Attracting Manifold for a Viscous Topology Transition

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An analytical method is developed describing the approach to a finite-time singularity associated with collapse of a narrow fluid layer in an unstable Hele-Shaw flow. Under the separation of time scales near a bifurcation point, a long-wavelength mode entrains higher-frequency modes, as described by a version of Hill's equation. In the slaved dynamics, the initial-value problem is solved explicitly, yielding the time and analytical structure of a singularity which is associated with the motion of zeros in the complex plane. This suggests a general mechanism of singularity formation in this system.

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One of the most fundamental questions underlying a broad class of hydrodynamic phenomena is: How do smooth initial conditions evolve to form finite-time singularities? Historically, two main classes of systems and phenomena have been investigated in this context [1]. The first involves distributions of vorticity evolving under Euler's equation [2]. The second, dating back to classical work of Rayleigh [3], concerns the motion of interfaces bounding masses of fluid undergoing fission [4–11], or in other problems of pattern formation [12]. An issue on which considerable progress has been made is the relevance of self-similar solutions that may describe the region asymptotically close to the topology transition. However, there is little understanding of how such special solutions emerge from the large scale dynamics.

We report here an approach to this *initial-value problem* associated with a topological rearrangement of fluid interfaces. Under the assumptions of lubrication theory, an approximation valid for long-wavelength deformations of thin layers, the method is developed for the Rayleigh-Taylor instability of two-phase Hele-Shaw flow [6,13]. This lubrication approximation has been the focus of a considerable body of recent theoretical work on thin film flows and the spreading of drops [5–9]. Very similar, and even identical, equations of motion describe phenomena as diverse as Marangoni convection [14], pattern formation in superconductors [15] and in biological systems [16], and oxidation of semiconductor surfaces [17].

The method exploits the separation of time scales occurring close to the first instability in a system of finite lateral extent, where the spectrum of modes is discrete. As in the study of normal form expansions near convective or lasing instabilities [18], the slaving of the high-frequency modes allows the derivation of a nonlinear evolution equation for the amplitude of the first unstable mode. It also allows an analytic approximation of the singular contribution from all other modes.

Dynamics and separation of time scales.—We begin with the equation of motion for the half thickness $h(x,t)$ of a thin layer of fluid in Hele-Shaw flow, bounded from above and below by mutually immiscible fluids. The equation of motion in rescaled form is [6]

$$h_t = -\partial_x(h[h_{xxx} + Bh_x]), \qquad (1)$$

where the Bond number $B = 2g\Delta\rho L^2/\sigma$ is the single dimensionless parameter characterizing the competition between the surface tension σ and buoyancy associated with density jump $\Delta\rho$ in a gravitational field g, and where L is the lateral width of the Hele-Shaw cell. An unstable density stratification corresponds to $B > 0$ [6]. The solution $h(x,t)$ can also be interpreted as the height of a fluid layer lying at a wall and beneath another fluid. Equation (1) is also relevant to recent experiments on the pinching of annular rings of fluid in the Hele-Shaw cell [10].

As in related earlier models with $B = 0$ [5,7], this partial differential equation (PDE) has a flux form $h_t + \nabla \cdot \mathbf{j} = 0$ arising from incompressibility, with $\mathbf{j} = h\mathbf{U}$ the hallmark of lubrication theory. The characteristic velocity $\mathbf{U} \sim -\nabla P$ arises from Darcy's law, and the pressure P is set by boundary conditions involving surface tension and gravity. In other contexts, the velocity has the more general form $\mathbf{U} \sim -h^m \nabla P$, such as in the spreading of drops ($m = 2$). Equation (1), with a different physical meaning from B, arises in the dynamics of the population density h of feeding herbivores [16], and also in the long-wavelength limit of the homogenized model of type-II superconductors [15], with h the local density of vortices.

It is sufficient for our purposes to consider Eq. (1) in a system of length 2π with periodic boundary conditions. If $h(x,t)$ is even and periodic, then it also describes a flow between two rigid walls at which the interface has a 90° contact angle. Linearizing about a flat interface $h = \bar{h}$, we obtain the growth rates

$$\nu(k) = \bar{h}(Bk^2 - k^4) \qquad (k = 0, 1, 2, \ldots). \qquad (2)$$

The number m of unstable modes scales as $\sqrt{B}$. If $B < 1$ all modes are linearly stable, whereas for $1 < B < 4$ the mode $k = 1$ is unstable, while those with $k \geq 2$ remain damped. Moreover, if B is tuned to be slightly larger than unity, say, $B = 1 + \epsilon$, then the first mode evolves on a time scale of order ϵ^{-1}, while the others rapidly equilibrate, thus being *slaved* to the first.

A contracting flow for $B = 1$.—Exactly at the bifurcation point $B = 1$ there is a *manifold* of steady-state solutions, $h_0(x) = \bar{h}(1 + a\cos x)$, for any $a < 1$. This manifold is also an attractor. Let $h = h_0 + \delta\zeta_x$ ($\delta \ll 1$), with ζ of zero mean. The linearized evolution about h_0 is $\zeta_t = -h_0\mathcal{L}_1\zeta_x$, where $\mathcal{L}_B = \partial_{xxx} + B\partial_x$, and it preserves $\langle\zeta\rangle = 0$. Consider then the norm

$$\mathcal{F} = \int_0^{2\pi} dx\, \frac{\zeta^2(x,t)}{2h_0(x)}$$

with

$$\mathcal{F}_t = -\sum_k k^2(k^2 - 1)|\hat{\zeta}_k|^2.$$

One may verify the inequalities $\mathcal{F}_t \geq \mathcal{F}_t(0)\exp(\pm Ht)$, where $H = 2\min_x\{h_0(x)\} > 0$ [19]. Since $\mathcal{F}_t \leq 0$, and further, $\mathcal{F}_t = 0$ iff ζ is entirely in the null space of $\mathcal{L}_1$ (and then ζ is a steady-state solution). One inequality from above then gives the bound $\mathcal{F}_t \geq \mathcal{F}_t(0)e^{-Ht}$, where $\mathcal{F}_t(0) \leq 0$. And so $\mathcal{F}_t \to 0$ as $t \to \infty$, which proves that the null space of $\mathcal{L}_1$ is attracting. Actually, $h_0 = \bar{h}(1 + a\cos kx)$ is a steady state for any integer k when $B = k^2$, but is unstable to subharmonic perturbations if $k > 1$.

Motion along the manifold for $B > 1$.—As the Bond number is increased slightly beyond unity, the first mode grows in time, but we expect that the amplitudes of higher modes will remain small. More generally, for larger B, we expect a finite number m of active modes including those that are linearly unstable. A natural approach then is to partition h into low (p) and high (q) modes. Let $\mathcal{P}_m$ project a periodic function onto its lower m modes and write

$$h = p + q \qquad (\mathcal{P}_m p = p,\ \ \mathcal{P}_m q = 0)\,. \tag{3}$$

Substituting this decomposition into (1), ignoring contributions of order q^2, invoking slaving of higher modes ($\partial_t q \simeq 0$), and integrating twice with respect to x, we obtain

$$pq_{xx} - p_x q_x + (p_{xx} + Bp)q = -\tilde{\tilde{p}}_t - \tilde{J}_p + C\,, \tag{4}$$

where $J_p = p\mathcal{L}_B p$ is the *flux* associated with the lower modes, C is an integration constant, and $\tilde{f} = \int^x dx' f(x')$. Since p is periodic in x, we find rather remarkably the computation of q reduced to the solution of an inhomogeneous Hill's equation [20]. Coupling to the partition constraints, $\mathcal{P}_m p_t = p_t$ and $\mathcal{P}_m q = 0$, gives a complete set of equations to determine q and p_t.

The *slaving approximation* (3) and (4) is particularly easy to analyze in the limit $B - 1 \equiv \epsilon \to 0$, for which there is only one active mode, thus $p = \bar{h}[1 + a(t)\cos x]$, and we rescale (4) with

$$B = 1 + \epsilon\,, \quad \tau = \epsilon\bar{h}t\,, \quad q = \epsilon Q\,. \tag{5}$$

At $\mathcal{O}(\epsilon)$ we obtain an inhomogeneous Ince equation [20],

$$(1 + a\cos x)Q_{xx} + a\sin x Q_x + Q = a_\tau\cos x - \tfrac{1}{2}(1 + a\cos x)^2 + C\,, \tag{6}$$

where $C = (1 + a^2/2)/2$ by the orthogonality of p and q.

Solvability and the amplitude equation—The solution of Eq. (6), found by variation of parameters, contains a secular term proportional to $x\sin x$. Its removal is the solvability condition that determines a_τ and yields the nonlinear equation of motion

$$a_\tau = \frac{a}{2}\Big(1 + \sqrt{1 - a^2}\Big). \tag{7}$$

For $a \ll 1$ this yields the exponential growth $a_\tau = a + \cdots$ of the linear stability result (2), but this behavior crosses over to a much different form in the nonlinear regime near pinching. Defining the function

$$f(a) = \frac{1 - \sqrt{1 - a^2}}{a^2} - \ln\!\left(\frac{1 - \sqrt{1 - a^2}}{a}\right), \tag{8}$$

with $a_0 = a(\tau = 0)$, we find $f(a_0) - f(a) = \tau$ by direct integration of (7). The singularity (or "pinch") time τ_p occurs when $a \nearrow 1$, so $f(a_0) - 1 = \tau_p$, and thus

$$t_p = \frac{f(a_0) - 1}{\bar{h}(B - 1)}\,. \tag{9}$$

For $a_0 \ll 1$, $t_p \sim \ln(2/a_0)/\bar{h}(B - 1)$, again consistent with the exponential growth of the linear instability. Near the touchdown, $a(t) = 1 - (t_p - t)/2 + \cdots$. Figure 1 shows excellent agreement between these asymptotic results and numerical studies of the lubrication PDE (1) for the pinch times $t_p(a_0)$, and for the minimum height $h_{\min} = \bar{h}[1 - a(t)]$.

The correction Q is found to be

$$Q(x) = \lambda_+\Big\{\sqrt{1 - a^2}\sin x\tan^{-1}\!\left(\frac{\lambda_-\sin x}{1 - \lambda_-\cos x}\right) - \frac{1}{2}(a + \cos x)\ln(1 - 2\lambda_-\cos x + \lambda_-^2) - a\left(\frac{3}{4}\lambda_-\cos x - \frac{1}{2}\right)\Big\}, \tag{10}$$

where $\lambda_\pm$ are the two real *zeros* of the quadratic $a\lambda^2 + 2\lambda + a = 0$, for which $\lambda_+\lambda_- = 1$ and $\lambda_- \leq 1$. As $a \nearrow 1$, $\lambda_- \to -1$, and thus within this analysis the interface curvature, through $Q''(x)$, develops a logarithmic singularity. This divergence can also be interpreted as the collision on the real axis of two singularities, located at $\pi \pm i\ln(\lambda_-)$ in the complex x plane.

A spectral cascade—The finite-time singularity can be understood by considering the spectrum of Q, writing

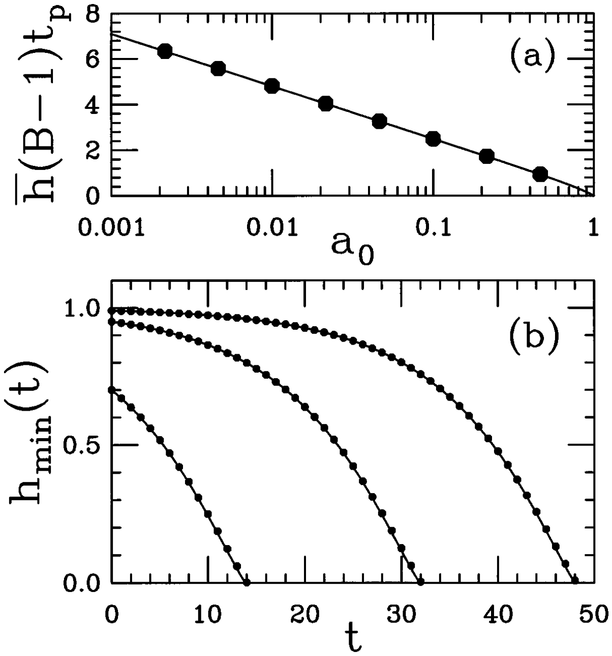

FIG. 1. Comparison between numerical solution of lubrication equation (1) and asymptotic analysis for $B \to 1$. (a) Singularity time versus initial amplitude a_0 from the numerical solution of Eq. (1) (solid circles) for $B = 1.05$, and from asymptotics in Eq. (9). (b) Minimum interface height as a function of time. Solid lines are the results of Eq. (7) with $a_0 = 0.01, 0.05, 0.30$ from top to bottom of figure, all with $B = 1.05$; solid circles show numerical results for those same initial conditions.

$Q = \sum_{k=2}^{\infty} Q_k \cos kx$. The recursion relation for the Q_k's obtained from Eq. (6) has the *exact* solution (for $k \geq 3$)

$$Q_k = C_+ \frac{\lambda_+^k}{k^3 - k} + C_- \frac{\lambda_-^k}{k^3 - k}, \tag{11}$$

where $C_\pm$ are constants determined by the inhomogeneous terms. Since $|\lambda_+| > 1$ for $a < 1$, the first term in (11) is the secular term eliminated by the solvability condition, and thus the power-law spectrum of Q is cut off by an exponential factor which tends to unity as $a \nearrow 1$. Full simulations show very good agreement, to very near the singularity time, between the form of the correction function (and its spectrum) with the asymptotic result (10). Figure 2(a) shows a comparison between the two in real space, and Fig. 2(b) shows the strong agreement between pointwise estimates $-\ln|\lambda| \simeq \ln(Q_k/Q_{k+1})$ for four wave numbers $k \gg 1$, illustrating the collapse of the analyticity strip width in accord with Eq. (7). At extremely small values of $h_{\min}$, the slaving assumptions should break down, and terms such as q_t cannot be neglected. Indeed, Bertozzi has noted that the ultimately negative divergence of Q_{xx}, while only logarithmic, is inconsistent with the existence of a single touchdown [21]. Her numerical studies following $h_{\min}$ down to $\mathcal{O}(10^{-30})$ suggest instead a saturation of the curvature.

Bifurcation of the singularities—Finally, we consider values of the Bond number well beyond the bifurcation point $B = 1$. Using an initial condition $h = \bar{h}(1 + a\cos x)$, with $a = 0.01$, Fig. 3 shows how the single *symmetric* touchdown at $x = \pi$ seen for $B \gtrsim 1$ bifurcates for $B \gtrsim 1.55$ into two *asymmetric* touchdowns. This behavior reflects the general increase in the number of active modes with increasing B. While the asymptotics

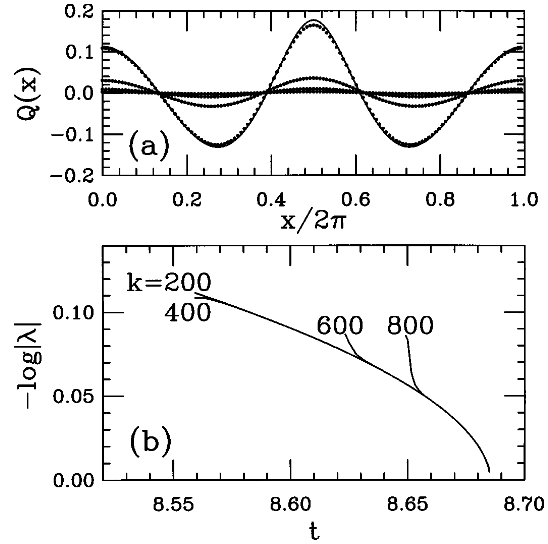

FIG. 2. (a) The function $Q(x)$ in Eq. (10) obtained from the asymptotic analysis (solid), compared with numerical solution of the full PDEs for $B = 1.05$ (dots). Results are for four times ranging from close to the initial condition to near the singularity time. (b) Collapse of the analyticity strip width as a function of time from numerical studies. For the largest two values of k shown, deviations from the common curve arise from these amplitudes lying initially beneath machine precision.

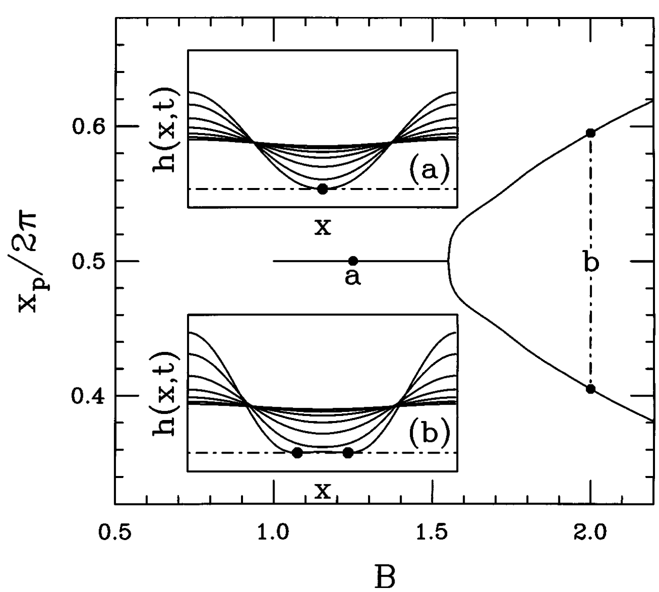

FIG. 3. Bifurcation diagram showing singularity locations versus Bond number. Insets (a) and (b) show interface evolution at $B = 1.25$ and 2.0.

developed for $B = 1 + \epsilon$ are not quantitatively valid for $\epsilon = \mathcal{O}(1)$, the slaving hypothesis with an increased number of modes leads to an appealing picture of how the singularities are generated in this regime.

The ansatz $p = \bar{h}[1 + a(t)\cos x + b(t)\cos 2x]$ is the simplest allowing for two singularities and leads to a spectrum of the form [19]

$$Q_k \sim \sum_{\nu=1}^{4} C_\nu \frac{\lambda_\nu^k}{k^3} \qquad (k \gg 1), \tag{12}$$

where $\{\lambda_\nu\}$ are the zeros of the quartic $b\lambda^4 + a\lambda^3 + 2\lambda^2 + a\lambda + b = 0$, defined by two independent quantities $\lambda_{1,2}$ as $\lambda_1, \lambda_1^{-1}, \lambda_2, \lambda_2^{-1}$. Except when any $|\lambda_{1,2}| = 1$, two of the four zeros lie within the unit circle in the complex λ plane, the other two lie outside. Elimination of the secular solutions associated with the latter two yields $a_t(t)$ and $b_t(t)$. As a and b evolve in time, the remaining zeros move toward the unit circle. In the first quadrant of the (a, b) parameter space the line $b = a - 1$, for $1 \leq a \leq 4/3$ defines those pinching configurations with a single touchdown (at $x = \pi$), while the curve $a = [8b(1 - b)]^{1/2}$ for $a > 4/3$ is the locus of configurations with two touchdowns. In the former case, $\lambda_{1,2}$ are real (as are $C_{1,2}$), and only one zero reaches the unit circle at the pinch time, thus producing only one singularity. Beyond the point $a = 4/3, b = 1/3$, λ_1 and λ_2 are complex conjugates of each other (with $C_1 = C_2^* = Re^{i\psi}$) and reach the unit circle simultaneously, producing two singularities.

It follows from this analysis that any ansatz for the active modes contained in p generates a set of zeros in the complex plane. Some of these singularities will move toward, although not all reach the unit circle as the pinch time is approached. Simulations of the full problem agree well with this picture. The numerical studies indicate that the nature of the singularities depends on the symmetry of the touchdowns; we observe a jump discontinuity in h_{xx} for asymmetric pinching, rather than the divergence seen with a symmetric pinch. This is associated with a rotation of phase ψ to $\pi/2$. We do not yet understand whether this behavior may be captured within the slaving approximation.

This analysis thus merges two previously separate concepts in dynamical systems described by PDEs. First is the coupling of slaved small scales to low-mode dynamics that recalls the reduction of a dissipative PDE to an inertial manifold [22]. Second is the motion of zeros in the complex plane as in the reduction of certain PDEs to "pole dynamics" [12,23].

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