Shape of the Ideal Icicle

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The growth of icicles is considered as a free-boundary problem. A synthesis of thin-film fluid dynamics, atmospheric heat transfer, and geometrical considerations leads to a nonlinear ordinary differential equation for the shape of a uniformly advancing icicle, the solution to which defines a family of shapes which compares very favorably with natural icicles. Away from the tip, the solution has a power-law form identical to that recently found for the growth of stalactites by precipitation of calcium carbonate. This analysis thereby explains why stalactites and icicles are so similar in form despite the vastly different physics and chemistry of their formation.

The formation of patterns in snow and ice has been a source of fascination since antiquity. As early as 1611, Johannes Kepler[1] sought a physical explanation for the beautiful forms of snowflakes. While attention has been lavished upon snowflakes ever since,[2] their wintry cousins, icicles, have remained largely ignored. The basic mechanisms of icicle growth are well known,[3–5] but there are few mathematical analyses describing their forms. For instance, icicle surfaces are typically covered with ripples a few centimeters in wavelength, but only recently[6–8] has theoretical work begun to address the underlying dynamic instability that produces them. On a more basic level, the familiar long, slender form of icicles has not been explained quantitatively. As one can see in Figure 1, icicles and stalactites – the iconic structures found in limestone caves[9] – can bear a striking resemblance, particularly insofar as they evince a slightly convex carrot-like form that is distinct from a cone. Of course visual similarity does not imply mechanistic similarity, but there is reason to think that a common mathematical structure might link the two phenomena.[10] Each involves an evolving solid structure enveloped by a thin film of liquid through which is transported a diffusing field (CO_2 for stalactites, latent heat of fusion for icicles). While the thickness of the film on stalactites limits the growth rate by precipitation of calcium carbonate[11–13], the growth rate of icicles is typically limited by heat transport through the surrounding air; we shall see, however, that the distinction between rate-limiting factors is (surprisingly) irrelevant to the asymptotic shape.

Recent work[14,15] examining stalactite growth as a free boundary problem established a novel geometrical growth law based on the coupling of thin film fluid dynamics and calcium carbonate chemistry. Numerical studies showed an attractor in the space of shapes whose analytical form was determined and found to compare very favorably with that of natural stalactites. Is there an analogous ideal shape for icicles? It is tempting to view icicle growth as a classic Stefan problem, as explored extensively for solidification from the melt.[16] There, growth is controlled by a quasi-static diffusive field and the growth rate is determined by a gradient of that variable. However, such systems generally lack the thin flowing film that separates the developing solid from its surroundings, and thus they do not conceptually match icicle growth. Exceptions occur, for instance, in the presence of surface premelting.[17] Other models have been considered[3–5] which are in good agreement with respect to the dependence of growth rates upon such factors as temperature and flow rate, but are not formulated as true free-boundary problems. One context in which progress has been made is the formation of "ice stalactites," hollow tubular structures formed below sea ice as salt is rejected during solidification,[18,19] but these formations are quite distinct from typical icicles. Here we suggest an approach to the problem of icicle growth which

FIG. 1: Icicles and stalactites. (a) A collection of icicles.[21] (b) Stalactites in Kartchner Caverns, Benson, AZ.

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synthesizes geometrical principles, heat flow in the water and atmosphere, and thin-film fluid dynamics to arrive at the existence of an ideal growing shape for icicles which compares well with observations. Interestingly, the shape far from the tip has the same mathematical form as that recently derived[14,15] for the growth of stalactites.

We first consider the water layer flowing down the surface of a growing icicle The volumetric flow rate Q over icicles is typically[3,20] on the order of tens of milliliters per hour (~ 0.01 cm^3/s), and icicle radii are usually in the range of $1-10$ cm. To understand the essential features of the flow, consider a cylindrical icicle of radius r, over the surface of which flows an aqueous film of thickness h (Fig. 2). Since $h \ll r$ over nearly the entire icicle surface, the velocity profile in the layer may be determined as that flowing on a flat surface. Furthermore, we expect the Reynolds number to be low enough that the Stokes approximation is valid. If y is a coordinate normal to the surface and θ is the angle that the tangent vector $\hat{\mathbf{t}}$ makes with respect to the horizontal, then the Stokes equation for gravity-driven flow is $\nu_w d^2u/dy^2 = g\sin\theta$, where g is the gravitational acceleration and ν_w=0.01 cm^2/s is the kinematic viscosity of water. Enforcing no-slip and stress-free boundary conditions at the solid-liquid and liquid-air interfaces, the thickness is

$$h = \left(\frac{3Q\nu_w}{2\pi g r\sin\theta}\right)^{1/3} . \tag{1}$$

Using typical flow rates and radii, we deduce a layer thickness that is tens of microns and surface velocities $u_s \simeq (gh^2/2\nu_w)\sin\theta$ below several mm/s, consistent with known values,[3,18] yielding $Re = 0.01-0.1$, well in the laminar regime as anticipated. At distances from the icicle tip comparable to the capillary length (several millimeters), the complex physics of pendant drop detachment takes over and the thickness law (1) ceases to hold.

If the icicle is growing, the volumetric flux Q must vary along the arclength s of the icicle as water is converted to ice. With the icicle profile described by $r(z)$ (Fig. 2) and the growth velocity normal to the ice at any point being v_g, Q varies along the surface as

$$\frac{dQ}{ds} = 2\pi r v_g \ , \tag{2}$$

the positive sign on the right-hand-side reflecting the choice of origin at the tip, with s increasing upward.

We seek a uniformly translating shape,[10,11] for which every point on the icicle must grow at a rate such that $v_g = v_t\cos\theta$, where v_t is the growth velocity of the tip, usually millimeters per hour ($\sim 10^{-4}$ cm/s).[3,18] Given the complexities of droplet detachment,[3] the tip velocity here will be considered a parameter of the theory. Substituting into (2) and using $dr = ds\cos\theta$, one finds that an exact integration may be performed, yielding

$$Q = Q_t + \pi r^2 v_t \ , \tag{3}$$

where Q_t is the flow rate at the icicle's tip. This result neglects evaporation.

Turning to heat transport, note that the curvature of the icicle surface is sufficiently small everywhere that the Gibbs-Thompson correction[21] to the melting temperature $T_{\rm m}$ is negligible. Thus, the temperature of the water at the ice-water interface is well-approximated as $T_{\rm m}$ along the entire icicle. Furthermore, since most icicles possess an unfrozen liquid core,[3−5] heat does not travel radially outward from the center of the icicle, as it would if the core were solid and the temperature inside were decreasing over time. Hence, any flux of heat present at the ice-water interface consists solely of latent heat being removed as the water changes phase. The issue of advective heat transport by the flowing water is addressed by considering the Peclet number $Pe = u_s h/\alpha_w$, where $\alpha_w \simeq 10^{-3}$ cm^2/s is the thermal diffusivity of water. Using our previous estimates for the flow velocity u_s and thickness, we find $Pe \simeq 0.1-1$, indicating that energy transport down the icicle is generally subordinate to conduction of heat across the water layer. The heat flux across the water, then, is $F_w = \kappa_w(T_{\rm m} - T_{\rm i})/h$, where κ_w is the heat conductivity of water and $T_{\rm i}$, the temperature at the air-water interface, is found below.

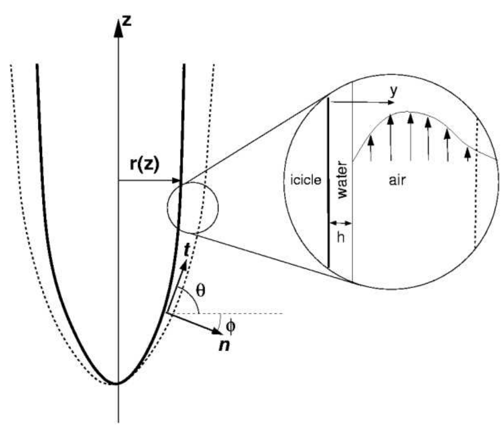

FIG. 2: Features of a hanging axisymmetric shape used in development of the theory. The flowing water layer, not to scale, is much thinner than the rising thermal boundary layer.

Once the heat has traversed the water layer, it must be transported through the air surrounding the icicle. As is well-known, objects warmer than their surroundings often create rising thermal boundary layers in the adjacent atmosphere due to the buoyancy of the heated surrounding air, a fact incorporated into an earlier study.[4] Similarity solutions for the coupled Navier Stokes and heat transport equations in the Boussinesq approximation can provide the basis for understanding this boundary layer. For instance, for a flat, vertical, isothermal plate, solutions show that the rising warm air is confined to a boundary layer whose thickness δ as a function of the vertical coordinate z is[22]

$$\delta = C\left(\frac{\nu_a^2 z}{g\beta\Delta T}\right)^{1/4} , \tag{4}$$

where C is a dimensionless constant that depends on the Prandt number of air (0.68) and is of order unity, $\nu_a \simeq 0.13$ cm^2/sec is the kinematic viscosity of air, $\beta \simeq 3.7 \times 10^{-3}$ K^{-1} is the volumetric coefficient of expansion for air, and ΔT is the temperature difference between the plate and the ambient temperature $T_{\rm a}$ far away.

We may use Eq. 4 to approximate the boundary layer thickness for our icicle for the following reasons. Firstly, using a temperature difference of ten degrees Kelvin, one finds a boundary layer thickness on the order of a centimeter, much greater than the thickness of the water layer on a typical icicle, but less than a typical icicle radius, so that flatness is approximated. Secondly, the peak velocity of the warm air in the layer is

$$u_p \simeq \frac{2}{3}\sqrt{g\Delta T\beta z} \ , \tag{5}$$

around $5-10$ cm/s, much greater than the downward water velocity, so the no slip condition used in the flat plate analysis is nearly attained. Thirdly, the atmospheric heat flux can be written as $F_a = \kappa_a(T_{\rm i} - T_{\rm a})/\delta$, where $\kappa_{\rm a}$ is the thermal conductivity of air, differing from the exact form only by the multiplication of an order one constant. If we equate this heat flux with that through the water layer, one finds that $T_{\rm i}$ is given by

$$T_{\rm i} = T_{\rm m} - (T_{\rm m} - T_{\rm a})\frac{h\kappa_a/\delta\kappa_w}{1 + h\kappa_a/\delta\kappa_w} \ . \tag{6}$$

On account of the vast difference in scale between h and δ mentioned above, the ratio $h\kappa_a/\delta\kappa_w \simeq 0.01$, so $T_{\rm i}$ is lower than $T_{\rm m}$ by only $10^{-3} - 10^{-2}K$. Hence, from the view of atmospheric heat transport, the icicle is essentially isothermal at $T_{\rm m}$. Finally, we account for the nonverticality of the icicle's surface by simply replacing g with $g\cos\phi$ in Eq. 4, a reasonable action given the slow variation of ϕ away from the icicle's tip.

At this point, then, we are in a position to derive a formula for the growth velocity v_g of the icicle's surface. We divide the heat flux as calculated through the atmospheric boundary layer by the latent heat of fusion per volume L of water (0.334 J/cm^3) to obtain the velocity

$$v_g = \frac{\kappa\Delta T}{LC}\left(\frac{g\cos\phi\beta\Delta T}{\nu_a^2 z}\right)^{1/4} = v_t\left(\frac{\ell\cos\phi}{z}\right)^{1/4} \ , \tag{7}$$

where v_t is, again, the velocity of the tip and ℓ is a characteristic length of about 10^{-1} cm. To find the equation governing the icicle profile, we scale the variables r and z both by ℓ, defining new variables ρ and ζ and enforce the condition for uniformly translating shapes $v_g = v_t \sin\phi$. After rewriting trigonometric functions in terms of the slope of the profile ρ', one finds the equation

$$(\rho')^2\,\zeta^{1/2} = \left[1 + (\rho')^2\right]^{3/4} \ . \tag{8}$$

Expanding for small ρ', as is valid away from the tip, gives an equation which can be exactly integrated to yield

$$\rho = \frac{4}{3}\left(\zeta^{1/2} + \frac{3}{2}\right)\sqrt{\zeta^{1/2} - \frac{3}{4}} \ . \tag{9}$$

Note that this shape far from the tip goes as $\rho \sim \zeta^{3/4}$ and therefore the thickness of the thermal boundary layer relative to the icicle radius scales as $\delta/r \sim \zeta^{-1/2}$ and the two-dimensional boundary layer calculation becomes ever more satisfactory further up the icicle, albeit slowly.

Before discussing the resultant shape in greater detail, we use a slightly different method to arrive at the same solution, this time calculating the growth velocity using the heat flux through the thin water layer. We first make the approximation that the temperature difference, $\Delta T_{\rm w}$, between $T_{\rm m}$ and $T_{\rm i}$ is constant. Of course, this is not valid near the tip of the icicle; however, it can be seen through a synthesis of (1), (3), (6), and (9) that for large ζ,

$$\Delta T_{\rm w} \simeq \left(\frac{2\nu_w\ell v_t^4}{g}\right)^{1/3}\frac{L}{\kappa_w} \ . \tag{10}$$

So, we may perform a similar calculation to the one above by using the relation for F_w and Eqs. 1 and 3 to give

$$\left(\frac{d\zeta}{d\rho}\right)^3 + \frac{d\zeta}{d\rho} - \frac{3}{4}\frac{\gamma + \rho^2}{\rho} = 0 \ , \tag{11}$$

where $\gamma = Q_t/(\pi v_t\ell^2)$. This differential equation supports power-law solutions in the asymptotic limit of large ρ and ζ', for the balance of terms is then $[\zeta'(\rho)]^3 \simeq \rho$, and the shape is given approximately by

$$\zeta \simeq \left(\frac{3}{4}\rho\right)^{4/3} \ . \tag{12}$$

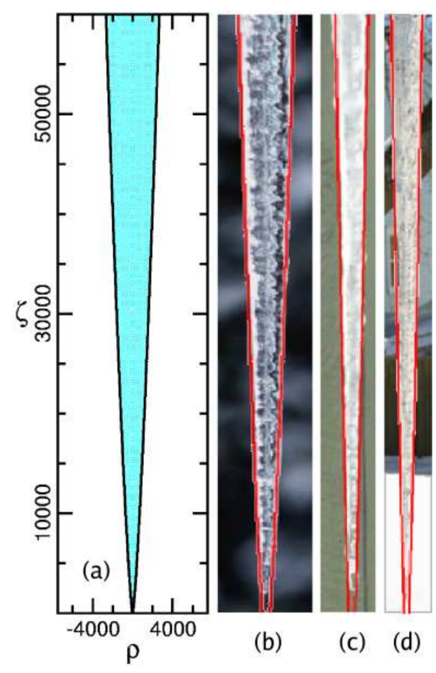

FIG. 3: Ideal shape of an icicle and comparison with natural icicles. (a) Ideal shape in dimensionless units of radius and height. (b)-(d) A selection of natural icicles,[23] each with the appropriately-scaled ideal form overlaid.

As must be the case, this asymptotic form is the same seen in Eq. 9 above, completing the logical circle. As promised, this asymptotic power law is identical to that found in the case of stalactites[14,15], finally explaining their strikingly similar appearances. Furthermore, if we evaluate this asymptotic form at some point on the surface (ρ^*, ζ^*) where the aspect ratio (length/width) is $A = \zeta^*/\rho^*$, then the shape can be rewritten as $\zeta/\zeta^* \simeq (\rho/\rho^*)^{4/3}$, a universal, self-similar form.

This ideal shape compares well with typical shapes of icicles. As an example, Figures 3b-d show overlays of the theoretical shape with images of natural icicles.[23] All that is necessary to make the comparison is to adjust the magnification and compare aspect ratios as discussed earlier. In fitting these, we have ignored the region just near the tip, where the droplet detachment obviously rounds out the shape. The excellent agreement obtained, in the absence of any material fitting parameters, suggests strongly the validity of this analysis. Systematic, controlled experiments on the growth of icicles are needed to check in detail various aspects of the theory, such as the significance of depletion and the assumption that a traveling shape is indeed an attractor of the dynamics.

Clearly, the scenario presented here, by which a free boundary dynamics for icicle growth is derived, contains a number of simplifications and approximations whose quantitative accuracy merits further study. Chief among these is the use of a boundary layer theory which assumes a flat and vertical surface. Both of these asumptions are justifiable only far away from the icicle's tip. A full numerical study would likely prove most illuminating. We expect the analysis presented here to serve as a basis for further understanding of ice structures, including axisymmetric perturbations such as the ripples so commonly found on icicles, as well as strongly non-axisymmetric forms such as the sheets which are analogous to "draperies" in limestone caves. In this regard, recent work on solidification on surfaces of arbitrary curvature[24] may prove quite relevant.

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