LETTERS

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Domain of convergence of perturbative solutions for Hele-Shaw flow near interface collapse

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Recent work [Phys. Fluids **10**, 2701 (1998)] has shown that for Hele-Shaw flows sufficiently near a finite-time pinching singularity, there is a breakdown of the leading-order solutions perturbative in a small parameter ϵ controlling the large-scale dynamics. To elucidate the nature of this breakdown we study the structure of these solutions at higher order. We find a finite radius of convergence that yields a new length scale exponentially small in ϵ. That length scale defines a ball in space and time, centered around the incipient singularity, inside of which perturbation theory fails. Implications of these results for a possible matching of outer solutions to inner scaling solutions are discussed. © *1999 American Institute of Physics.* [S1070-6631(99)00910-1]

Pinching singularities in Hele-Shaw flows arise typically either through intrinsic instabilities[1–4] or through driving via boundary conditions.[5–8] This class of dynamical systems is most readily studied through equations of motion for interfaces between fluids, and recasting the problem of singularities as one of interface collision. In this context, numerical data[4,7] suggest for both cases that the type of singularity may be very sensitive to the details of the large-scale forcing. It has also been observed that the scaling properties of the interface $h(x,t)$ can change qualitatively at remarkably small distances from the singularity point (x_p,t_p). There has been no analytic explanation for this. There is also a growing body of evidence supporting the existence of similarity solutions for the innermost region, of the type $h(x,t)\sim(t_p-t)^{\alpha}F((x-x_p)/(t_p-t)^{\beta})$, but no clear understanding either of how to connect such solutions to the outer flow far from the singularity or of the origin of the inner length scale.

Given the highly nonlinear nature of these problems of interface motion, it is perhaps not surprising that the singularity structure would depend in a very nontrivial way on the large-scale forcing. Nevertheless, we recently found[4] that a lowest-order *perturbative* solution to one particular problem, the Rayleigh–Taylor instability in Hele-Shaw flow, actually provided an accurate and fairly detailed description of the approach to interface touchdown. Intriguingly, close to the singularity we observed a divergence between the perturbative results and numerical studies, traceable to an internal inconsistency in the asymptotic perturbative results. An understanding of this has been lacking.

Here we study in detail the structure of the perturbation theory at higher order and find that the previously seen discrepancies and inconsistencies actually stem from the existence of a finite radius of convergence. This result then implies that there is a ball around the incipient singularity, *exponentially small* in a parameter controlling the large-scale flow, within which perturbation theory breaks down. We conjecture that this fact may underlie the aforementioned sensitivity to large-scale forcing, that appears to be qualitatively different from the three-dimensional case.[9] Moreover, this will perhaps provide a clue as to how to define the length scale at which outer flows match to the inner similarity solutions that are by now well-understood from local expansions around the singular point.[6,7]

Figure 1 shows the physical situation of interest: an interface in a gravitational field g separating two fluids, the heavier on top. Let $h(x,t)$ be the interface height above a bounding plane; a singularity will occur when $h\to 0$ at a "pinching" point x_p. By suitably rescaling space and time and introducing the Bond number $B\propto\Delta\rho g/\sigma$, where $\Delta\rho$ is the density difference between the fluids and σ the surface tension, Darcy's Law for a thin layer can be shown[3,4] to reduce to

$$h_t=-\partial_x(h(h_{xxx}+Bh_x)). \tag{1}$$

We are particularly interested in the case of system of finite lateral extent (say, 2π), for which $B=1$ is the threshold

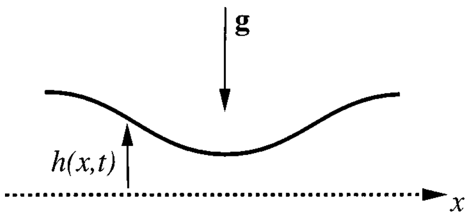

FIG. 1. Thin layer of fluid of height $h(x,t)$ in a Hele-Shaw cell, trapped between a wall and a heavier fluid.

of instability. A perturbative theory can be constructed near this point by setting

$$\begin{aligned} &B=1+\epsilon, \quad t=T/\epsilon, \\ &h=q^{(0)}+\epsilon q^{(1)}+\epsilon^2 q^{(2)}+\cdots. \end{aligned} \tag{2}$$

The scaling of time is motivated by the linear stability analysis,[4] and reveals that ϵ corresponds to a singular perturbation in time (i.e., multiplying the highest time derivative), while associated with a regular perturbation in space. As is often found in singular perturbations, we would anticipate the emergence of a new *time* scale controlled by ϵ. However, the coupling of space and time within the lubrication PDE implies an associated length scale as well.

The flux form of the governing equation allows a first integration,

$$\tilde{h}_t=-h(h_{xxx}+Bh_x), \tag{3}$$

where we define[4]

$$\tilde{f}(x,t)=\int dx f(x,t), \tag{4}$$

for any function f. Substituting from Eq. (2) into Eq. (3) and grouping terms by powers of ϵ we obtain the first few equations in the hierarchy:

$$\begin{aligned} &\mathcal{O}(\epsilon^0): q_{3x}^{(0)}+q_x^{(0)}=0, \\ &\mathcal{O}(\epsilon^1): q_{3x}^{(1)}+q_x^{(1)}=-q_x^{(0)}-\tilde{q}_T^{(0)}\frac{1}{q^{(0)}}, \\ &\mathcal{O}(\epsilon^2): q_{3x}^{(2)}+q_x^{(2)}=-q_x^{(1)}-\tilde{q}_T^{(1)}\frac{1}{q^{(0)}}+\tilde{q}_T^{(0)}\frac{q^{(1)}}{q^{(0)2}}, \\ &\mathcal{O}(\epsilon^3): q_{3x}^{(3)}+q_x^{(3)}=-q_x^{(2)}-\tilde{q}_T^{(2)}\frac{1}{q^{(0)}}+\tilde{q}_T^{(1)}\frac{q^{(1)}}{q^{(0)2}} \\ &\qquad\qquad -\tilde{q}_T^{(0)}\left[\frac{q^{(1)2}}{q^{(0)3}}-\frac{q^2}{q^{(0)2}}\right]. \end{aligned} \tag{5}$$

This yields the lowest-order solution

$$q^{(0)}=1+a(T)\cos(x), \tag{6}$$

which touches down at $x_p=\pi$ as $a\to 1$. The time dependence of the mode amplitude $a(T)$ and the first-order correction $q^{(1)}$ were determined by a solvability condition at the next order.[4]

Proceeding to higher orders, we find the general right-hand side of Eq. (5) at order n to be

$$f^{(n)}\equiv -\sum_{k=0}^{n-1}\frac{\tilde{q}_T^{(n-1-k)}}{q^{(0)}} \times \sum_{\substack{j_1,\ldots,j_k=1 \\ \Sigma_{i=1}^k i j_i=k}}^{k}\left(\sum_{i=1}^{k} j_i\right)!\prod_{i=1}^{k}\frac{(-q^{(i)}/q^{(0)})^{j_i}}{j_i!}.$$

Note that the left-hand side of each equation in Eq. (5) is a total derivative that integrates up to $q_{xx}^{(n)}+q^{(n)}$, whose homogeneous solutions are $\sin(x)$ and $\cos(x)$, with Wronskian $W=1$. The method of variation of parameters yields the general solution

$$\begin{aligned} q^{(n)}(x,T)=&\cos(x)\int^x d\xi \sin(\xi)\tilde{f}^{(n)}(\xi,T) \\ &-\sin(x)\int^x d\xi\cos(\xi)\tilde{f}^{(n)}(\xi,T). \end{aligned} \tag{7}$$

Thus far, these perturbative formulas are exact. Since here we are only interested in the neighborhood of the touchdown, we shall focus on the dominant terms at each order that lead to a singularity.[10] To that end, we recall[4] that the first-order solution $q^{(1)}$ has a logarithmic singularity in its second derivative. Introducing $\chi=x-\pi$, the distance from the pinch point, and the time $\tau=T_p-T$ from the singularity, the mode amplitude $a(\tau)$ behaves as

$$a(\tau)\simeq 1-\tfrac{1}{2}\tau+\cdots, \quad (\tau\geqslant 0). \tag{8}$$

Then, the two first orders have the asymptotic forms

$$\begin{aligned} &q^{(0)}\simeq \tfrac{1}{2}(\tau+\chi^2), \\ &q^{(1)}\simeq -\tfrac{1}{4}(\tau-\chi^2)\ln(\tau+\chi^2). \end{aligned} \tag{9}$$

The higher-order corrections consistent with these leading-order terms are obtained using the asymptotic form of Eq. (7):

$$q^{(n)}(\chi,\tau)\simeq \int^{\chi} d\xi\,\xi\tilde{f}^{(n)}(\xi,\tau)-\chi\int^{\chi} d\xi\tilde{f}^{(n)}(\xi,\tau). \tag{10}$$

Systematic analytic calculation up to seventh order reveals a simple procedure to compute the most singular contribution at a given order, once those at previous orders are known. The steps are as follows: (i) every term on the right-hand-side of the hierarchy Eq. (5) contributes to the leading order behavior of $\tilde{f}^{(n)}$, with successive terms having a common factor of $\ln^{n-1}(\tau+\chi^2)$ multiplying a polynomial in τ and χ; (ii) a similar pattern results after performing the two integrations in Eq. (10)—the first integral is $K(\tau+\chi^2)\ln^n(\tau+\chi^2)$, while the second is $2K\mathrm{X}\ln^n(\tau+\chi^2)$, with K a calculable constant. Collecting together terms, we obtain at second order

$$q^{(2)}\simeq +\tfrac{1}{8}(\tau-\chi^2)\ln^2(\tau-\chi^2), \tag{11}$$

or, in general, for $n\geqslant 1$,

$$q^{(n)}\simeq(-1)^n a_n(\tau-\chi^2)\ln^n(\tau+\chi^2), \tag{12}$$

where we have computed analytically

$$a_3=\frac{1}{12}, \quad a_4=\frac{13}{192}, \quad a_5=\frac{59}{960}, \tag{13}$$

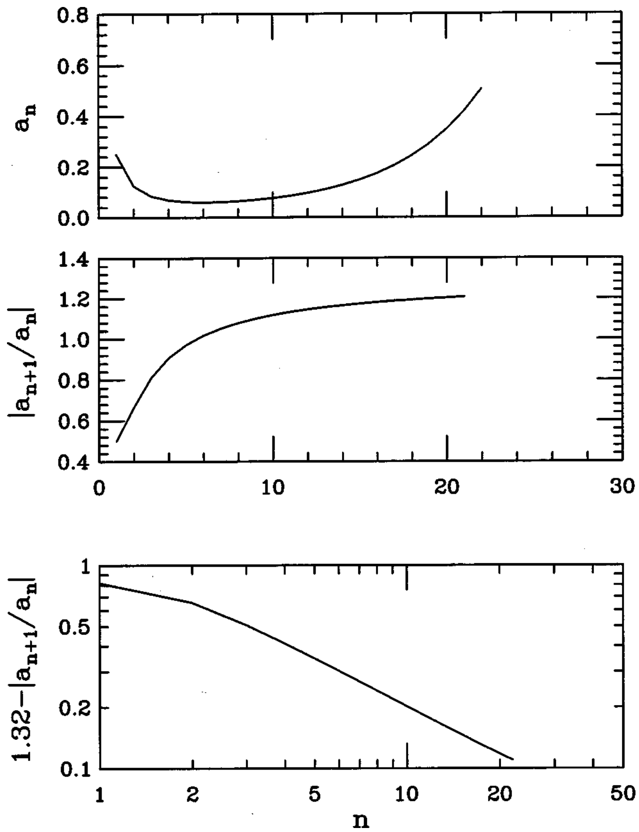

FIG. 2. Top to bottom: coefficients, ratios, and convergence properties of the series $\{a_n\}$, obtained from a numerical solution of the recursion relation.

$$a_6 = \frac{172}{2880}, \quad a_7 = \frac{4901}{80\,640}. \tag{14}$$

Using these results and the expression for $\tilde{f}$, one can construct a *recursion relation* for the coefficients a_n. The complexity of evaluating this relation at high order centers around the combinatorics of the constrained sums. We have not succeeded in finding the general term a_n. However, we have solved this recursion relation numerically for coefficients up to order a_{22}. Figure 2(a) displays those coefficients, while Fig. 2(b) displays the successive ratios a_{n+1}/a_n as a function of n. We fitted the approach to an asymptote as an inverse power of n and deduced the approximate numerical value of the ratio as $n \to \infty$,

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| \equiv \mathcal{R} \simeq 1.32. \tag{15}$$

Figure 2(c) illustrates the fit.

The fact that $\mathcal{R} \neq 0$ implies that the second spatial derivative of $h(x,t)$,

$$h_{xx} \simeq 2 \sum_{n=0}^{\infty} (-1)^{n+1} a_n [\epsilon \ln(\tau + \chi^2)]^n, \tag{16}$$

fails to converge if

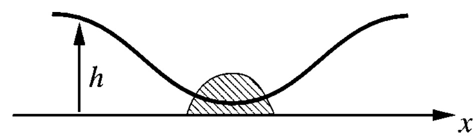

FIG. 3. Ball of nonconvergence of perturbation theory.

$$\tau + \chi^2 \leqslant \exp\left(-\frac{1}{\epsilon \mathcal{R}} \right). \tag{17}$$

Since τ is proportional to the minimum height of the interface and χ measures the spatial distance from the touchdown point, we see from Eq. (17) that there is a ball centered at $\tau = \chi = 0$ inside of which perturbation theory breaks down (Fig. 3). Remarkably then, the bifurcation parameter ϵ, which controls the large-scale flow, defines a new length scale in the problem.

It is reasonable to conjecture that: (a) the breakdown of perturbation theory connected to a feature of the large-scale flow would persist even in situations in which the touchdown is driven by far-field boundary conditions instead of an intrinsic instability,[5–7] and (b) this new length scale may be the matching point for outer flows and inner self-similar solutions. Solving these two issues may finally explain how smooth initial conditions lead to finite-time singularities.[3]

ACKNOWLEDGMENTS

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