Mapping of the Relativistic Kinetic Balance Equations onto the Klein-Gordon and Second-Order Dirac Equations

Adriana I Pesci†, Raymond E Goldstein†‡, and Hermann Uys†

†Department of Physics, University of Arizona, Tucson, AZ 85721
‡Program in Applied Mathematics, University of Arizona, Tucson, AZ 85721

Abstract. We have previously analyzed a mapping of the first two kinetic balance equations derived from the Boltzmann equation. Here we extend this mapping to the relativistic case. The essence of this mapping consists of applying a Fourier transform to the momentum coordinate of the distribution function. This procedure introduces a natural parameter η with units of angular momentum. In the non-relativistic case the ansatz of either separability, or separability and additivity, imposed on the probability distribution function produces mappings onto the Schrödinger equation and the Pauli equation respectively. The case leading to the Schrödinger operator corresponds to an irrotational flow, while the ansatz leading to the Pauli equation corresponds to a fluid with non-zero vorticity. In this work we show that the relativistic mappings lead to the Klein-Gordon equation in the irrotational case and to the second-order Dirac equation in the rotational case.

Submitted to: *Nonlinearity*

PACS numbers: 03.65.Ta, 05.20.Dd, 45.05.+x, 03.65.Sq

E-mail: pesci@physics.arizona.edu, gold@physics.arizona.edu, uys@physics.arizona.edu

1. Introduction

As we have shown in previous works [1, 2] it is possible, depending on the requirements imposed on the Fourier transform of the one particle probability function, to map the first balance equations obtained from the classical Boltzmann equation onto either the Schrödinger equation or the Pauli equation for a particle with spin 1/2. We have also shown in our previous work that the rules to obtain the hydrodynamic averages associated with the fluid equations obtained from the balance equations read like the postulates of quantum mechanics, provided the parameter η introduced in our mapping as a consequence of the Fourier transform is identified with $\hbar$. It seems natural to wonder if analogous mappings with equivalent ansatzes for the one particle probability function can be produced that lead to the Klein-Gordon and Dirac equations. As we will show this is indeed the case. When we use the ansatz corresponding to irrotational flow it is rather straightforward to obtain the Klein-Gordon equation by following closely the steps that lead to the Schrödinger equation in the non-relativistic case. Obtaining the Dirac equation is somewhat more complex, and instead of obtaining it in standard form we are lead to the second-order version of the operator. As we will see later in this work, the reason for this outcome relates to the fact that a Lagrangian formalism is the natural framework for rotational flows. As explained previously by Feynman [3, 4] the Klein-Gordon equation is easily handled within the Lagrangian formalism, but the Dirac equation is very hard to represent directly in this framework and one is always lead first to its second-order version. This particular issue is also connected to the fact that it is possible to use a simple two-component spinor for the wave function as long as we work with the second-order Dirac equation. These facts, even though of great importance when finding the map, do not have any bearing on the results since the full equivalence of the first- and second-order formulations of the Dirac equation was proven by Feynman and Gell-Mann [5].

2. The Mapping

We start as usual by noting that the motion of an ensemble of N particles governed by the relativistic Liouville equation can be recast in a hierarchy of non-linear partial differential equations (PDEs) for the reduced probability functions defined as follows:

$$f_N(x_1, p_1, \ldots, x_N, p_N) = \frac{D}{\int_\Omega D d\Omega} \ , \tag{1}$$

and for $1 \leq j < N$

$$f_j(\mathbf{x}^N, \mathbf{p}^N) = \int_\Omega f_N(\mathbf{x}^N, \mathbf{p}^N) \prod_{l=j+1}^{N} dx_l dp_l \ , \tag{2}$$

where D represents the number density of points in phase space, Ω is the volume in phase space and $(\mathbf{x}^N, \mathbf{p}^N) = (x_1, p_1, \ldots, x_N, p_N)$. These functions correspond to the probability of finding the subsystem of $j < N$ particles in the phase volume $\prod_{l=1}^{j} dx_l dp_l$ about the state $(x_1, p_1, \ldots, x_j, p_j)$. The N PDEs generated are known as the BBKGY hierarchy [6], the first two members of which (i.e. the equations for f_1 and f_2) determine the kinetic and potential energy of an aggregate of particles, and have a crucial role in fluid dynamics. These equations can be decoupled in several different ways by introducing a particular ansatz for the probability functions. Since we are interested in the relationship between the Klein-Gordon and Dirac equations and the relativistic fluid equations we adopt the ansatz that leads to the relativistic Boltzmann equation for $f_1 \equiv f$:

$$p^\mu \frac{\partial f}{\partial x^\mu} + G^\mu(x^\mu, p^\mu) \frac{\partial f}{\partial p^\mu} = C(f) \ , \tag{3}$$

where $C(f)$ is the collision integral, G_μ represents the external force averaged over the coordinates of all particles except one and we have dropped the subindex 1 since there is now a single set of coordinates. Here we are interested in two particular cases of G^μ,

$$G^\mu = \frac{e}{c} F^{\mu\nu} p_\nu \ , \tag{4}$$

which is the electromagnetic force, where $F^{\mu\nu}$ is the electromagnetic tensor, and

$$G^{\mu} = \frac{e}{c}F^{\mu\nu}p_{\nu} + \frac{e}{2c}N_{\alpha\beta}\partial^{\mu}F^{\alpha\beta} \tag{5}$$

which corresponds to the dipole interaction, where $N^{\alpha\beta}$ is the magnetic moment tensor. We will use these expressions in the case of irrotational and rotational flow respectively. For convenience we use (5) to calculate the conservation laws that lead to the fluid equations. When the case of irrotational flow arises (Klein-Gordon mapping) we shall ignore the last term of (5), so that we recover (4).

The conservation laws of the system are then obtained by averaging (3) with respect to p (conservation of number of particles), multiplying by p_{σ} and averaging over p (conservation of momentum) and multiplying by $p^{\sigma}p_{\sigma}$ and averaging over p (conservation of energy). In all three cases the right hand side vanishes and thus the first two balance equations read:

$$\int_{-\infty}^{+\infty} dp\; p^{\mu}\frac{\partial f}{\partial x^{\mu}} = 0 \;, \tag{6}$$

and

$$\int_{-\infty}^{+\infty} dp\; p_{\sigma}\left(p^{\mu}\frac{\partial f}{\partial x^{\mu}} + G^{\mu}(x,p)\frac{\partial f}{\partial p^{\mu}}\right) = 0 \;. \tag{7}$$

Here we have assumed that any surface terms vanish due to the convergence properties of f. We now introduce into (6) and (7) the following representation for f,

$$f(x,p,t) = \frac{1}{(2\pi\eta)^4}\int_{-\infty}^{+\infty}\exp\left(-i\frac{p_{\mu}y^{\mu}}{\eta}\right)\hat{f}(x,y,t)dy \;, \tag{8}$$

and $\hat{f}(x,y,t)$ is given by

$$\hat{f}(x,y,t) = \int_{-\infty}^{+\infty}\exp\left(i\frac{p_{\mu}y^{\mu}}{\eta}\right)f(x,p,t)dp \;. \tag{9}$$

With these definitions and some straightforward algebra, Eqs. (6) and (7) become [1]

$$\lim_{y\to 0}\; \frac{\eta}{i}\frac{\partial}{\partial x^{\mu}}\frac{\partial \hat{f}}{\partial y_{\mu}} = 0 \tag{10}$$

and

$$\lim_{y\to 0}\left[-\eta^2\frac{\partial}{\partial x^{\mu}}\cdot\left(\frac{\partial^2\hat{f}}{\partial y_{\mu}\partial y_{\sigma}}\right) - \frac{e\eta}{ic}F^{\sigma\mu}\frac{\partial \hat{f}}{\partial y^{\mu}} - \frac{e}{2c}N_{\alpha\beta}\partial^{\sigma}F^{\alpha\beta}\hat{f}\right] = 0 \;.$$

These two limits correspond to the following averages:

$$\begin{aligned}\lim_{y\to 0}\hat{f} &= \lim_{y\to 0}\int_{-\infty}^{+\infty}\exp\left(i\frac{p_{\mu}y^{\mu}}{\eta}\right)f(x,p,t)dp \\ &= \int_{-\infty}^{+\infty} f(x,p,t)\;\; dp = \frac{\rho(x,t)}{m} \;,\end{aligned} \tag{11}$$

$$\begin{aligned}\lim_{y\to 0}\frac{\partial\hat{f}}{\partial y^{\sigma}} &= \lim_{y\to 0}\frac{\partial}{\partial y^{\sigma}}\int_{-\infty}^{+\infty}\exp\left(i\frac{p_{\mu}y^{\mu}}{\eta}\right)f(x,p,t)dp \\ &= \frac{i}{\eta}\int_{-\infty}^{+\infty} p^{\sigma}f(x,p,t)dp \\ &= \frac{i}{\eta}\rho(x,t)u^{\sigma}(x,t) \;,\end{aligned} \tag{12}$$

where we have defined the mean 4-velocity u as the average, over the momentum only, of p/m, with m the rest mass. We can see from these expressions that $\hat{f}$ is the generating function for the averages with respect to p. Replacing these values into the balance equations we obtain the fluid equations:

$$\partial_\mu(\rho u^\mu) = 0 \ . \tag{13}$$

and

$$\lim_{y\to 0} -\eta^2 \frac{\partial}{\partial x^\mu}\left(\frac{\partial^2 \hat{f}}{\partial y_\mu \partial y^\sigma}\right) - \frac{e}{c}\rho F_{\sigma\mu}u^\mu - \frac{e}{2mc}\rho N_{\alpha\beta}\partial^\sigma F^{\alpha\beta} = 0 \ .$$

The tensor in equation (14) has been evaluated in great detail for the non-relativistic case [1, 2]. Since the calculation in the relativistic case is identical, except for the use of 4-vector notation, we will only summarize the procedure. First we introduce the canonical change of variables $y = x' - x''$ and $x = (x' + x'')/2$, which satisfies the following relationships:

$$\begin{aligned} x' &= x + \frac{y}{2} \ , \quad x'' = x - \frac{y}{2} \\ \frac{\partial}{\partial y} &= \frac{1}{2}\left(\frac{\partial}{\partial x'} - \frac{\partial}{\partial x''}\right) \ , \quad \frac{\partial}{\partial x} = \left(\frac{\partial}{\partial x'} + \frac{\partial}{\partial x''}\right) \ . \end{aligned} \tag{14}$$

Notice that the limit $y \to 0$ corresponds to $x' \to x''$ and $x' = x'' \equiv x$. It is at this point that we need to make some assumptions on the properties of $\hat{f}$ to continue our calculation. We concentrate on two different ansatzes:

a) $\hat{f}$ is fully separable in the variables x' and x'',

$$\hat{f}(x', x'', t) = h'(x', t)h''(x'', t) \ , \tag{15}$$

and

b) $\hat{f}$ is the sum of two separable functions of the variables x' and x'',

$$\hat{f}(x', x'', t) = h'(x', t)h''(x'', t) + g'(x', t)g''(x'', t) \ . \tag{16}$$

It is important to notice that $\hat{f}$ must be real in the limit $y \to 0$: therefore case (a), i.e. full separability, corresponds to h' and h'' being complex conjugates of each other, leaving only two independent real functions, and case (b), i.e. sum of two separable functions, corresponds to h' and g' being complex conjugates of h'' and g'' respectively leaving four independent real functions. Thus, adopting the ansatz of case (a) will give us enough degrees of freedom to treat irrotational flows in the context of the fluid equations (13) and (14), while adopting the ansatz of case (b) will give us enough degrees of freedom to work with vortical flows with the same equations.

As has been mentioned before [1] these solutions, were they to exist, are just a very particular subset of the solutions to the original Boltzmann equation and their importance lie in the fact that they are the only ones that lead to the Klein-Gordon and Dirac operators respectively.

3. The Klein-Gordon equation.

In order to obtain the Klein-Gordon equation we proceed by imposing ansatz (a) on $\hat{f}$, then calculate the limit $y \to 0$ of the tensor in (14), using (4) for G^μ.

As shown previously [1], the limit $y \to 0$ with the ansatz of case (a) corresponds to

$$\lim_{y\to 0} \frac{\partial^2}{\partial y_\mu \partial y^\sigma} m\hat{f}(x', x'', t) = \frac{1}{4}\rho \frac{\partial^2 \ln\rho}{\partial x_\mu \partial x^\sigma} - \rho \frac{m^2}{\eta^2} u^\mu u_\sigma \tag{17}$$

where we have defined

$$\begin{aligned} \lim_{y\to 0} \sqrt{m}\ h'(x', t) &= \psi(x, t) \\ \lim_{y\to 0} \sqrt{m}\ h''(x'', t) &= \psi^*(x, t) \end{aligned} \tag{18}$$

and the density ρ can be easily verified to be given by $\rho = \psi^*\psi$. Replacing this result into (14) with the appropriate expression for G^μ we obtain from

$$\lim_{y\to 0} \left[-\eta^2 \frac{\partial}{\partial x^\mu} \left(\frac{\partial^2 \hat{f}}{\partial y_\mu \partial y_\sigma} \right) - \frac{e\eta}{ic} F^{\sigma\mu} \frac{\partial}{\partial y^\mu} \hat{f} \right] = 0 \ , \tag{19}$$

the Euler equation that together with continuity reads [7]

$$\begin{aligned} &\partial_\mu(\rho u^\mu) \quad = 0 \\ &m u^\mu \partial_\mu u_\sigma - \frac{\eta^2}{2m} \partial_\sigma \left(\frac{\partial_\mu \partial^\mu \rho^{1/2}}{\rho^{1/2}} \right) - \frac{e}{c} F_{\sigma\nu} u^\nu = 0 \ , \end{aligned} \tag{20}$$

where we have reduced the second equation of this pair making use of continuity and the identity

$$\frac{1}{\rho}\partial_\mu \left(\rho \frac{\partial^2 \ln\rho}{\partial x_\mu \partial x^\sigma} \right) = 2\partial_\sigma \left(\frac{\partial_\mu \partial^\mu \rho^{1/2}}{\rho^{1/2}} \right) \ . \tag{21}$$

Since we are working under the assumption that the Euler equations we have generated correspond to an irrotational flow, it is natural to introduce the generalized average 4-velocity $m u^\mu = \partial^\mu S + (e/c)A^\mu$. Taking advantage of the expression for the electromagnetic tensor as a function of the vector potential, $F_{\sigma\mu} = (\partial_\sigma A_\mu - \partial_\mu A_\sigma)$, and defining a new function $R = \rho^{1/2}$, the fluid equations can be rewritten as

$$2\partial_\mu R \left(\partial^\mu S + \frac{e}{c} A^\mu \right) + R \partial_\mu \left(\partial^\mu S + \frac{e}{c} A^\mu \right) = 0 \tag{22}$$

and

$$R \left(\partial^\mu S + \frac{e}{c} A^\mu \right) \left(\partial_\mu S + \frac{e}{c} A_\mu \right) - \eta^2 \partial_\mu \partial^\mu R + K m^2 = 0 \ , \tag{23}$$

where we have a yet undetermined constant K as a consequence of having integrated the second equation once. Now multiplying the first equation by (i/η) the second one by $-1/\eta^2$ and adding them it is easy to see that we can use the standard Hopf-Cole transformation $\Omega = \ln R + (i/\eta)S = \ln\Psi$ to rewrite the pair of equations (22) and (23) as

$$\left[\left(i\eta\partial_\mu - \frac{e}{c} A_\mu \right)^2 - K m^2 \right] \Psi = 0 \ . \tag{24}$$

It is apparent that the choices $K = c^2$ and $\eta = \hbar$ would make (24) the Klein-Gordon equation.

4. The Dirac equation

Here we consider the case of vortical flows. As mentioned above, the outcome of this mapping leads to the second-order Dirac equation. In the non-relativistic case, vortical flows can only be treated in all generality by using a Lagrangian formalism [2]. This is also true in the relativistic case. Thus, following closely the steps for the non-relativistic calculation, the relativistic mapping begins by invoking ansatz (b) for our function $\hat{f}$. Then we evaluate the tensor in the momentum balance equation including the dipole interaction term, i.e. using the expression (5) for the external force, and recast the whole expression as the Euler fluid equations. Next, we introduce the action corresponding to the fluid equations and perform the variation to prove that indeed this Lagrangian density corresponds to the equations of motion. Finally, we introduce a change of variables and when the variation is performed on the new variables the resulting equation of motion indeed is the second-order Dirac equation when η is set equal to $\hbar$.

Under ansatz (b), the tensor of (14) in the limit $y \to 0$ has the value [2]

$$\begin{aligned}
&\lim_{y\to 0} \frac{\partial}{\partial y^\mu}\frac{\partial}{\partial y_\sigma} m\hat{f}(x', x'', t) = \\
&\lim_{x',x''\to x} \frac{1}{4}\left(\frac{\partial}{\partial x'^\mu} - \frac{\partial}{\partial x''^\mu}\right)\left(\frac{\partial}{\partial x'_\sigma} - \frac{\partial}{\partial x''_\sigma}\right) m\left[h'h'' + g'g''\right] \\
&= \frac{1}{4}\left[\rho\frac{\partial^2 \ln\rho}{\partial x^\mu \partial x_\sigma} - 4\rho\frac{m^2}{\eta^2}u_\mu u^\sigma - \rho\frac{\partial\Sigma_i}{\partial x^\mu}\frac{\partial\Sigma_i}{\partial x_\sigma}\right]
\end{aligned} \tag{25}$$

where we have defined

$$\begin{aligned}
\lim_{y\to 0}\sqrt{m}\ h'(x', t) &= \psi_1(x, t) \\
\lim_{y\to 0}\sqrt{m}\ h''(x'', t) &= \psi_1^*(x, t) \\
\lim_{y\to 0}\sqrt{m}\ g'(x', t) &= \psi_2(x, t) \\
\lim_{y\to 0}\sqrt{m}\ g''(x'', t) &= \psi_2^*(x, t)\ ,
\end{aligned} \tag{26}$$

introduced the notation

$$\Sigma_i = \frac{\psi^\dagger\sigma_i\psi}{\psi^\dagger\psi}\ , \tag{27}$$

where

$$\psi = \begin{pmatrix}\psi_1 \\ \psi_2\end{pmatrix}\ , \quad \psi^\dagger = (\psi_1^*, \psi_2^*)\ , \tag{28}$$

and σ_i are the Pauli matrices, and then used (11) and (12) for the density ρ and the mean 4-velocity u_μ, which can be rewritten as a function of ψ as $\rho = \psi^\dagger\psi$ and $u^\mu = -(\eta/2mi)(\psi^\dagger\partial_\mu\psi - \psi\partial_\mu\psi^\dagger)$. Expression (25), even though formally correct, is not very useful in its present form since Σ, as presented in (27), reads like a 3-vector (notice that all three components Σ_i are real). This problem can be easily solved by introducing the 4-spinor

$$\Psi = \begin{pmatrix}\psi \\ -\psi\end{pmatrix}\ , \quad \Psi^\dagger = \left(\psi^\dagger, -\psi^\dagger\right)\ . \tag{29}$$

These definitions allow us also to rewrite (25) in a much more convenient and instructive way

$$\lim_{y\to 0}\frac{\partial}{\partial y^\mu}\frac{\partial}{\partial y_\sigma}m\hat{f}(x',x'',t) = \frac{1}{4}\left[\rho\frac{\partial^2\ln\rho}{\partial x^\mu\partial x_\sigma} - 4\rho\frac{m^2}{\eta^2}u_\mu u^\sigma - \frac{1}{4}\rho\frac{\partial M_{\alpha\beta}}{\partial x^\mu}\frac{\partial M^*_{\alpha\beta}}{\partial x_\sigma}\right] \tag{30}$$

where $\rho = (1/2)\Psi^\dagger\Psi$ and $u_\mu = -(\eta/2mi)(1/2)(\Psi^\dagger\partial_\mu\Psi - \Psi\partial_\mu\Psi^\dagger)$. The tensor $M_{\alpha\beta}$ is defined as

$$\begin{aligned} M_{\alpha\alpha} &= 0 \\ M_{\alpha\beta} &= \frac{1}{2}i\frac{\Psi^\dagger\left[\gamma_\alpha\gamma_\beta - \gamma_\alpha\gamma_\beta\right]\Psi}{\Psi^\dagger\Psi}\ , \end{aligned} \tag{31}$$

where the γ_μ are the gamma matrices,

$$\gamma_t = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \gamma_i = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix} \tag{32}$$

The explicit expression for the tensor M is then

$$M = \begin{pmatrix} 0 & \Sigma_z & -\Sigma_y & i\Sigma_z \\ -\Sigma_z & 0 & \Sigma_x & i\Sigma_y \\ \Sigma_y & -\Sigma_x & 0 & i\Sigma_z \\ -i\Sigma_x & -i\Sigma_y & -i\Sigma_z & 0 \end{pmatrix} \tag{33}$$

Introducing these definitions into Eqs. (13) and (14), with the force G_μ given by (5), we obtain after some lengthy but straightforward algebra the following fluid equations

$$\begin{aligned} &\partial_\mu(\rho u^\mu) = 0 \\ &m u^\mu\partial_\mu u_\sigma - \frac{\eta^2}{2m}\partial_\sigma\left(\frac{\partial_\mu\partial^\mu\rho^{1/2}}{\rho^{1/2}}\right) \\ &+\frac{\eta^2}{4m\rho}\partial_\mu\left(\frac{\rho}{4}\frac{\partial M_{\alpha\beta}}{\partial x^\mu}\frac{\partial M^*_{\alpha\beta}}{\partial x_\sigma}\right) \\ &-\frac{e}{c}F_{\sigma\nu}u^\nu - \frac{e\eta}{4mc}M_{\alpha\beta}\partial_\sigma F_{\alpha\beta} = 0\ . \end{aligned} \tag{34}$$

where once again we have used identity (21) and continuity and identified the tensor M with the magnetic moment tensor through the relationship $N = (\eta/2)M$. The motivation to make this identification between M and N lies in the fact that both quantities transform in the same manner and have the exact number of powers of $\eta/2$ to ascribe to them units of angular momentum. Moreover, from equation (33) we see that there are only three non-zero elements in M that are different from each other. Since these three elements are equal to the non-relativistic components of the vector angular momentum in three dimensions it is natural to think of $(\eta/2)M$ (and also N) as the relativistic tensor that corresponds to the axial 3-vector $\mathbf{\Sigma}$. As we have shown before [2] it is possible to express these quantities in a more physical context by introducing the Clebsch variables ζ and ω [8]. As a consequence of representing $\mathbf{\Sigma}$ with the Pauli matrices, ζ corresponds to the z-component of the vector $\mathbf{\Sigma}$ and ω corresponds to the azimuthal angle, i.e, the canonical conjugate variable of Σ_z. Then, as a function of the components of $\mathbf{\Sigma}$ (or equivalently the elements of the tensor M), the expressions for ζ and ω are

$$\begin{aligned} \zeta &= \frac{\eta}{2}\Sigma_z \\ \omega &= \tan^{-1}\left(\frac{\Sigma_x}{\Sigma_y}\right) \ . \end{aligned} \tag{35}$$

If we also introduce the angle θ to represent the angle that $\mathbf{\Sigma}$ makes with the z-axis the three distinct elements of M can be expressed as

$$\begin{aligned} \Sigma_x &= \sin\theta\sin\omega \\ \Sigma_y &= \sin\theta\cos\omega \\ \Sigma_z &= \cos\theta \end{aligned} \tag{36}$$

We can now rewrite the last term in equation (30) as a function of the Clebsch variables. After some algebra it is easy to verify that

$$\frac{\partial M_{\alpha\beta}}{\partial x^\mu}\frac{\partial M^*_{\alpha\beta}}{\partial x_\sigma} = \frac{\partial_\mu\zeta\partial^\sigma\zeta}{q} + \frac{4}{\eta^2}q\partial_\mu\omega\partial^\sigma\omega \ , \tag{37}$$

where we have defined q as

$$q(\zeta) \equiv q = \frac{\eta^2}{4} - \zeta^2 \ , \tag{38}$$

a function of ζ only.

Now we show that the action that corresponds to the equations of motion (34) is

$$\begin{aligned} \mathcal{A} = & -\int d^4x\left[\frac{\rho}{2m}\left(\partial_\mu S + \zeta\partial_\mu\omega + \frac{e}{c}A_\mu\right)^2 - mc^2\rho \right. \\ & + \frac{\eta^2}{8m}\frac{\partial_\mu\rho\partial^\mu\rho}{\rho} + \frac{\eta^2}{8m}\left(\frac{(\partial_\mu\zeta)^2}{q} + \frac{4}{\eta^2}q(\partial_\mu\omega)^2\right) \\ & \left. - \frac{e\eta}{4mc}\rho M_{\alpha\beta}F^{\alpha\beta}\right] \ , \end{aligned} \tag{39}$$

where we have already replaced into (39) the expression $u_\mu = \frac{1}{m}(\partial_\mu S + \zeta\partial_\mu\omega + \frac{e}{c}A_\mu)$ which is the result of the variation of $\mathcal{A}$ with respect to u^μ i.e. $\delta\mathcal{A}/\delta u^\mu$. The remaining variations are given by:

$$\begin{aligned} \frac{\delta\mathcal{A}}{\delta\omega} &: \ u^\mu\partial_\mu\zeta + \frac{1}{m}\partial_\mu\left(\rho q\partial^\mu\omega\right) + \frac{e\eta}{4mc}\frac{\partial M_{\alpha\beta}}{\partial\omega}F_{\alpha\beta} = 0 \\ \frac{\delta\mathcal{A}}{\delta\zeta} &: \ u^\mu\partial_\mu\omega - \frac{\eta^2}{8m}\frac{q'}{q}(\partial_\mu\zeta)^2 + \frac{1}{2m}q'(\partial_\mu\omega)^2 \\ & \quad - \frac{\eta^2}{4m}\partial_\mu\left(\rho\frac{\partial_\mu\zeta}{q}\right) - \frac{e\eta}{4mc}\frac{\partial M_{\alpha\beta}}{\partial\zeta}F_{\alpha\beta} = 0 \\ \frac{\delta\mathcal{A}}{\delta S} &: \ \partial_\mu\left[\rho\frac{1}{m}(\partial^\mu S + \zeta\partial^\mu\omega + \frac{e}{c}A^\mu)\right] \equiv \partial_\mu(\rho u^\mu) = 0 \\ \frac{\delta\mathcal{A}}{\delta\rho} &: \ \frac{1}{2m}(\partial_\mu S + \zeta\partial_\mu\omega + \frac{e}{c}A_\mu)^2 - \frac{\eta^2}{2m}\frac{\partial^\mu\partial_\mu\rho^{1/2}}{\rho^{1/2}} \\ & \quad - mc^2 + \frac{\eta^2}{8m}\left(\frac{(\partial_\mu\zeta)^2}{q} + \frac{4}{\eta^2}q(\partial_\mu\omega)^2\right) \\ & \quad - \frac{e\eta}{4mc}M_{\alpha\beta}F^{\alpha\beta} = 0 \ . \end{aligned} \tag{40}$$

Now we take the derivative $\partial_\sigma(\delta\mathcal{A}/\delta\rho)=0$ and substitute $u_\mu=\frac{1}{m}(\partial_\mu S+\zeta\partial_\mu\omega-\frac{e}{c}A_\mu)$. Then use continuity and replace the values of the variations with respect to ω and ζ to obtain finally (34). Notice also that continuity is simply given by the variation of $\mathcal{A}$ with respect to S, $\delta\mathcal{A}/\delta S=0$.

Finally to show that the action $\mathcal{A}$ also gives rise to the second-order Dirac equation we introduce the following expression for the 4-spinor Ψ [9]:

$$\Psi=\begin{pmatrix}\psi\\ -\psi\end{pmatrix}\quad\text{with }\psi=R\mathrm{e}^{iS/\eta}\begin{pmatrix}\cos\frac{\theta}{2}\ \mathrm{e}^{i\omega/2}\\ i\sin\frac{\theta}{2}\ \mathrm{e}^{-i\omega/2}\end{pmatrix}\tag{41}$$

In this representation the action (39) becomes

$$\begin{aligned}\mathcal{A}_\Psi=&-\frac{1}{2}\int d^4x\left[\frac{1}{2m}\left(i\eta\frac{\partial\Psi^\dagger}{\partial x^\mu}+\frac{e}{c}\Psi^\dagger A_\mu\right)\right.\\ &\times\left(-i\eta\frac{\partial\Psi}{\partial x_\mu}+\frac{e}{c}A^\mu\Psi\right)-mc^2\Psi^\dagger\Psi\\ &\left.-\frac{e\eta}{2mc}\left(\frac{1}{2}M_{\alpha\beta}F^{\alpha\beta}\right)\Psi^\dagger\Psi\right]\end{aligned}\tag{42}$$

Then taking the variation with respect to $\Psi^\dagger$ [9] we obtain

$$\left[\left(-i\eta\frac{\partial}{\partial x^\mu}+\frac{e}{c}A_\mu\right)^2-m^2c^2-\frac{e\eta}{2c}\sigma_{\alpha\beta}F^{\alpha\beta}\right]\Psi=0\tag{43}$$

where $\sigma_{\alpha\beta}=(i/2)(\gamma_\alpha\gamma_\beta-\gamma_\beta\gamma_\alpha)$. As with the Klein-Gordon equation, if we choose to identify $\eta=\hbar$ then Eq. (43) reads

$$\left[\left(i\hbar\frac{\partial}{\partial x^\mu}-\frac{e}{c}A_\mu\right)^2-\frac{e\eta}{2c}\sigma_{\alpha\beta}F^{\alpha\beta}\right]\Psi=m^2c^2\Psi\tag{44}$$

which is the second-order Dirac equation in the Feynman-Gell-Mann formulation [5].

5. Conclusions

In this work we have performed the relativistic extension of previous results connecting the Boltzmann equation to the Schrödinger and Pauli operators. We have found that the Fourier transform of the one-particle distribution function of the classic relativistic Boltzmann equation with respect to the momentum variable can be mapped either onto the Klein-Gordon or the Dirac equations. As in the non-relativistic case, the first part of the mapping leads to a set of Euler equations for a compressible fluid. From them, the analysis of irrotational flows coupled to the ansatz of separability applied to the one-particle probability function leads to the Klein-Gordon equation for particles with no spin. A similar analysis for rotational flows and the ansatz of separability and addition leads to the Dirac equation for particles with spin $\eta/2$. The rules to calculate the averages of physical quantities in the p-conjugate space are the four vector versions of the rules found in the non-relativistic case and which read like the postulates of quantum mechanics with η replaced by $\hbar$ [1].

There is a very interesting consequence to the ansatzes (a) and (b) imposed on $\hat{f}$. As emphasized previously, these solutions form a very small subset of all possible solutions to the balance equations. Returning to the original Boltzmann equation, we see that its right hand side is the collision integral that includes a first approximation to the two-particle probability function $f_2(x_1, p_1, x_2, p_2)$ constructed as a combination of products of one-particle probability functions. Independent of the approximation used, f_2 must be symmetric under the exchange of coordinates $1 \leftrightarrow 2$, so $f_2(x_1, p_1, x_2, p_2) = f_2(x_2, p_2, x_1, p_1)$ [6]. When the approximation for f_2 used in the Boltzmann equation is adopted this symmetry will also hold true for its Fourier transform $\hat{f}_2(x_1, y_1, x_2, y_2)$. If now we invoke the separability condition $\hat{f}_2 = \Psi^\dagger(x_1'', x_2'')\Psi(x_1', x_2')$ [1], and note that we are working with identical particles, the functions Ψ must be either symmetric or antisymmetric under the exchange of variables $1 \leftrightarrow 2$ so that $\hat{f}_2$ will be symmetric. It may be possible to prove that the case which maps onto the Klein-Gordon equation requires Ψ to be symmetric, while the Dirac case requires antisymmetry. Such a proof would need a detailed study of the Bogoliubov hypothesis that leads to the Boltzmann equation.

One last issue that we would like to mention relates to the Proca equations that govern particles of spin 1 or higher. Since these can be developed from the Dirac equation [10] it seems reasonable to think that there might be a new ansatz for $\hat{f}$ that would lead to the equations for higher spin. Unfortunately, we have not yet been able to find a satisfactory derivation that would settle the issue either way. Perhaps these issues should be the subject of further work.

Acknowledgments

We are grateful to Wim van Saarloos for his hospitality during our stay at the Lorentz Institute and also for his very helpful comments.

References

[1] Pesci A and Goldstein R 2004 Mapping of the classic kinetic balance equations onto the Schrödinger equation *Nonlinearity* **17**, 000-000 (preceding paper)
[2] Pesci A, Goldstein R and Uys H 2004 Mapping of the classic kinetic balance equations onto the Pauli equation *Nonlinearity* **17**, 000-000 (preceding paper)
[3] Feynman R 1948 Space-time approach to non-relativistic quantum mechanics *Rev. Mod. Phys* **20**, 367-387
[4] Feynman R 1951 An operator calculus having applications in quantum electrodynamics *Phys. Rev.* **84**, 108-110
[5] Feynman R and Gell-Mann M 1958 Theory of the Fermi interaction *Phys. Rev.* **109**, 193-198.
[6] Liboff R L 1998 *Kinetic Theory. Classical, Quantum and Relativistic Descriptions* (New York: Wiley-Interscience)
[7] Bialynicky-Birula, I 1971 Relativistic quantum mechanics of dyons. Exact solution *Phys. Rev. D* **3**, 2413-15
[8] Clebsch A 1859 Ueber die integration der hydrodynamischen gleichungen *J. Reine Angew. Math.* **56**, 1-10
[9] Schiller R 1962 Quasi-classical theory of a relativistic spinning electron *Phys. Rev.* **128**, 1402-12
[10] Greiner W 2000 *Relativistic Quantum Mechanics. Wave Equations* (New York: Springer)