeslets, rotlets, etc.), some of which we use in constructing our outer expansion. They have also compared some of their exact solutions with those of slender-body theory to determine its accuracy.

The rest of this paper consists of the following sections: §2, inner expansion; §3, outer expansion; §4, inner expansion of outer expansion; §5, outer expansion of inner expansion; §6, matching; §7, iterative solution of integral equation; §8, drag coefficients; §9, applications; §10, twisting and dilating bodies; appendix.

We shall use the body length L as the unit of length.

2. Inner expansion

Let $\mathbf{x} = \mathbf{x}_0(s)$, $0 \leqslant s \leqslant 1$, be the centre-line C of a body of length 1, with s being arc length along C. Let $\mathbf{v}(s)$ be the translational velocity of the body surface at s, let $\omega(s)$ be its angular velocity about C, let $a(s) \ll 1$ be the radius of the body cross-section at s, and let $\dot{a}(s)$ be the radial velocity of the body surface. Finally, let $\mathbf{u}(\mathbf{x})$ be the velocity of the fluid surrounding the body and let $\mathbf{u}_0(\mathbf{x})$ be the fluid velocity in the absence of the body. Then $\mathbf{u}(\mathbf{x})$ must satisfy the Stokes equations for slow flow, be equal to $\mathbf{v}(s)+\omega(s)\,\hat{\boldsymbol{\theta}}+\dot{a}(s)\,\hat{\boldsymbol{\rho}}$ on the body and tend to $\mathbf{u}_0(\mathbf{x})$ far from the body. Here $\hat{\boldsymbol{\theta}}$ and $\hat{\boldsymbol{\rho}}$ are unit vectors in the circumferential and radial directions in the plane normal to C at s.

In the fluid near a point $\mathbf{x}_0(s)$ on C, the flow is essentially that around a translating, rotating, dilating circular cylinder. Therefore the leading terms in its inner expansion are

$$\begin{aligned}\mathbf{u}(\mathbf{x}) \sim\ & \mathbf{v}(s)+\mathbf{i}\beta(s)\log\frac{\rho}{a(s)}+\mathbf{j}(s)\left[\log\frac{\rho}{a(s)}+\frac{1}{2}\left(1-\frac{a^2(s)}{\rho^2}\right)\right]\\ & -(\mathbf{j}\cos^2\theta+\mathbf{k}\sin\theta\cos\theta)\,\gamma(s)\left(1-\frac{a^2(s)}{\rho^2}\right)\\ & +(-\mathbf{j}\sin\theta+\mathbf{k}\cos\theta)\frac{\omega(s)a^2(s)}{\rho}+(\mathbf{j}\cos\theta+\mathbf{k}\sin\theta)\frac{a(s)\,\dot{a}(s)}{\rho}.\end{aligned} \qquad (1)$$

The orthogonal unit vectors $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ depend upon the point s, with $\mathbf{i} = \mathbf{x}_{0s}$ being tangential to C, $\mathbf{j}$ being in the direction of the component normal to C of the relative velocity of the body and the fluid near it, and $\mathbf{k} = \mathbf{i}\times\mathbf{j}$. The direction $\mathbf{j}(s)$, as well as the coefficients $\beta(s)$ and $\gamma(s)$, are to be determined. The distance ρ and the angle θ are the polar co-ordinates of $\mathbf{x}$ in the plane normal to C, with θ measured from the $\mathbf{j}$ direction.

The first term on the right side of (1) is the translational velocity of the body, the second term is due to the component of relative velocity along the body, the third and fourth terms are due to the component of relative velocity normal to the body in the $\mathbf{j}$ direction, the fifth term is due to rotation and the sixth term is due to dilation. All these terms are those for uniform motion of a circular cylinder.

3. Outer expansion

Far from the body we can represent $\mathbf{u}$ as $\mathbf{u}_0$ plus the flow due to a distribution of Stokeslets with density $\boldsymbol{\alpha}(s)$, rotlets with density $\Omega(s)\,\mathbf{i}(s)$ and sources with density $\delta(s)$, (See Chwang & Wu 1975; or Lighthill 1975, chap. 3.) Thus the leading terms in the outer expansion of $\mathbf{u}$ are

$$\mathbf{u}(\mathbf{x}) \sim \mathbf{u}_0(\mathbf{x}) + \int_0^1 \left[\frac{\boldsymbol{\alpha}}{R} + \frac{\mathbf{RR}\,.\,\boldsymbol{\alpha}}{R^3} + \frac{\Omega\mathbf{i}\times\mathbf{R}}{R^3} + \frac{\delta\mathbf{R}}{R^3}\right] ds'. \tag{2}$$

Here $\mathbf{R}(s') = \mathbf{x} - \mathbf{x}_0(s')$ and $R = |\mathbf{R}|$. The force exerted by the fluid on a Stokeslet of strength $\boldsymbol{\alpha}\,ds$ is $-8\pi\mu\boldsymbol{\alpha}\,ds$, while that exerted on a rotlet or source is zero, where μ is the viscosity of the fluid. Therefore the total force $\mathbf{F}$ exerted by the fluid on the body is

$$\mathbf{F} \sim -8\pi\mu L \int_0^1 \boldsymbol{\alpha}(s)\,ds. \tag{3}$$

The torque exerted by the fluid on a rotlet of strength $\boldsymbol{\Omega}(s)\,ds$ is $-8\pi\mu\boldsymbol{\Omega}\,ds$ while that exerted on a Stokeslet or source is zero. Therefore the total torque $\mathbf{T}$ exerted by the fluid on the body is

$$\mathbf{T} \sim -8\pi\mu L^2 \int_0^1 [\mathbf{R}(s)\times\boldsymbol{\alpha}(s) + \Omega(s)\,\mathbf{i}(s)]\,ds. \tag{4}$$

4. Inner expansion of outer expansion

To determine $\boldsymbol{\alpha}$, β, γ, δ, Ω and $\mathbf{j}$, we must match (1) and (2). For simplicity we shall first omit twisting and dilation by setting $\omega = \dot{a} = 0$ in (1). Then we can set $\Omega = 0$ and $\delta = 0$ in (2) and (4), since (1) and (2) can be matched when we do so. In order to match we must evaluate (2) for $\mathbf{x}$ near $\mathbf{x}_0(s)$ on C. This evaluation is done in the appendix, with the result

$$\mathbf{u}(\mathbf{x}) \sim \mathbf{u}_0(s) - [4\alpha_1(s)\,\mathbf{i} + 2\alpha_2(s)\,\mathbf{j} + 2\alpha_3(s)\,\mathbf{k}]\log\rho + \mathbf{u}_f(s). \tag{5}$$

Here $\mathbf{u}_0(s) = \mathbf{u}_0[\mathbf{x}_0(s)]$ while α_1, α_2 and α_3 are the components of $\boldsymbol{\alpha}$.

The function $\mathbf{u}_f(s)$ in (5) is the velocity at $\mathbf{x}_0(s)$ due to that part of the body away from $\mathbf{x}_0(s)$, i.e. outside a neighbourhood of $\mathbf{x}_0(s)$. It is given by the finite part of the integral in (2) with $\boldsymbol{\Omega} = 0$ and $\delta = 0$, and is defined as follows:

$$\begin{aligned}\mathbf{u}_f(s) &= \rlap{\,-}\int_0^1 \left[\frac{\boldsymbol{\alpha}}{R} + \frac{\mathbf{RR}\,.\,\boldsymbol{\alpha}}{R^3}\right] ds' \\ &= \lim_{\rho\to 0}\left\{\int_0^1 \left[\frac{\boldsymbol{\alpha}}{R} + \frac{\mathbf{RR}\,.\,\boldsymbol{\alpha}}{R^3}\right] ds' + [4\alpha_1(s)\,\mathbf{i} + 2\alpha_2(s)\,\mathbf{j} + 2\alpha_3(s)\,\mathbf{k}]\log\rho\right\}.\end{aligned} \tag{6}$$

In the last line of (6), $\mathbf{R}$ is the vector from $\mathbf{x}_0(s')$ to a point at a distance ρ from $\mathbf{x}_0(s)$. In the appendix it is shown that, with the $\log\rho$ term given in (6), the limit is finite.

5. Outer expansion of inner expansion

We next evaluate (1) for $\mathbf{x}$ far from C, i.e. for $\rho \gg a$. To do so we merely omit the ρ^{-2} term, since then (1) will match with (5).

6. Matching

Now we equate the two expressions for $\mathbf{u}$ given by (1) with the ρ^{-2} term omitted and with $\omega = \dot{a} = 0$, and by (5). From the coefficients of $\log \rho$ we get

$$\mathbf{i}\beta + \mathbf{j}\gamma = -4\alpha_1 \mathbf{i} - 2\alpha_2 \mathbf{j} - 2\alpha_3 \mathbf{k}. \tag{7}$$

The terms independent of ρ yield

$$\mathbf{v} - (\mathbf{i}\beta + \mathbf{j}\gamma)\log a + \tfrac{1}{2}\gamma\mathbf{j} - \gamma(\mathbf{j}\cos^2\theta + \mathbf{k}\sin\theta\cos\theta) = \mathbf{u}_0 + \mathbf{u}_f \tag{8}$$

From (7) it follows that

$$\beta(s) = -4\alpha_1(s), \quad \gamma(s) = -2\alpha_2(s), \quad \alpha_3(s) = 0. \tag{9}$$

By using (9) and (7) in (8) we obtain the following integral equation for $\boldsymbol{\alpha}(s)$:

$$2\alpha_1(s)\,\mathbf{i} + \alpha_2(s)\,\mathbf{j} = \frac{1}{2\log a(s)}\Big\{\mathbf{u}_0(s) - \mathbf{v}(s) + \alpha_2(s)\,\mathbf{j} \\ - 2\alpha_2(s)\,(\mathbf{j}\cos^2\theta + \mathbf{k}\sin\theta\cos\theta) + \int_0^1 \left[\frac{\boldsymbol{\alpha}(s')}{R} + \frac{\mathbf{R}\mathbf{R}.\boldsymbol{\alpha}(s')}{R^3}\right] ds'\Big\}. \tag{10}$$

We can consider separately the $\mathbf{i}$ and $\mathbf{j}$ components of (10) and recombine them to get

$$\boldsymbol{\alpha}(s) = \frac{1}{4\log a(s)}\,[2\mathbf{I} - \mathbf{i}(s)\,\mathbf{i}(s)].\Big\{\mathbf{u}_0(s) - \mathbf{v}(s) + \alpha_2(s)\,\mathbf{j} \\ - 2\alpha_2(s)\,(\mathbf{j}\cos^2\theta + \mathbf{k}\sin\theta\cos\theta) + \int_0^1 \left[\frac{\mathbf{I}}{R} + \frac{\mathbf{R}\mathbf{R}}{R^3}\right].\boldsymbol{\alpha}(s')\,ds'\Big\}. \tag{11}$$

Here $\mathbf{I}$ is the identity matrix. This equation can be shown to be equivalent to the equation obtained by combining equations (47) and (48) of Hancock (1953).

The finite part of the integral in (11) is the sum of the two integrals given by (A 12) and (A 18) with $\Omega = \delta = 0$. We can use them to replace the finite part of the integral by an ordinary integral. In doing so we find that the term $2\hat{\boldsymbol{\rho}}\hat{\boldsymbol{\rho}}.\boldsymbol{\alpha}$ in (A 18), in which $\hat{\boldsymbol{\rho}}$ is a unit vector pointing from $\mathbf{x}_0(s)$ to $\mathbf{x}$, exactly cancels the term $-2\alpha_2(\mathbf{j}\cos^2\theta + \mathbf{k}\sin\theta\cos\theta)$ in (11). Then (11) becomes

$$\boldsymbol{\alpha}(s) = \frac{1}{4\log a(s)}\,[2\mathbf{I} - \mathbf{i}(s)\,\mathbf{i}(s)].\Big\{\mathbf{u}_0(s) - \mathbf{v}(s) + \alpha_2(s)\,\mathbf{j} + \boldsymbol{\alpha}(s)\log[4s(1-s)] \\ + \alpha_1(s)\,\mathbf{i}(s)\,[\log(4s(1-s)) - 2] \\ + \int_{-s}^{1-s}\left[\frac{\boldsymbol{\alpha}(s+t)}{R_0} - \frac{\boldsymbol{\alpha}(s)}{|t|} + \frac{\mathbf{R}_0\mathbf{R}_0.\boldsymbol{\alpha}(s+t)}{R_0^3} - \frac{\alpha_1(s)\,\mathbf{i}(s)\,t^2}{|t|^3}\right] dt\Big\}. \tag{12}$$

Here $\mathbf{R}_0 = \mathbf{x}_0(s) - \mathbf{x}_0(s+t)$ and $\mathbf{j}$ is in the direction of the component normal to $\mathbf{i}$ of the expression in curly brackets on the right side of (12). Apart from the term $\alpha_2\,\mathbf{j}$, this expression is the velocity of the fluid relative to C.

Equation (12) is the main result of our analysis in the case of a non-twisting, non-dilating body. Since $-8\pi\mu\boldsymbol{\alpha}(s)$ is the force per unit length exerted by the fluid on the body, it is essentially an equation for this force distribution. Once (12) has been solved for $\boldsymbol{\alpha}(s)$, all the quantities in the inner expansion (1) and in the outer expansion (2) are known in terms of it, when $\omega = \dot{a} = 0$.

7. Iterative solution of integral equation

By setting $\boldsymbol{\alpha} = 0$ on the right side of (12), we obtain a lowest-order or zeroth approximation $\boldsymbol{\alpha}^{(0)}(s)$ to $\boldsymbol{\alpha}(s)$, given by

$$\boldsymbol{\alpha}^{(0)}(s) = \frac{1}{4\log a(s)}[2\mathbf{I} - \mathbf{i}(s)\,\mathbf{i}(s)]\,.\{\mathbf{u}_0(s) - \mathbf{v}(s)\}. \tag{13}$$

The first iterate $\boldsymbol{\alpha}^{(1)}(s)$, obtained by using $\boldsymbol{\alpha}^{(0)}(s)$ in the integral in (12), is

$$\begin{aligned}\boldsymbol{\alpha}^{(1)}(s) = \boldsymbol{\alpha}^{(0)}(s) + \frac{1}{4\log a(s)}[2\mathbf{I} - \mathbf{i}(s)\,\mathbf{i}(s)]\,.\Big\{&\alpha_2^{(0)}(s)\,\mathbf{j} + \boldsymbol{\alpha}^{(0)}(s)\log[4s(1-s)] \\ &+ \mathbf{i}(s)\,\alpha_1^{(0)}(s)\,\{\log[4s(1-s)] - 2\} \\ &+ \int_{-s}^{1-s}\left[\frac{\boldsymbol{\alpha}^{(0)}(s+t)}{R_0} - \frac{\boldsymbol{\alpha}^{(0)}(s)}{|t|} + \frac{\mathbf{R}_0\mathbf{R}_0\,.\,\boldsymbol{\alpha}^{(0)}(s+t)}{R_0^3} - \frac{\alpha_1^{(0)}(s)\,\mathbf{i}(s)\,t^2}{|t|^3}\right]dt\Big\}.\end{aligned} \tag{14}$$

The force density $-8\pi\mu\boldsymbol{\alpha}^{(1)}(s)$ can be shown to agree exactly with the result (6.2) of Cox (1970), which is in a quite different form.

8. Drag coefficients

Various authors have attempted to represent the force per unit length at the point $\mathbf{x}_0(s)$ on a slender body as a multiple of the relative velocity $\mathbf{u}_0(s) - \mathbf{v}(s)$ at s. If $\mathbf{u}_0 - \mathbf{v}$ is independent of s this is possible, and the coefficient of proportionality is a matrix $\mathbf{C}_D$ called the drag-coefficient matrix. However, if $\mathbf{u}_0(s) - \mathbf{v}(s)$ varies with s, then the force density at s is not determined by $\mathbf{u}_0 - \mathbf{v}$ at the point s alone. Instead it depends upon the entire function $\mathbf{u}_0(s) - \mathbf{v}(s)$. Therefore there is no local or pointwise proportionality between force density and relative velocity. If we wish to represent the force density in terms of the relative velocity, we must introduce a non-local drag-coefficient operator C_D.

To illustrate this, let us consider the approximation (14) for $\boldsymbol{\alpha}^{(1)}(s)$, so that the force density is $-8\pi\mu\boldsymbol{\alpha}^{(1)}(s)$. By substituting (13) into (14), we can write this in the form $-8\pi\mu\boldsymbol{\alpha}^{(1)}(s) = C_D[\mathbf{u}_0(s) - \mathbf{v}(s)]$. The operator C_D clearly consists of a multiplicative part plus an integral operator. When $\mathbf{u}_0(s) - \mathbf{v}(s)$ is a constant, C_D becomes a matrix.

9. Applications

When the curve C is a straight line segment then $\mathbf{x}_0(s) = \mathbf{i}s$, $\mathbf{R}_0 = \mathbf{i}t$ and $R_0 = |t|$, with $\mathbf{i}$ a constant. Then (12) can be simplified to the following:

$$\boldsymbol{\alpha}(s) = \frac{1}{4\log a(s)}[2\mathbf{I}-\mathbf{ii}].\Big\{\mathbf{u}_0(s)-\mathbf{v}(s)+\alpha_2(s)\,\mathbf{j}+\boldsymbol{\alpha}(s)\log[4s(1-s)] + \alpha_1(s)\,\mathbf{i}\{\log[4s(1-s)]-2\}+\int_{-s}^{1-s}[\mathbf{I}+\mathbf{ii}].\frac{[\boldsymbol{\alpha}(s+t)-\boldsymbol{\alpha}(s)]}{|t|}dt\Big\}. \quad (15)$$

If $\mathbf{u}_0(s)-\mathbf{v}(s) = \mathbf{i}U(s)$ is parallel to C, then we assume that $\boldsymbol{\alpha}(s) = \mathbf{i}\alpha_1(s)$, and (15) becomes

$$\alpha_1(s) = \frac{1}{2\log a(s)}\left\{\frac{U_1(s)}{2}+\alpha_1(s)\{\log[4s(1-s)]-1\}+\int_{-s}^{1-s}[\alpha_1(s+t)-\alpha_1(s)]\frac{dt}{|t|}\right\}. \quad (16)$$

If, instead, $\mathbf{u}_0(s)-\mathbf{v}(s) = \mathbf{j}U_2(s)$ with $\mathbf{j}$ constant, so that the relative velocity is normal to C, then we assume that $\boldsymbol{\alpha}(s) = \alpha_2(s)\,\mathbf{j}$. In this case (15) becomes

$$\alpha_2(s) = \frac{1}{2\log a(s)}\left\{U_2(s)+\alpha_2(s)\{[\log 4s(1-s)]+1\}+\int_{-s}^{1-s}[\alpha_2(s+t)-\alpha_2(s)]\frac{dt}{|t|}\right\}. \quad (17)$$

For a circular cylinder moving as a rigid body along its axis, we have a = constant and U_1 = constant. Then the iterative solution of (16) yields for the second iterate

$$\alpha_1^{(2)}(s) = \frac{U_1}{4\log a}\left\{1+\frac{1}{2\log a}\{\log[4s(1-s)]-1\}+\frac{1}{(2\log a)^2}\{\log[4s(1-s)]-1\}^2 + \frac{1}{(2\log a)^2}\int_{-s}^{1-s}\{\log[(s+t)(1-s-t)]-\log[s(1-s)]\}\frac{dt}{|t|}\right\}. \quad (18)$$

The total force $F_1\,\mathbf{i}$, obtained by integrating $-8\mu\pi\alpha_1^{(2)}(s)$, is also in the direction $\mathbf{i}$ and is given in dimensional form by

$$F_1 = 2\pi\mu L U_1\left\{\left[\log\frac{L}{a}-\tfrac{3}{2}+\log 2-\left(1-\frac{\pi^2}{12}\right)\Big/\log\frac{L}{a}\right]^{-1}+O\left[\left(\frac{1}{\log L/a}\right)^4\right]\right\}. \quad (19)$$

The integral in (18) does not contribute to the force, nor would the last term in (16) contribute no matter what α_1 was.

For a cylinder moving normal to its axis, $\alpha_2^{(2)}$ is given by (18) with $\frac{1}{2}U_1$ replaced by U_2 and with $\log[4s(1-s)]-1$ replaced by $\log[4s(1-s)]+1$. Then the force $F_2\,\mathbf{j}$ is found to be

$$F_2 = 4\pi\mu L U_2\left\{\left[\log\frac{L}{a}-\tfrac{1}{2}+\log 2-\left(1-\frac{\pi^2}{12}\right)\Big/\log\frac{L}{a}\right]^{-1}+O\left[\left(\frac{1}{\log L/a}\right)^4\right]\right\}. \quad (20)$$

The results (19) and (20) extend the previous results mentioned in the introduction.

10. Twisting and dilating bodies

Let us now return to the general case of a body which can twist and dilate in addition to translating, bending and stretching. Then the inner and outer expansions of $\mathbf{u}$ are given by (1) and (5), and the outer expansion of the inner expansion is given by (1) with the ρ^{-2} term omitted. The inner expansion of the outer expansion is evaluated in the appendix with the result

$$\begin{aligned}\mathbf{u}(\mathbf{x}) \sim\ & \mathbf{u}_0(s) - [4\alpha_1(s)\,\mathbf{i} + 2\alpha_2(s)\,\mathbf{j} + 2\alpha_3(s)\,\mathbf{k}]\log\rho \\ & + 2\Omega\rho^{-1}(-\mathbf{j}\sin\theta + \mathbf{k}\cos\theta) - \Omega(c_3\mathbf{j} - c_2\mathbf{k})\log\rho \\ & + 2\Omega\rho^{-1}(\mathbf{j}\cos\theta + \mathbf{k}\sin\theta) + \{-\mathbf{i}\delta' + \mathbf{j}c_2\delta + \mathbf{k}c_3\delta\}\log\rho + \mathbf{u}_f(s).\end{aligned} \tag{21}$$

The functions $c_2(s)$ and $c_3(s)$ are defined by $\mathbf{i}_s(s) = c_2\mathbf{j} + c_3\mathbf{k}$ and $\mathbf{u}_f$ is defined by

$$\begin{aligned}\mathbf{u}_f(s) = ⨍_0^1 \left[\frac{\boldsymbol{\alpha}}{R} + \frac{\mathbf{RR}.\boldsymbol{\alpha}}{R^3} + \frac{\Omega\mathbf{i}\times\mathbf{R}}{R^3} + \frac{\delta\mathbf{R}}{R^3}\right] ds' = \lim_{\rho\to 0}\left\{\int_0^1 \left[\frac{\boldsymbol{\alpha}}{R} + \frac{\mathbf{RR}.\boldsymbol{\alpha}}{R^3} + \frac{\Omega\mathbf{i}\times\mathbf{R}}{R^3} + \frac{\delta\mathbf{R}}{R^3}\right] ds'\right. \\ + [4\alpha_1(s)\,\mathbf{i} + 2\alpha_2(s)\,\mathbf{j} + 2\alpha_3(s)\,\mathbf{k}]\log\rho - 2\Omega\rho^{-1}(-\mathbf{j}\sin\theta + \mathbf{k}\cos\theta) \\ \left. + \Omega(c_3\mathbf{j} - c_2\mathbf{k})\log\rho - 2\delta\rho^{-1}(\mathbf{j}\cos\theta + \mathbf{k}\sin\theta) + \log\rho[\mathbf{i}2\delta' - \mathbf{j}c_2\delta - \mathbf{k}c_3\delta]\right\}.\end{aligned} \tag{22}$$

We now equate the expressions for $\mathbf{u}$ in (1) and (21), omitting the ρ^{-2} term in (1). From the coefficients of $\log\rho$ we get

$$\mathbf{i}\beta + \mathbf{j}\gamma = -4\alpha_1\mathbf{i} - 2\alpha_2\mathbf{j} - 2\alpha_3\mathbf{k} - \Omega(c_3\mathbf{j} - c_2\mathbf{k}) + (-2\mathbf{i}\delta' + \mathbf{j}c_2\delta + \mathbf{k}c_3\delta). \tag{23}$$

From the terms independent of ρ we get

$$\mathbf{v}(s) - (\mathbf{i}\beta + \mathbf{j}\gamma)\log a + \tfrac{1}{2}\gamma\mathbf{j} - \gamma(\mathbf{j}\cos^2\theta + \mathbf{k}\sin\theta\cos\theta) = \mathbf{u}_0(s) + \mathbf{u}_f(s). \tag{24}$$

From the coefficients of ρ^{-1} we obtain

$$\Omega(s) = \tfrac{1}{2}\omega(s)\,a^2(s), \quad \delta(s) = \tfrac{1}{2}a(s)\,\dot{a}(s). \qquad (25), (26)$$

Next we get from (23)

$$\beta = -4\alpha_1 - 2\delta', \quad \gamma = -2\alpha_2 - \Omega c_3 + \delta c_2, \quad \alpha_3 = \tfrac{1}{2}(\Omega c_2 + \delta c_3). \tag{27}$$

Upon substituting (22) and (23) into (24) we obtain

$$\begin{aligned}2\alpha_1\mathbf{i} + \alpha_2\mathbf{j} + \alpha_3\mathbf{k} = & -\tfrac{1}{2}\Omega(c_3\mathbf{j} - c_2\mathbf{k}) - \mathbf{i}\delta' + \tfrac{1}{2}\mathbf{j}c_2\delta + \tfrac{1}{2}\mathbf{k}c_3\delta \\ & + \frac{1}{2\log a(s)}\left\{\mathbf{u}_0(s) - \mathbf{v}(s) - \tfrac{1}{2}\gamma\mathbf{j} + \gamma(\mathbf{j}\cos^2\theta + \mathbf{k}\sin\theta\cos\theta)\right. \\ & \left. + ⨍_0^1 \left[\frac{\boldsymbol{\alpha}}{R} + \frac{\mathbf{RR}.\boldsymbol{\alpha}}{R^3} + \frac{\Omega\mathbf{i}\times\mathbf{R}}{R^3} + \frac{\delta\mathbf{R}}{R^3}\right] ds'\right\}.\end{aligned} \tag{28}$$

This is the integral equation for $\boldsymbol{\alpha}(s)$ in the general case. The functions α_3, β, γ, δ and Ω are given by (21)–(23), while $\mathbf{j}$ is in the direction of the component normal to $\mathbf{i}$ of the expression in curly brackets in (28), which is the relative velocity of the body and the fluid, except for the term $-\tfrac{1}{2}\gamma\mathbf{j}$. Furthermore, the integral equation can be solved by iteration as before.

The finite part of the integral in (28) is the sum of the integrals given by (A 12) and (A 18). Therefore, by means of those equations, it can be replaced by the ordinary integrals plus the additional terms given on the right sides of those equations.

This work was supported in part by the National Science Foundation under Grant MPS74-19234 and the Air Force Office of Scientific Research under Grant AFOSR-76-2884.

Appendix

Let $\mathbf{I}(\mathbf{x})$ denote the integral in (2), which we wish to evaluate for $\mathbf{x}$ near $\mathbf{x}_0(s)$. To do so we set $\mathbf{x} = \mathbf{x}_0(s)+\boldsymbol{\rho}$, where $\boldsymbol{\rho}.\mathbf{x}_{0s}(s) = 0$ because $\mathbf{x}_0(s)$ is the point on C nearest to $\mathbf{x}$. We also set $s' = s+t$ and integrate with respect to t. Then we can write $\mathbf{I}(\mathbf{x})$ in the form

$$\mathbf{I}(\mathbf{x}) = \int_{-s}^{1-s}\left[\frac{\mathbf{h}_1(t)}{R}+\frac{\mathbf{h}_3(t,\boldsymbol{\rho})}{R^3}\right]dt. \tag{A 1}$$

Here $\mathbf{h}_1(t) = \boldsymbol{\alpha}(s+t)$, $\mathbf{h}_3(t,\boldsymbol{\rho})$ is the sum of the last three numerators in (2) and $\mathbf{R}(t,\rho) = \mathbf{x}-\mathbf{x}_0(s+t) = \boldsymbol{\rho}+\mathbf{x}_0(s)-\mathbf{x}_0(s+t)$. If $\mathbf{R}_0(s,t) = \mathbf{x}_0(s)-\mathbf{x}_0(s+t)$ then

$$\mathbf{h}_3(t,\boldsymbol{\rho}) = [\boldsymbol{\rho}+\mathbf{R}_0]\,[\boldsymbol{\rho}+\mathbf{R}_0].\boldsymbol{\alpha}(s+t)+\Omega\mathbf{i}\times[\boldsymbol{\rho}+\mathbf{R}_0]+\delta[\boldsymbol{\rho}+\mathbf{R}_0]. \tag{A 2}$$

Next we write R^2 in the form

$$R^2(t,\boldsymbol{\rho}) = \rho^2+2\boldsymbol{\rho}.\mathbf{R}_0+R_0^2 = \rho^2+t^2c^2(t,\boldsymbol{\rho}). \tag{A 3}$$

Here we have introduced $c^2(t,\boldsymbol{\rho})$, defined by

$$c^2(t,\boldsymbol{\rho}) = t^{-2}\{\mathbf{R}_0^2+2\boldsymbol{\rho}.\mathbf{R}_0\}. \tag{A 4}$$

By writing $\mathbf{x}_0(s+t)$ as a Taylor series in t, and using this series to expand $\mathbf{R}_0$ in (A 4), with $\mathbf{x}_{0s}(s) = \mathbf{i}(s)$, we obtain

$$c^2(t,\boldsymbol{\rho}) = 1-\boldsymbol{\rho}.\mathbf{i}_s+O(t). \tag{A 5}$$

Now let us consider the integral of $\mathbf{h}_n(t,\boldsymbol{\rho})\,R^{-n}(t,\boldsymbol{\rho})$ for any positive integer n and any $\mathbf{h}_n$ with n continuous derivatives. As $\boldsymbol{\rho}$ tends to zero, R tends to $tc(t,0)$ and the integral becomes infinite, provided that $t = 0$ is in the range of integration. To isolate the divergent part we write the integral as follows:

$$\int_{-s}^{1-s}\frac{\mathbf{h}_n(t,\boldsymbol{\rho})}{R^n(t,\boldsymbol{\rho})}dt = \int_{-s}^{1-s}\left\{\frac{\mathbf{h}_n(t,\boldsymbol{\rho})}{[t^2c^2(t,\boldsymbol{\rho})+\rho^2]^{\frac{1}{2}n}}-\frac{\sum_{j=0}^{n-1}\frac{t^j}{j!}\frac{d^j}{dt^j}[\mathbf{h}_n(t,\boldsymbol{\rho})\,c^{-n}(t,\boldsymbol{\rho})]_{t=0}}{(t^2+\rho^2)^{\frac{1}{2}n}}\right\}dt$$
$$+\int_{-s}^{1-s}\frac{\sum_{j=0}^{n-1}\frac{t^j}{j!}\frac{d^j}{dt^j}[\mathbf{h}_n(t,\boldsymbol{\rho})\,c^{-n}(t,\boldsymbol{\rho})]_{t=0}}{(t^2+\rho^2)^{\frac{1}{2}n}}dt. \tag{A 6}$$

The first integral on the right side of (A 6) is finite at $\rho = 0$. To see this we merely set $\rho^2 = 0$ in the denominators and put both numerators over t^n. The new numerator is just $\mathbf{h}_n c^{-n}$ minus the first n terms in its Taylor series in t, so it begins with t^n, which cancels the denominator. Thus the integrand is finite at $\rho = 0$ for all t, and therefore the integral is finite at $\rho = 0$.

It follows that the singular part of the integral on the left side of (A 6) is given by the second integral on the right. But that integral can be evaluated explicitly, and thus the singular part can be found explicitly. For $n = 1$ and $n = 3$, the following integrals, in which $t > 0$, are needed:

$$\int_0^t (t^2+\rho^2)^{-\frac{1}{2}}\,dt = \log[t+(t^2+\rho^2)^{\frac{1}{2}}]-\log\rho$$
$$= -\log\rho+\log 2t+O(\rho^2), \tag{A 7}$$

$$\int_0^t (t^2+\rho^2)^{-\frac{3}{2}}\,dt = t\rho^{-2}(t^2+\rho^2)^{-\frac{1}{2}} = \rho^{-2}-\tfrac{1}{2}t^{-2}+O(\rho^2), \tag{A 8}$$

$$\int_0^t t(t^2+\rho^2)^{-\frac{3}{2}}\,dt = \rho^{-1}-(t^2+\rho^2)^{-\frac{1}{2}} = \rho^{-1}-t^{-1}+O(\rho^2), \tag{A 9}$$

$$\int_0^t t^2(t^2+\rho^2)^{-\frac{3}{2}}\,dt = \log[t+(t^2+\rho^2)^{\frac{1}{2}}]-t(t^2+\rho^2)^{-\frac{1}{2}}-\log\rho$$
$$= -\log\rho+\log 2t-1+O(\rho^2). \tag{A 10}$$

For $n = 1$ the last term in (A 6) is just $\mathbf{h}_1(0,\boldsymbol{\rho})\,c^{-1}(0,\boldsymbol{\rho})$ multiplied by the integral in (A 7) with limits $-s$ and $1-s$. The part of this integral which is singular as $\boldsymbol{\rho}$ tends to zero is just $-2\log\rho$. Furthermore, from (A 5), $c(0,0) = 1$, and if $\mathbf{h}_1(t) = \boldsymbol{\alpha}(s+t)$ then $\mathbf{h}_1(0) = \boldsymbol{\alpha}(s)$. Thus the singular part of the last term in (A 6) is $-2\boldsymbol{\alpha}(s)\log\rho$. If we subtract this part from the integral on the left of (A 6), the resulting difference has a finite limit as ρ tends to zero. We call this limit the finite part of the integral and write it as follows:

$$⨍_{-s}^{1-s}\frac{\boldsymbol{\alpha}(s+t)}{R(t,\boldsymbol{\rho})}\,dt = \lim_{\rho\to 0}\left[\int_{-s}^{1-s}\frac{\boldsymbol{\alpha}}{R}\,dt+2\boldsymbol{\alpha}(s)\log\rho\right]. \tag{A 11}$$

Another expression for the finite part of the integral can be obtained by using (A 6) for the integral on the right side of (A 11). Then we can just set $\rho = 0$ in the first integral on the right of (A 6) and use (A 7) to evaluate the second integral. In this way we get

$$⨍_{-s}^{1-s}\frac{\boldsymbol{\alpha}(s+t)}{R(t,\rho)}\,dt = \int_{-s}^{1-s}\left\{\frac{\boldsymbol{\alpha}(s+t)}{R_0(s,t)}-\frac{\boldsymbol{\alpha}(s)}{|t|}\right\}dt+\boldsymbol{\alpha}(s)\log[4s(1-s)]. \tag{A 12}$$

For $n = 3$ the integrals (A 8)–(A 10) enter the last term in (A 6). From (A 2) and (A 5) their coefficients are

$$\mathbf{h}_3(0,\boldsymbol{\rho})\,c^{-3}(0,\boldsymbol{\rho}) = [\boldsymbol{\rho}\boldsymbol{\rho}\,.\,\boldsymbol{\alpha}(s)+\Omega(s)\,\mathbf{i}(s)\times\boldsymbol{\rho}+\delta(s)\boldsymbol{\rho}]\,[1-\mathbf{i}_s(s)\,.\,\boldsymbol{\rho}]^{-\frac{3}{2}}$$
$$= [\Omega\mathbf{i}\times\boldsymbol{\rho}+\delta\boldsymbol{\rho}]\,[1+\tfrac{3}{2}\mathbf{i}_s\,.\,\boldsymbol{\rho}]+\boldsymbol{\rho}\boldsymbol{\rho}\,.\,\boldsymbol{\alpha}+O(\rho^3), \tag{A 13}$$

$$\frac{d}{dt}[\mathbf{h}_3(t,\boldsymbol{\rho})\,c^{-3}(t,\boldsymbol{\rho})]_{t=0} = -\mathbf{i}\delta+O(\rho), \tag{A 14}$$

$$\frac{1}{2}\frac{d^2}{dt^2}[\mathbf{h}_3c^{-3}]_{t=0} = \mathbf{i}\mathbf{i}\,.\,\boldsymbol{\alpha}+\frac{\Omega}{2}\,\mathbf{i}\times\mathbf{i}_s-\frac{\delta}{2}\mathbf{i}_s+O(\rho). \tag{A 15}$$

Now (A 8) and (A 13) yield the singular term $2\rho^{-2}(\Omega\mathbf{i}\times\boldsymbol{\rho}+\delta\boldsymbol{\rho})$, (A 9) and (A 14) yield no singular term because (A 9) has an odd integrand, and (A 10) and

(A 15) yield $-(2\mathbf{ii}.\boldsymbol{\alpha}+\Omega\mathbf{i}\times\mathbf{i}_s-\delta\mathbf{i}_s-2\delta_s\mathbf{i})\log\rho$. Upon combining these singular terms with the term $-2\boldsymbol{\alpha}(s)\log\rho$ from the integral of $\mathbf{h}_1/R$, we obtain as the singular part of $\mathbf{I}$ the following:

$$2\rho^{-2}(\Omega\mathbf{i}\times\boldsymbol{\rho}+\delta\boldsymbol{\rho})-(2\boldsymbol{\alpha}+2\mathbf{ii}.\boldsymbol{\alpha}+\Omega\mathbf{i}\times\mathbf{i}_s-\delta\mathbf{i}_s-2\delta_s\mathbf{i})\log\rho. \qquad \text{(A 16)}$$

By subtracting (A 16) from $\mathbf{I}$ and letting ρ tend to zero, we get the finite part of $\mathbf{I}$, denoted by $\mathbf{u}_f$ in (6) and (22). Then $\mathbf{I}$ is the sum of the singular part, the finite part and terms which tend to zero with ρ. The expression (A 16) is given in (21) and is used in (22) to define the finite part of $\mathbf{I}$. With $\delta=\Omega=0$, it is used in (5) and (6).

We can obtain another expression for the finite part of the integral of $\mathbf{h}_3/R^3$ by using (A 6) as we did in deriving (A 12). The singular part of the integral is given by (A 16) without the term $-2\boldsymbol{\alpha}(s)\log\rho$. Thus the finite part is defined by

$$\begin{aligned}&\int\!\!\!\!\!\!-_{s}^{1-s}[\mathbf{RR}.\boldsymbol{\alpha}+\Omega\mathbf{i}\times\mathbf{R}+\delta\mathbf{R}]\,R^{-3}\,dt\\&\quad=\lim_{\rho\to 0}\left[\int_{-s}^{1-s}\mathbf{h}_3R^{-3}\,dt-2\rho^{-2}(\Omega\mathbf{i}\times\boldsymbol{\rho}+\delta\boldsymbol{\rho})+(2\alpha_1\mathbf{i}+\Omega\mathbf{i}\times\mathbf{i}_s-\delta\mathbf{i}_s-2\delta_s\mathbf{i})\log\rho\right].\end{aligned} \qquad \text{(A 17)}$$

Now we use (A 6) for the integral on the right side of (A 17), setting $\rho=0$ in the first integral on the right side of (A 6). We also use (A 8)–(A 10) and (A 13)–(A 15) to evaluate the limit as ρ tends to zero of the last integral in (A 6). In this way we get

$$\begin{aligned}&\int\!\!\!\!\!\!-_{-s}^{1-s}[\mathbf{RR}.\boldsymbol{\alpha}+\Omega\mathbf{i}\times\mathbf{R}+\delta\mathbf{R}]\,R^{-3}\,dt\\&\quad=\int_{-s}^{1-s}\left[\frac{\mathbf{R}_0\mathbf{R}_0.\boldsymbol{\alpha}(s+t)+\Omega(s+t)\,\mathbf{i}(s+t)\times\mathbf{R}_0+\delta(s+t)\,\mathbf{R}_0}{R_0^3(s,t)}\right.\\&\qquad\left.-\frac{-\mathbf{i}(s)\,\delta(s)\,t+(\alpha_1\mathbf{i}+\tfrac{1}{2}\Omega\mathbf{i}\times\mathbf{i}_s-\tfrac{1}{2}\delta\mathbf{i}_s-\delta_s\mathbf{i})\,t^2}{|t|^3}\right]dt+3\mathbf{i}_s.\hat{\boldsymbol{\rho}}(\Omega\mathbf{i}\times\hat{\boldsymbol{\rho}}+\delta\hat{\boldsymbol{\rho}})+2\hat{\boldsymbol{\rho}}\hat{\boldsymbol{\rho}}.\boldsymbol{\alpha}\\&\qquad-\mathbf{i}\delta\left(\frac{1}{s}-\frac{1}{1-s}\right)+\left[\alpha_1\mathbf{i}+\frac{\Omega}{2}\mathbf{i}\times\mathbf{i}_s-\frac{\delta}{2}\mathbf{i}_s-\delta_s\mathbf{i}\right]\{\log[4s(1-s)]-2\}.\end{aligned} \qquad \text{(A 18)}$$

Here $\hat{\boldsymbol{\rho}}=\boldsymbol{\rho}/\rho$ is a unit vector in the direction of $\boldsymbol{\rho}$. The finite part of $\mathbf{I}$ is obtained by adding (A 12) to (A 18).

REFERENCES

Batchelor, G. K. 1970 *J. Fluid Mech.* **44**, 419.

Chwang, A. T. 1975 *J. Fluid Mech.* **72**, 17.

Chwang, A. T. & Wu, T. Y. 1974 *J Fluid Mech.* **63**, 607.

Chwang, A. T. & Wu, T. Y. 1975 *J. Fluid Mech.* **67**, 787.

Cox, R. G. 1970 *J. Fluid Mech.* **44**, 791.

Geer, J. F. 1976 *J. Fluid. Mech.* (in the press.)

Hancock, G. J. 1953 *Proc. Roy. Soc.* A **217**, 96.

Lighthill, M. J. 1975 *Mathematical Biofluiddynamics*. Philadelphia: SIAM.

Tillett, J. P. K. 1970 *J. Fluid Mech.* **44**, 401.

Tuck, E. O. 1964 *J. Fluid Mech.* **18**, 619.

Van Dyke, M. 1964 *Perturbation Methods in Fluid Mechanics*. Academic.