Paul Langevin's 1908 paper "On the Theory of Brownian Motion" ["Sur la théorie du mouvement brownien," C. R. Acad. Sci. (Paris) 146, 530–533 (1908)]

introduced by Don S. Lemons[a)]
Department of Physics, Bethel College, North Newton, Kansas 67117

translated by Anthony Gythiel
Department of History, Wichita State University, Wichita, Kansas 67260-0045

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We present a translation of Paul Langevin's landmark paper. In it Langevin successfully applied Newtonian dynamics to a Brownian particle and so invented an analytical approach to random processes which has remained useful to this day.

I. LANGEVIN, EINSTEIN, AND MARKOV PROCESSES

In 1908, three years after Albert Einstein initiated the modern study of random processes with his ground breaking paper on Brownian motion,[1] Paul Langevin (1872–1946), a French physicist and contemporary of Einstein, devised a very different but likewise successful description of Brownian motion.[2] Both descriptions have since been generalized into mathematically distinct but physically equivalent tools for studying an important class of continuous random processes.

Langevin's work, like Einstein's, remains current and is widely referenced and discussed.[3] Yet, while Einstein's paper is readily available in English,[4] Langevin's is not. Here we present a translation of this important primary source.

Langevin's approach to Brownian motion is, in his own words, "infinitely more simple" than Einstein's. Indeed, his paper is apparently more simple and for this reason is attractive as an introduction to the subject. While Einstein, starting from reasonable hypotheses, derived and solved a partial differential equation (i.e., a Fokker–Planck equation) governing the time evolution of the probability density of a Brownian particle, Langevin applied Newton's second law to a representative Brownian particle. In this way Langevin invented the "F=ma" of stochastic physics now called the "Langevin equation."

Today it is clear that the apparent simplicity of Langevin's approach was purchased at the cost of forcing into existence new mathematical objects with unusual properties. While Langevin manipulated these objects (Gaussian white noise and the stochastic differential equation) cautiously and intuitively, their formal properties have now been developed and widely applied. Thus Langevin's 1908 paper inspired new mathematics as well as new physics.

The Langevin equation and the Fokker–Planck equation both describe the physics of continuous, Markov (i.e., memoryless stochastic) processes. In fact, Einstein and Langevin used their respective methods to derive the same result: that the root-mean-squared displacement of a Brownian particle (imagine, say, a perfume particle in a still room) increases with the square root of the time. Nonetheless, Langevin's analysis of Brownian motion was slightly more general and more correct than Einstein's. In particular, Langevin introduced a stochastic force (his phrase is "complementary force") pushing the Brownian particle around in velocity space, while Einstein worked completely in configuration space. This is to say, in modern terminology, Langevin described the Brownian particle's velocity as an Ornstein–Uhlenbeck process and its position as the time integral of its velocity, while Einstein described its position as a driftless Wiener process. The former is a covering theory for the latter and reduces to it in a special "coarse-graining" limit.[5]

II. LANGEVIN'S WORK AND LIFE

Langevin is, probably, best known for his still standard theoretical model of para- and diamagnetism. During World War I he did early work on sonar and he was an enthusiastic advocate of the then new ideas in relativity. Einstein said of him "...It seems to me certain that he would have developed the special theory of relativity if that had not been done elsewhere, for he had clearly recognized the essential points."[6]

Langevin loved teaching and excelled at it. A married man with four children, he had an affair in 1911 with the recently widowed Marie Curie which was publicized by scandal mongering newspapers. He subsequently challenged his chief tormentor, the editor Téry, to a duel. Although the challenge was accepted and the combatants met on a sports field, no shots were fired because Téry did not want, as he said, "to deprive French science of a precious mind."[7] In the prelude to World War II Langevin became a vocal anti-fascist and peace activist. Eventually he joined the French communist party. He was arrested by the Nazis after their invasion of France in 1940, was briefly imprisoned by the Vichy government, and finally escaped to Switzerland. Thus, near the end of his life, he personally experienced, as it were, the chaos of Brownian motion into which the whole of Europe was thrown. He died in 1946 and was buried with high honors conferred by the French government.

III. THE TEXT

Langevin's note is divided into three untitled parts. His analysis of Brownian motion proper begins in the first sentence of part II with the phrase "...and, furthermore, that it is easy to give a demonstration..." and continues to the end of part II. This analysis is self-contained, constitutes the bulk of his paper, and will be of most interest to physicists today. However, the careful reader may also note that Langevin's characterization of his sources in parts I and III is problematic.

At issue is the correct form and quantitative verification of $\overline{\Delta_x^2}$: the mean-square displacement of a Brownian particle. In part I Langevin refers to two papers of Einstein's in which the latter derives the functional form of $\overline{\Delta_x^2}$ reported in Langevin's equation (1). Langevin's own analysis in part II also generates Eq. (1). This much is clear.

Smoluchowski, on the other hand, using yet different methods ''...has obtained for $\overline{\Delta_x^2}$ an expression of the same form as (1) but which differs from it by the coefficient 64/17.'' Does Smoluchowski's theory predict a value of $\overline{\Delta_x^2}$ larger by a factor of 64/27 or smaller by a factor of 27/64 than that predicted by the Einstein/Langevin formula (1)? If the translation is here slightly ambiguous, it only reflects a similar ambiguity in Langevin's French. Yet the natural reading is that Smoluchowski's prediction is larger than Einstein's and Langevin's by a factor of 64/27. Indeed, an inspection of Smoluchowski's paper confirms this interpretation. We mention this detail because it leads to the problem.

In part III we find that the only experimental results available to Langevin with which to compare theory are those of Svedberg, and these, apparently, ''...differ from those given by formula (1) only by approximately the ratio of 1 to 4.'' Again, the natural interpretation is that Svedberg's measurements are consistent with a value of $\overline{\Delta_x^2}$ one-fourth the size of that predicted by the Einstein/Langevin formula (1). Such divergences among theories and experiment are, perhaps, unexceptional in a new field. However, Langevin goes on to say, in the second half of the sentence quoted above, that Svedberg's experimental results are ''...closer to the ones calculated with M. Smoluchowski's formula.'' How can this be? Smoluchowski predicts a mean square displacement $\overline{\Delta_x^2}$ larger while Svedberg measures a $\overline{\Delta_x^2}$ smaller than that of the Einstein/Langevin formula (1), yet Svedberg's results are supposed to be closer to those predicted by Smoluchowski! Evidently, Langevin misstates the case. In just what way and for what reason, we are unsure.

If we must fault Langevin's exposition, we admire his physics. In the first place Langevin found that even if Smoluchowski's method is sound his execution of it was mistaken. Langevin corrected Smoluchowski's calculation and found that it too leads to formula (1) without the suspicious factor 64/27. Langevin also discerned that Svedberg's measurement of $\overline{\Delta_x^2}$ was not direct and that the Brownian particles the latter observed were probably too small to invoke Stokes's formula upon which formula (1) depended. Fortunately, Langevin had more confidence in his and Einstein's well motivated and well executed theories than in the supposed convergence of flawed theory and flawed experiment.

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[a]Electronic mail: dlemons@bethelks.edu

[1]Albert Einstein, ''On the movement of small particles suspended in stationary liquids required by the molecular-kinetic theory of heat,'' Ann. Phys. **17**, 549–60 (1905), appearing in *The Collected Papers of Albert Einstein*, English translation by Anna Beck (Princeton U.P., Princeton, NJ, 1989), Vol. 2, pp. 123–134.

[2]Paul Langevin, ''Sur la théorie du mouvement brownien,'' C. R. Acad. Sci. (Paris) **146**, 530–533 (1908).

[3]See, for example, relevant monographs by C. W. Gardiner, *Handbook of Stochastic Methods* (Springer-Verlag, New York, 1990), pp. 6–8; N. G. Van Kampen, *Stochastic Processes in Physics and Chemistry* (North Holland, New York, 1992), pp. 219 ff; and D. T. Gillespie, *Markov Processes; An Introduction for Physical Scientists* (Academic, New York, 1992), pp. 138 ff.

[4]See also the collection *Albert Einstein, Investigations on the Theory of the Brownian Movement* (Dover, New York, 1956), pp. 1–18.

[5]Daniel T. Gillespie, ''Exact numerical simulation of the Ornstein–Uhlenbeck process and its integral,'' Phys. Rev. E **54**, 2084–2091 (1996).

[6]*Dictionary of Scientific Biography*, edited by Charles Coulston Gillispie (Scribners, New York, 1971), Vol. VII, p. 10.

[7]Françoise Giroud, *Marie Curie: A Life*, translated by Lydia Davis (Holmes and Meier, New York, 1986), pp. 183–184.

PHYSICS--On the Theory of Brownian Motion
A note from M. P. Langevin, presented by M. Mascart.

I. The very great theoretical importance presented by the phenomena of Brownian motion has been brought to our attention by M. Gouy.(1) We are indebted to this physicist for having clearly formulated the hypothesis which sees in this continual movement of particles suspended in a fluid an echo of molecular-thermal agitation, and for having demonstrated this experimentally, at least in a qualitative manner, by showing the perfect permanence of Brownian motion, and its indifference to external forces when the latter do not modify the temperature of the environment.

A quantitative verification of this theory has been made possible by M. Einstein(2), who has recently given a formula that allows one to predict, at the end of a given time τ, the mean square $\overline{\Delta_x^2}$ of displacement Δ_x of a spherical particle in a given direction x as the result of Brownian motion in a liquid as a function of the radius a of the particle, of the viscosity μ of the liquid, and of the absolute temperature T. This formula is:

$$\overline{\Delta_x^2} = \frac{RT}{N}\frac{1}{3\pi\mu a}\tau \tag{1}$$

where R is the perfect gas constant relative to one gram-molecule and N the number of molecules in one gram-molecule, a number well known today and around 8×10^{23}.

M. Smoluchowski(3) has attempted to approach the same problem with a method that is more direct than those used by M. Einstein in the two successive demonstrations he has given of his formula, and he has obtained for $\overline{\Delta_x^2}$ an expression of the same form as (1) but which differs from it by the coefficient 64/27

II. I have been able to determine, first of all, that a correct application of the method of M. Smoluchowski leads one to recover the formula of M. Einstein *precisely*, and, furthermore, that it is easy to give a demonstration that is infinitely more simple by means of a method that is entirely different.

The point of departure is still the same: The theorem of the equipartition of the kinetic energy between the various degrees of freedom of a system in thermal equilibrium requires that a particle suspended in any kind of liquid possesses, in the direction x, an average kinetic energy $\frac{RT}{2N}$ equal to that of a gas molecule of any sort, in a given direction, at the same

temperature. If $\xi=\frac{dx}{dt}$ is the speed, at a given instant, of the particle in the direction that is considered, one therefore has for the average extended to a large number of identical particles of mass m

$$m\overline{\xi^2}=\frac{RT}{N}. \tag{2}$$

A particle such as the one we are considering, large relative to the average distance between the molecules of the liquid, and moving with respect to the latter at the speed ξ, experiences a viscous resistance equal to $-6\pi ma\xi$ according to Stokes' formula. In actual fact, this value is only a mean, and by reason of the irregularity of the impacts of the surrounding molecules, the action of the fluid on the particle oscillates around the preceding value, to the effect that the equation of the motion in the direction x is

$$m\frac{d^2x}{dt^2}=-6\pi\mu a\frac{dx}{dt}+X. \tag{3}$$

About the complementary force X, we know that it is indifferently positive and negative and that its magnitude is such that it maintains the agitation of the particle, which the viscous resistance would stop without it.

Equation (3), multiplied by x, may be written as:

$$\frac{m}{2}\frac{d^2x^2}{dt^2}-m\xi^2=-3\pi\mu a\frac{dx^2}{dt}+Xx. \tag{4}$$

If we consider a large number of identical particles, and take the mean of the equations (4) written for each one of them, the average value of the term Xx is evidently null by reason of the irregularity of the complementary forces X. It turns out that, by setting $z=\overline{\frac{dx^2}{dt}}$,

$$\frac{m}{2}\frac{dz}{dt}+3\pi\mu az=\frac{RT}{N}.$$

The general solution

$$z=\frac{RT}{N}\frac{1}{3\pi\mu a}+Ce^{-\frac{6\pi\mu a}{m}t}$$

enters a *constant regime* in which it assumes the constant value of the first term at the end of a time of order $m/6\pi\mu a$ or approximately 10^{-8} seconds for the particles for which Brownian motion is observable.

One therefore has, at a constant rate of agitation,

$$\overline{\frac{dx^2}{dt}}=\frac{RT}{N}\frac{1}{3\pi\mu a};$$

hence, for a time interval τ,

$$\overline{x^2}-\overline{x_0^2}=\frac{RT}{N}\frac{1}{3\pi\mu a}\tau.$$

The displacement Δ_x of a particle is given by

$$x=x_0+\Delta_x,$$

and, since these displacements are indifferently positive and negative,

$$\overline{\Delta_x^2}=\overline{x^2}-\overline{x_0^2}=\frac{RT}{N}\frac{1}{3\pi\mu a}\tau;$$

thence the formula (1).

III. A first attempt at experimental verification has just been made by M. T. Svedberg(4), the results of which differ from those given by formula (1) only by about the ratio 1 to 4 and are closer to the ones calculated with M. Smoluchowski's formula.

The two new demonstrations of M. Einstein's formula, one of which I obtained by following the direction begun by M. Smoluchowski, definitely rule out, it seems to me, the modification suggested by the latter.

Furthermore, the fact that M. Svedberg does not actually measure the quantity $\overline{\Delta_x^2}$ that appears in the formula and the uncertainty of the real diameter of the ultramicroscopic granules he observed call for new measurements. These, preferably, should be made on microscopic granules whose dimensions are easier to measure precisely and for which the application of the Stokes formula, which neglects the effects of the inertia of the liquid, is certainly more legitimate.

FOOTNOTES

[translators note: In the original, footnote numbering started anew on each page; here, in order to avoid confusion, numbering is sequential throughout the paper.]

1. Gouy, Journ. de Phys., 2e série, t. VII, 1888, p. 561; Comptes rendus, t. CIX, 1889, p. 102.

2. A. Einstein, Ann. d. Physik, 4e série, t. XVII, 1905, p. 549; Ann. d. Physik, 4e série, t. XIX, 1906, p. 371.

3. M. von Smoluchowski, Ann. d. Physik, 4e série, t. XXI, 1906, p. 756.

4. T. Svedberg, Studien zer Lehre von den kolloïden Lösungen. Upsala, 1907.

DEFINING THE DYNE

A parrot-like learning of stereotyped phrases is apt to produce calamitous results, as was the case with what I once read in an examination paper as the definition of a ''dyne'': ''A dyne is that force which, when placed one centimeter away from a magnetic pole of exactly similar strength, repels it with the force of one dyne.''

W. F. G. Swann, ''The Teaching of Physics,'' Am. J. Phys. **19**(3), 182–187 (1951).