

Point Processes and the Statistics of Cellular Neighborhoods in Simple Multicellular Organisms

Anand Srinivasan,^{*} Steph S.M.H. Höhn,[†] and Raymond E. Goldstein[‡]

Department of Applied Mathematics and Theoretical Physics, Centre for Mathematical Sciences,
University of Cambridge, Wilberforce Road, Cambridge CB3 0WA, United Kingdom

(Dated: November 20, 2023)

Recent work on several distinct multicellular organisms has revealed a hitherto unknown type of biological noise; rather than a highly regular arrangement, cellular neighborhood volumes, obtained by Voronoi tessellations of the cell locations, are broadly distributed and consistent with gamma distributions. We propose an explanation for those observations in the case of the alga *Volvox*, whose somatic cells are embedded in an extracellular matrix (ECM) they export. We show that the simplest models for stochastic ECM generation in one and two dimensions are point processes whose Voronoi tessellations are demonstrably governed by gamma distributions. These results highlight a universal consequence of intrinsic biological noise on the architecture of certain tissues.

Some of the simplest multicellular organisms have an architecture consisting of tens, hundreds, or thousands of cells arranged within an extracellular matrix (ECM), a network of proteins and biopolymers secreted by the cells. They often adopt regular geometric forms: the linear chains and rosettes of choanoflagellates [1, 2], the sheets or spheres of green algae [3], and the cylinders of sponges [4]. While the arrangement of cells within the ECM appears regular, recent work [5] has revealed a hitherto undocumented form of disorder found by assigning neighborhoods to the cells through a Voronoi tessellation based on the cell centers. Strikingly, both the lab-evolved organism “snowflake yeast” [6] (a ramified multicellular form found by repeated rounds of selection for sedimentation speed) and the green alga *Volvox carteri* display broad distributions of Voronoi volumes accurately fit by gamma distributions. These observations are central to an understanding of a general question in developmental biology: *How do cells produce structures external to themselves in an accurate and robust manner?*

Volvox (Fig. 1(a)) is one of the simplest systems to understand the statistical fluctuations in the generation of an ECM. The adult consists of $\sim 10^3$ somatic cells embedded at the surface of a transparent ECM, a thin shell $\sim 500 \mu\text{m}$ in diameter and $\sim 30 \mu\text{m}$ thick; the organism is $\sim 90\%$ ECM. Daughter colonies develop from germ cells at the inner ECM surface; repeated rounds of binary division produce a raft of cells held together by cytoplasmic bridges that remain after incomplete cytokinesis. Following “embryonic inversion” that turns the raft inside-out [7], daughter colonies enlarge (at fixed somatic cell number) by generation of the ECM via export of proteins from those cells, expanding the colony radius to the final size over the course of a day, during which the widely-distributed neighborhood volumes appear.

For *Volvox*, the general issue above becomes the question of how ECM is generated such that the spheroidal

form is maintained during the dramatic enlargement of the colony and yet the cellular neighborhood volumes are broadly distributed. A biological answer might invoke cell-cell signaling in response to mechanical forces as a mechanism to coordinate growth and would ascribe the distribution of neighborhood volumes to imperfections in that process. Surprisingly, the novel problem of cellular neighborhood distributions is so little-studied that we do not even understand quantitatively the feedback-free case, which serves as a benchmark for any analysis of the role of correlations. We note that in granular physics, gamma distributions have been argued to arise from maximizing entropy when space is divided up into a set of subvolumes constrained to add up to a fixed total volume [8]. But this then begs the question of why biological systems should follow a maximum-entropy principle.

Here, we examine a range of idealized models for distributions of cells within a thin growing ECM. Regardless of the process that generates ECM, the assumption that cell counts within any two disjoint regions are independent random variables—along with three technical and physically reasonable assumptions about the vanishing probability of realizing particular configurations [9]—is the defining feature of a *Poisson* point process [11]. In static and dynamic one-dimensional models, where cells sit in a circular ECM, we show analytically that Voronoi segments obey gamma distributions. In two dimensions we describe strong numerical evidence for consistency with gamma-type distributions and conjecture an analytical form. The following is a non-technical summary, with details provided in the Supplementary Material [9].

Our analysis is motivated by light-sheet measurements of somatic cell positions in adult *V. carteri* [5]. Figure 1(a) shows a darkfield image of *V. carteri* and a section of the Voronoi tessellation around the cells. The area distribution of Voronoi partitions across 6 organisms is shown in Fig. 1(b) along with a maximum-likelihood fit to the affine-transformed k -gamma distribution,

$$p(v) = \frac{1}{\bar{v} - v_c} \frac{k^k x^{k-1}}{\Gamma(k)} \exp(-kx), \quad x = \frac{v - v_c}{\bar{v} - v_c}, \quad (1)$$

where $\bar{v}$ is the mean and v_c is the somatic cell area, the

^{*} as3273@cam.ac.uk

[†] sh753@cam.ac.uk

[‡] R.E.Goldstein@damtp.cam.ac.uk

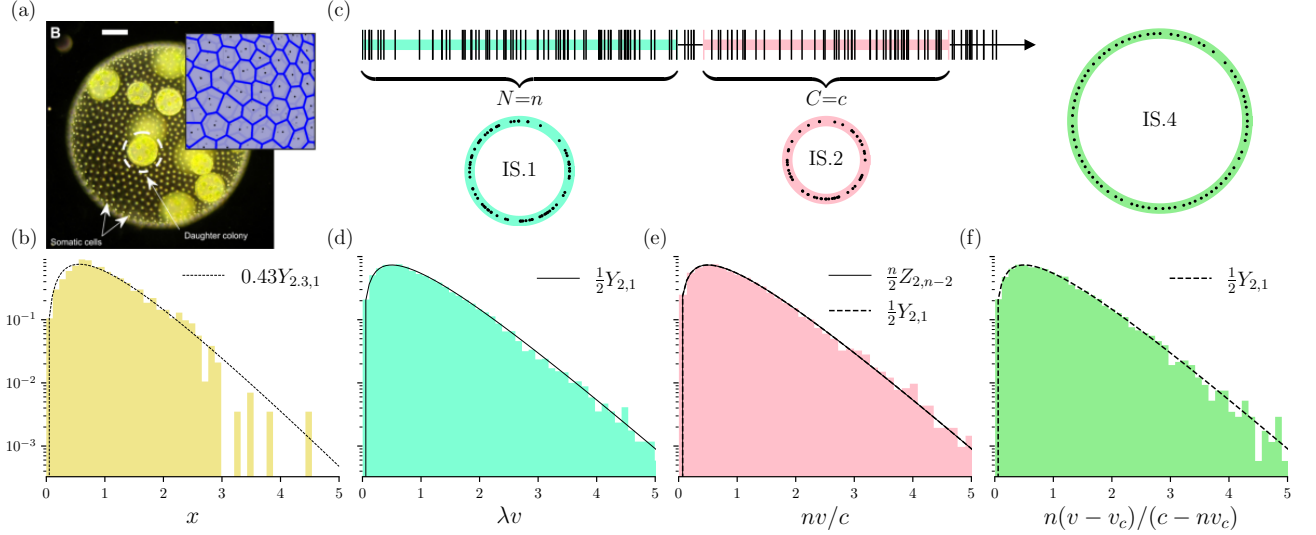


FIG. 1. The green alga *Volvox* and one-dimensional models for cell positions. (a) Adult *V. carteri* with cell types labelled. Inset shows section of Voronoi tessellation around somatic cells (visible as dots centered in each Voronoi polygon). (b) Voronoi cell areas v approximately follow the translated gamma distribution (1) [5], computed by a maximum-likelihood fit. (c) The Poisson process on $[0, \infty)$, where cell locations are indicated by vertical line segments and intercalating ECM is colored. Conditioning on the cell count or length leads to two different realizations of circular processes, IS.1 and IS.2, respectively, whose numerically obtained empirical distribution are shown in (d) and (e). Dashed and solid lines indicate approximate or exact density functions, respectively. (f) Empirical distribution for model IS.4, with no cell overlaps, compared to analytical result.

lower bound on v due to the non-overlapping of cells. We wish to explain the origin of the gamma distribution itself and its validity in the form translated by v_c .

Notation. In the following, random variables are denoted by capital letters $X_{i;\alpha,\beta,\dots}$ accompanied by indices i and parameters $\alpha, \beta, \dots$. Variables $i \neq j$ are independent unless otherwise noted. The letters $X_{i;\lambda}$, $Y_{i;k,\lambda}$, $Z_{i;\alpha,\beta}$, and $N_{i;\lambda}$ are reserved for exponential, gamma, beta, and Poisson random variables respectively. $X \stackrel{\text{pdf}}{\sim} f_X(x)$ indicates that X has the probability density function $f_X(x)$, with $X \stackrel{\text{cdf}}{\sim} F_X(x)$ indicating the same for cumulative density. $X_i \xrightarrow{d} Y$ indicates X_i converges to Y in distribution. Models are named by the ECM dimension d (I or II) and the property of being static (S) or dynamic (D), as summarized in Table I.

TABLE I. Summary of models

Model	Description
IS.0	fixed N on the half-line
IS.1	fixed N, variable C on the circle
IS.2	fixed N, fixed C on the circle
IS.3	variable N, fixed C on the circle
IS.4	fixed N and C with minimum spacing on the circle
ID.1	Brownian motion on the circle
ID.2	noncolliding Brownian motion on the unit circle
ID.3	noncolliding Brownian motion on a growing circle
IIS.1	variable N on the periodic unit square
IIS.2	variable N with minimum spacing on the unit sphere
IID.1	fixed N, Brownian motion on the unit sphere

One-dimensional equilibrium models. We first consider static models in which cells sit within a one-dimensional ECM, and view the arclength of ECM generated by each cell as a random variable. As in the Gibbs, microcanonical, and grand canonical ensembles, we consider three types of random configurations of cells in the ECM: (i) fixed cell counts $N = n$, with random intercell spacings and circumferences; (ii) fixed circumferences $C = c$ and counts $N = n$, with random positions $\{R_i\}_{i=1}^N$; and (iii) fixed circumferences $C = c$, with random cell counts N and positions. Like the convergence of the ensembles in the thermodynamic limit, we show that the same gamma distribution arises in the large- n, c limits of these cases.

Model IS.0 (on the half-line) —Consider a semi-infinite *Volvox* modelled as a homogeneous Poisson process of rate λ on the half-line $[0, \infty)$, as in Fig. 1(c). The sorted cell positions $R_1, R_2, R_3, \dots$ have exponentially distributed spacings $X_{i;\lambda}$. The Voronoi lengths $V_i = (X_{i;\lambda} + X_{i+1;\lambda})/2$ are proportional to the sum of exponentially-distributed random variables with the same rate, which, by a convolution [9], are gamma-distributed with $k = 2$,

$$V_i = \frac{1}{2}Y_{2;\lambda} \stackrel{\text{pdf}}{\sim} 4\lambda^2 v e^{-2\lambda v}. \quad (2)$$

This result already shows explicitly the deep link between the Voronoi construction and gamma distributions.

Model IS.1 (fixed N, variable C on the circle) —Now consider a circular *Volvox*. Select a fixed number $N_{;\lambda} = n + 1$ of successive points $\{R_i\}_{i=k}^{k+n+1}$ from the half-line process in IS.0 and form a circle by identifying the first

and last points, as in Fig. 1(c). This yields a variable circumference C which is n -gamma distributed, i.e. $C = Y_{n,\lambda}$, and Voronoi lengths V_i which by the same convolution argument follow the identical $k = 2$ gamma distribution as in IS.0. Fig. 1(d) shows a numerical demonstration of this using 10^4 realizations.

Model IS.2 (fixed N , fixed C)—Let us now constrain the circumference $C = c$ in IS.1 as follows. Take the points $\{R_i\}_{i=1}^{n+1}$, assume wlog $R_1 = 0$, and form the normalized sequence $cR_1/R_{n+1}, cR_2/R_{n+1}, \dots, c$. These are the order statistics of $n - 1$ uniform random variables on the interval $(0, c)$ (see §4.1 of [12]), as seen in Fig. 1(c). Identifying the first and last points as in IS.1, the arclengths $\{\ell_i\}_{i=1}^n$ are the ratio of i.i.d. exponential random variables to their sum,

$$\ell_i = c \frac{X_{i;\lambda}}{\sum_{j=1}^n X_{j;\lambda}}. \quad (3)$$

Thus, the corresponding Voronoi lengths $\{V_i\}$ are

$$V_i = \frac{c}{2} \frac{Y_{1;2,\lambda}}{Y_{1;2,\lambda} + Y_{2;n-2,\lambda}} = \frac{c}{2} Z_{;2,n-2} \quad (4)$$

$$\stackrel{\text{pdf}}{\sim} \frac{4v}{c(n-1)(n-2)} \left(1 - \frac{2v}{c}\right)^{(n-3)}, \quad (5)$$

where $v \in [0, \frac{c}{2}]$, which is computed using the joint distribution $(Y_1 + Y_2, Y_1/(Y_1 + Y_2))$ [9]. As can be seen, V_i is $\alpha = 2$ beta-distributed, taking values in $[0, \frac{c}{2}]$, which is the second order statistic of $n - 1$ uniform random variables. Taking the limits $n, c \rightarrow \infty$ with density $\rho = m/c = (n - 2)/c$ fixed—a kind of thermodynamic limit—we obtain, by taking the pointwise limit of cumulative density functions [9],

$$V_i = \frac{m}{2\rho} Z_{2;m} \xrightarrow{n \rightarrow \infty} \frac{1}{2} Y_{;2,\rho}, \quad (6)$$

converging in distribution to the $k = 2$ gamma distribution (with cell density/rate ρ) as in IS.0 & IS.1.

Model IS.3 (variable N , fixed C)—In contrast to IS.1, let us now constrain the circumference $C = c$, leaving the count $N_{;c\lambda}$ a Poisson-distributed random variable. Using IS.2, the distribution of V_i can be computed by marginalizing the joint distribution $(Z_{;2,n-1}, N_{;\lambda})$ over n —the *compound beta-Poisson distribution*—which is a $k = 2$ gamma distribution (see SM [9]). A simpler approach is to place an observation window $[x, x + c]$ at random on the half-line process and note that the inter-point spacings ℓ_i follow a truncated exponential distribution:

$$\ell_i \stackrel{\text{pdf}}{\sim} \frac{\lambda e^{-\lambda x}}{1 - e^{-\lambda c}}, \quad x \in [0, c] \quad (7)$$

Taking $c \rightarrow \infty$, leaving the rate λ fixed (analogous again to the thermodynamic limit), we have $\ell_i \xrightarrow{d} X_{i;\lambda}$, with V_i converging to a $k = 2$ -gamma distribution as in IS.0.

Thus far, we have considered models in which cells in the *Volvox* ECM, viewed as the final configuration of

some stochastic growth process, occur as points in space that are independently scattered. Gamma distributions govern their Voronoi tessellations in various limits, independent of the constraints applied to the scattering.

Model IS.4 (fixed N , fixed C , no overlaps)—Starting with fixed $C = c$ and $N_{;c\lambda} = n$ per IS.2, now suppose that points have a finite “diameter” v_c . In $d = 1$, the shape of a segment ℓ_i is invariant of the allocation v_c ; hence, we may start with v_c -length segments and add the random spacings of a subinterval of length $(c - nv_c)$. Using IS.2, V_i therefore is a v_c -shifted beta random variable, converging in the limits $n, c \rightarrow \infty$ with fixed density $\rho = (n - 2)/(c - nv_c)$ to a shifted gamma random variable,

$$V_i = \frac{n - 2}{2\rho} Z_{;2,n-2} + v_c \xrightarrow{n \rightarrow \infty} \frac{1}{2} Y_{;2,\rho} + v_c. \quad (8)$$

Therefore, (8) has the distributional form

$$f_{Y_{(k,\lambda)} + v_c}(v) = \frac{\lambda^k}{\Gamma(k)} (v - v_c)^{k-1} e^{-\lambda(v - v_c)}, \quad (9)$$

which is precisely the empirical law (1) found for *Volvox* with $\lambda = k/(\bar{v} - v_c)$. These *hard-core point processes* (including the Matérn and Gibbs processes, 5.4-5.5 [14]) relax the independence property characterizing Poisson processes by allowing energetic interactions—and therefore correlations—between points.

One-dimensional dynamic models. In IS.1-IS.4, fluctuations in Voronoi segments arise from random cell positions representing static configurations, with different combinations of fixed and free random variables. Focusing now on a single organism with fixed $N_{;c(0)\lambda} = n$ cells on a time-varying circumference $c(t)$, we now consider fluctuations in Voronoi segments arising from random motions of cells during ECM generation, represented by identically distributed stochastic processes $(\{\ell_t^{(i)}\}_{i=1}^n)_{t \geq 0}$. Physically, the random motion of cell locations arises from stochastic ECM production. Initially, let us consider a simplified view in which the ECM is treated like a fluid, and cell motions obey diffusive dynamics without constraints on inter-cell distances. Gamma distributions are shown to arise once again in the large- t and n limits.

Model ID.1 (Brownian motion on the circle)—Let $\{R_t^{(i)}\}_{i=1}^n$ be n Brownian motions (BM) on a circle of radius r given by

$$R_t^{(j)} = r \exp(i\theta_t^{(j)}), \quad (10)$$

with $\theta_t^{(j)}$ standard BMs on $\mathbb{R}$ and initial conditions for the probability density $p^{(j)}(0) = \delta_{R_0^{(j)}}$. The time-dependent probability density $p^{(i)}(t)$ tends exponentially in L^2 to the uniform distribution, hence $R_t^{(i)}$ converge in distribution to i.i.d. uniform r.v.s. Therefore, in large t the configuration approaches Model IS.2, and V_i are beta-distributed (4) in large t and gamma-distributed in large t, n (6)—see Fig. 2(a1,a2). Yet, the sample paths of (10) almost surely intersect, unphysically reordering the cells.

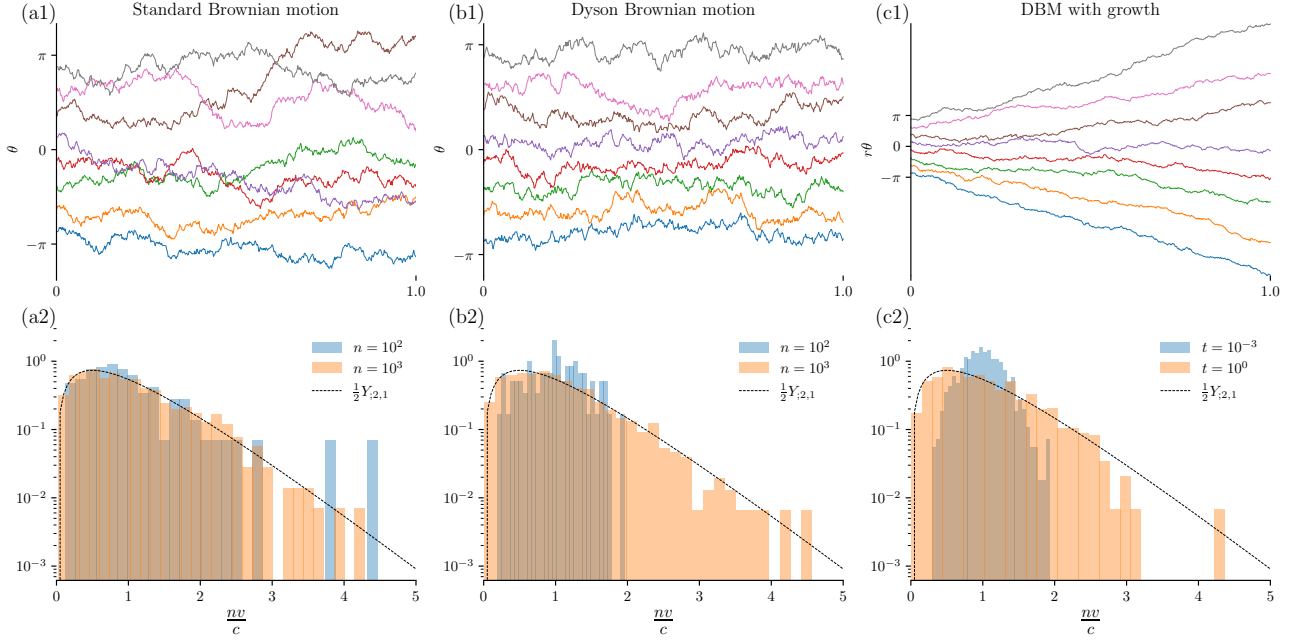


FIG. 2. Dynamic models of cellular positions during growth converge to gamma distributions in various limits. (a1,a2) Standard Brownian motion in angular coordinates converges to beta-distributed segments in large t , and gamma-distributed segments in large n . (b1,b2) Dyson Brownian motion in angular coordinates converges to gamma-distributed segments at large t and n . Non-conformance to gamma distributions is observed at low n due to pair-repulsion. (c1,c2) Dyson Brownian motion on a growing domain satisfying particular growth constraints converges in large- t and n to gamma-distributed segments.

Model ID.2 (noncolliding Brownian motion)—Let $\{R_t^{(i)}\}_{i=1}^n$ be samples of the conditional distributions of n circular Brownian motions whose angles are in ascending order modulo 2π , thereby lying in the set

$$D_n = \{x \in \mathbb{R}^n \mid x_1 < \dots < x_n < x_1 + 2\pi\}, \quad (11)$$

a construction known as *noncolliding Brownian motion* or Brownian motion within the *Weyl chamber* D_n [15]. In [16] eq. 4.1, it is shown that the conditional fluctuations are Gaussian plus a singular r^{-1} pair-repulsion,

$$d\theta_t^{(i)} = \sigma dB_t^{(i)} + \frac{\sigma^2}{2} \sum_{j \neq i} \cot\left(\frac{\theta_t^{(i)} - \theta_t^{(j)}}{2}\right) dt. \quad (12)$$

A physical interpretation of (12) is that of a gas confined to the unit circle with the pair-potential [9]

$$W = - \sum_{j < k} \log |\exp(i\theta_k) - \exp(i\theta_j)|, \quad (13)$$

constituting a simple model of non-colliding cell motion. Eq. 12 is precisely the eigenvalue dynamics $\lambda_t^{(j)} = \exp(i\theta_t^{(j)})$ of a Brownian motion U_t on the unitary group $\mathbb{U}(n)$, known as *Dyson Brownian motion* [9, 17].

Being confined to D_n (11), the positions $R_t^{(i)}$ do not converge to uniform r.v.s on the circle as in I.5—compare Figs. 2(a1) and 2(b1). The stationary distribution of

(12) is the circular unitary ensemble (CUE) [17],

$$\rho_*(\theta_1, \dots, \theta_n) = \frac{1}{Z_n} \prod_{j < k} |\exp(i\theta_j) - \exp(i\theta_k)|^2. \quad (14)$$

We use this result to derive the Voronoi length distribution in the large- t, n limits. Let $\{\theta^{(i)}\}_{i=1}^n$ be a sample of the stationary distribution (14), the spectrum of a uniform sample of $\mathbb{U}(n)$; the empirical distribution $\mu_\theta(n) = n^{-1} \sum_{i=1}^n \delta_{\theta^{(i)}}$ converges almost surely in large n (Theorem 3, [18]) to the uniform distribution on the unit circle. Thus, the spacings $\theta_t^{(i+1)} - \theta_t^{(i)}$ converge in large- t and n (in either order [9]) to the spacings of the order statistics of n uniform random variables as in IS.2. Taking $n \rightarrow \infty$ with density $\rho = (n-2)/c$ fixed, V_i converge to $k=2$ gamma random variables as in (6).

Convergence to gamma laws depending on particle count n is shown for BM and DBM in Figs. 2(a2,b2) [9]; in contrast to BM, DBM spacings lose the long tail at low n due to repulsion (14).

Model ID.3 (noncolliding Brownian motion, growth)—To account for growth of the ECM, let the radius $r(t)$ in ID.2 be time-dependent, scaling the configuration as $R_t^{(i)} = r(t) \exp(i\theta_t^{(i)})$ with $\theta_t^{(i)}$ given by (12) as shown in Fig. 2(c1). Applying Itô's lemma, $R_t^{(i)}$ satisfies the stochastic differential equation

$$dR_t^{(i)} = \dot{r} \exp(i\theta_t^{(i)}) dt + iR_t^{(i)} d\theta_t^{(i)} - \frac{r}{2} \theta_t^{(i)} dt. \quad (15)$$

Substituting $d\theta_t^{(i)}$ from (12), the diffusion constant for (15) is $D = r^2\sigma^2/2$. Requiring that the lateral diffusion is time invariant implies $\dot{D} = 0$, and thus that the standard deviation of the radius-normalized dynamics (12) should decay as $\sigma(t) \propto r(t)^{-1}$. Exponential convergence of (12) to the stationary solution (14) is ensured by a growth condition on $r(t)$. By standard Fourier arguments (e.g. §2.2, [19], and [9]), solutions to the time-dependent diffusion equation for the probability density of $p(t)$ of the unitary Brownian motion U_t satisfy, for some constant $C_n > 0$ depending only on n ,

$$d(p(t), \nu) \leq d(p(0), \nu) \exp\left(-C_n \int_0^t \sigma(s)^2 ds\right), \quad (16)$$

with d the L^2 metric and ν the uniform (Haar) measure on the unitary group $U(n)$. Therefore, if $r(t)$ satisfies:

$$\lim_{t \rightarrow \infty} \int_0^t \frac{1}{r(s)^2} ds = \infty, \quad r(0) > 0 \quad (17)$$

the RHS in (16) vanishes in large t , and global exponential stability is assured. Then, as in ID.2, the positions $R_t^{(i)}$ approach a random uniform spacing of a circle of radius $r(t)$ in large n , resulting in gamma-distributed Voronoi segments V_i . Rapid convergence in t to a gamma law (with $D = 0.1, n = 1000, \dot{r} = 1$) is shown in Fig. 2(c2). Formally, (17) is not satisfied by such a linear growth law, yet we observe empirical convergence to a gamma law on the order of a unit of (scaled) time.

Condition (17) can be understood by a Péclet number relating drift and diffusion timescales in growth. Considering the radial drift velocity v_r in (15), let

$$\text{Pe} = \frac{\tau_d}{\tau_{v_r}} = \frac{2L\dot{r}}{r^2\sigma^2} \propto L\dot{r} \quad (18)$$

with L a test length section. In the limit $\text{Pe} \rightarrow 0$, condition (17) is satisfied; when $\text{Pe} \rightarrow \infty$, the exponential multiplier in the bound (16) approaches 1, “freezing” the angles $\theta_t^{(i)}$ to initial conditions. Finally, conditions $\dot{D} = 0$, $\text{Pe} = 0$, and (17) cannot simultaneously be satisfied.

Let us now consider the ECM as a solid, with cell motions arising from stochastic growth of the inter-cell segments. As a simple starting point, suppose that the time-varying intercell lengths $\ell_t^{(i)}$ are stochastic processes with i.i.d. nonnegative increments G_t for $t = 0, 1, \dots$, representing a strictly additive growth process in discrete time. Naturally if G_t have finite second moment, by the central limit theorem,

$$\frac{\ell_t^{(i)}}{\sqrt{t}} - \sqrt{t}\mathbb{E}[G_t] \xrightarrow{t \rightarrow \infty} W_{0, \text{Var}(G_t)}, \quad (19)$$

where W is a Gaussian random variable. For example, exponentially distributed fluctuations $G_t = X_{i,\lambda}$, as in adding the spacings of T successive Poisson processes,

results in $(k = T)$ -gamma-distributed segments ℓ_i , converging in large T to a translated Gaussian random variable. This convergence to a Gaussian distribution restates the point made earlier [5] regarding the surprising appearance of gamma distributions and, in light of the results from static models above, indicates that not every random growth process produces snapshots of the configurations that conform to gamma distributions.

Model ID.4 (maximum-entropy growth rates)—Let the segments grow linearly in time as $\dot{\ell}_i = G_{i;\mu}$, with i.i.d. growth rates G_i which are positive continuous random variables with some common mean growth rate μ . If the distribution of G_i maximizes entropy subject to the mean and nonnegativity constraint—perhaps more interpretable as uncertainty about cellular behavior than global maximum-entropy assumptions [8]—then $G_{i;\mu} = X_{i;1/\mu}$ is an exponential random variable of rate $\lambda = 1/\mu$ (see [9]). Then, for any time t , the normalized configuration

$$\frac{\ell_i(t)}{\sum_{j=1}^n \ell_j(t)} = \frac{\ell_i(0) + tG_i}{\sum_{j=1}^n \ell_j(0) + tG_j} \xrightarrow{d} \frac{X_{i;1/\mu}}{\sum_{j=1}^n X_{j;1/\mu}} \quad (20)$$

converges in large t to a uniform spacing as in IS.2, (3) and therefore has beta-distributed Voronoi lengths (4) converging to gamma-distributed lengths in large n (6).

Two-dimensional equilibrium and dynamic models. In dimension $d \geq 2$, known analytical results concerning Voronoi tessellations of point processes are limited to lower-dimensional facets, such as edge (2D) and face (3D) distributions (see [23] and §4.4, [24]). The difficulty in higher dimensions arises primarily from the loss of uniqueness for shapes satisfying geometric properties—such as fixed measure (length, area, volume)—for which one must first consider distributions over shapes, then marginalize over the level-sets satisfying a scalar geometric property, such as aspect ratio. For this reason, in the following models we present primarily numerical results (with partial analytical arguments where applicable), and present a validation of the numerical method in the Supplementary Material [9].

Model IIS.1 (periodic unit square)—Consider a homogeneous Poisson point process on the unit square $[0, 1]^2$ with periodic boundary conditions, denoted $\mathbb{T}^2$. On general d -dimensional domains, the point process is specified by an *intensity measure* $\lambda(A)$ for subsets $A \subseteq \mathbb{T}^2$ in which the count $N_{A;\lambda(A)}$ is a Poisson random variable of rate $\lambda(A)$ (see [9] for sampling methods). A *homogeneous* Poisson process—one whose intensity λ is constant on sets A of constant measure—is realizable by sampling the total count $N_{\cdot;\lambda(\mathbb{T}^2)}$ and assigning the positions $\{R_i\}_{i=1}^n$ conditional on $N = n$ as i.i.d. uniform random variables. Figure 3 shows numerical simulations for $k = 1000$ trials with intensity $\lambda(\mathbb{T}^2) = 10^3$, which is on the order of the number of somatic cells in *Volvox carter*.

The periodic Voronoi tessellation on $\mathbb{T}^2$, shown in the small- n example in Figure 3(a1)-(d1), is constructed by copying $\{R_i\}$ in four quadrants around $\mathbb{T}^2$ and selecting the sub-tessellation corresponding to the original points.

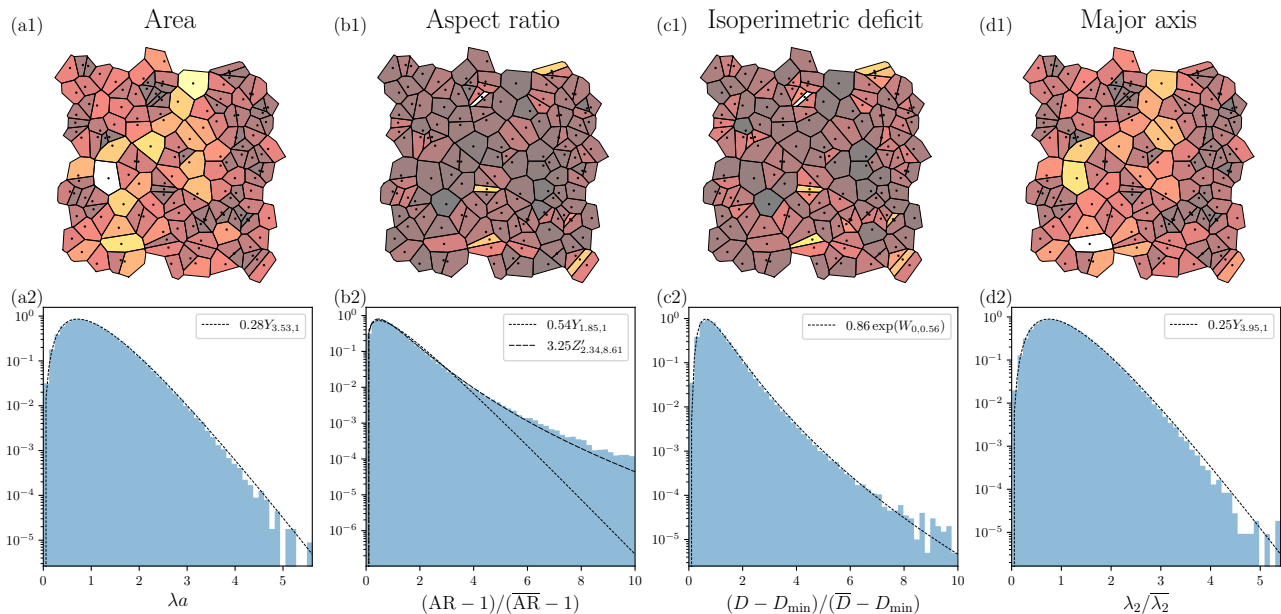


FIG. 3. Poisson-Voronoi tessellation on the periodic unit square. Panels (a1) - (d1) show the Voronoi cells colored by four geometric quantities defined for closed plane curves (see [9]). Panels (a2) - (d2) show empirical and maximum-likelihood estimations of gamma (denoted Y), lognormal (denoted $\exp(W)$), and beta prime (denoted Z') distributions where applicable. Poisson-Voronoi tessellations do not have gamma-distributed aspect ratios, instead following an approximate beta-prime law.

Areas a , nondimensionalized as λa , conform to a gamma random variable with $k \approx 3.5$, as in Fig. 3(a2). The isoperimetric deficit $D = L/\sqrt{4\pi a} - 1$ (see [9] and [37]) with L the perimeter, conforms after an appropriate rescaling to a log-normal random variable with $\sigma \approx 0.6$, shown in Fig. 3(c2). The aspect ratio AR of a Poisson-Voronoi tessellation does not conform to a gamma distribution (Fig. 3(b2)), in contrast to prior studies [20] finding gamma-distributed aspect ratio in a diverse range of densely-packed cell types and inert matter. Instead, AR conforms approximately to a beta-prime distribution, which naturally arises as the ratio of independent gamma random variables [9]. Figure 3(d2) shows that the major axis length is gamma-distributed; while we do not in general expect the major and minor axis to be independent random variables, we show in the SM [9] that both axes are gamma-distributed, with estimated parameters within the same order of magnitude as the estimated beta-prime distribution for aspect ratio.

Model IIS.2 (sphere with hard-core repulsion)—Consider a homogeneous Poisson process of rate λ , *thinned* (see [9]) to some rate $\lambda' < \lambda$ such that all points $\{R_i\}_{i=1}^n$ have a minimum pairwise distance $d(R_i, R_j) \geq r_d$ where $d(\cdot, \cdot)$ is the geodesic distance on the sphere. This generalizes IS.4 to the two-dimensional case, in which the geometric realization of such a *hard-core repulsion* is not unique, a fact intimately linked to the existence of an ordering of points in 1 dimension. Realizing a minimum spacing requires *dependent thinning*, as the inclusion of a particular point R_i depends upon the locations of nearest-neighbors R_j ,

and an *independently thinned* Poisson process is again a Poisson process ([9]). A commonly used procedure is the Matérn Type-II thinning rule (see [9] and 5.4, [14]):

$$\{R'_i\} = \{R_i \mid M_i < M_j \forall j \neq i, d(R_i, R_j) \leq r_d\} \quad (21)$$

where M_i are i.i.d. uniform random variables called *marks*. Heuristically, (21) imposes an upper bound on the aspect ratio, as the “minor axis” has a lower-bounded arclength r_d . We expect this to result in vanishing tails in both aspect ratio and isoperimetric deficit, confirmed by numerical simulation with $k = 100$ trials of $\lambda = 2000$, with r_d twice the apothem of a regular spherical hexagon (see [9]) of area $4\pi/\lambda$. The convergence to approximate gamma laws in area, aspect ratio, and anisotropy-adjusted isoperimetric deficit is illustrated in Figure 4. Notably, the isoperimetric deficit is no longer well-approximated by a log-normal law as in the Poisson case (Figure 3).

Model IID.1 (Brownian motion on the sphere)—Analogous to ID.1, consider a fluid ECM in which cells $R_t^{(i)}$ undergo diffusive motion without constraints on the inter-cell distances. Identically to the 1-dimensional case, $R_t^{(i)}$ convergence exponentially in distribution to iid uniform random variables on the sphere, which is the conditional distribution of a Poisson process. The resulting statistics on the sphere (Figure 4 (a1)-(d1)) closely follow those of the flat torus (Figure 3), with only area well-approximated by a gamma law.

A conjectured distribution. In dimension $d \geq 2$, exact distributions for the cell measures (areas, volumes, and so on) of Delaunay triangulations (denoted D_i) of a Poisson

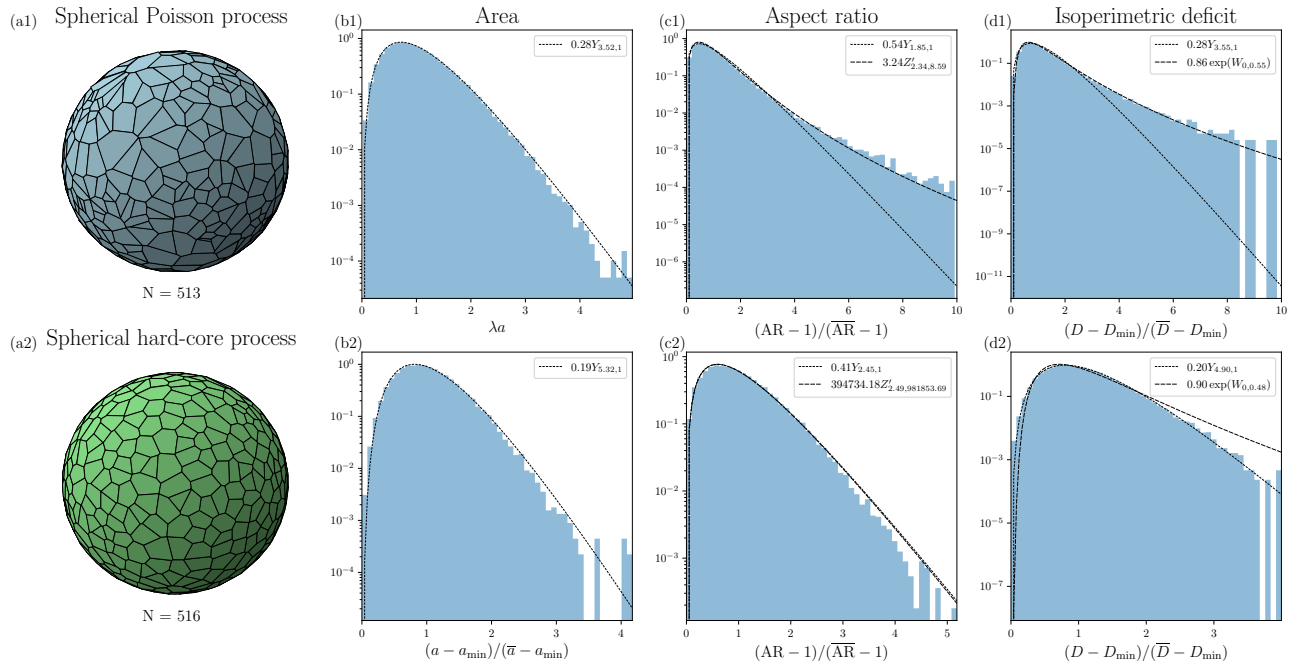


FIG. 4. Spherical point processes with a hard-core repulsion demonstrate a transition to gamma laws in geometric features, such as aspect ratio and isoperimetric deficit, not normally found in Poisson-Voronoi tessellations. Panel (a2) shows a realization of a Matérn Type-II process with the minimum distance r_d set to twice the apothem of a regular spherical hexagon of area $4\pi/\lambda$.

point process are known [22], with the distribution in $\mathbb{R}^2$ given by a modified Bessel function of the second kind,

$$f_{D_i}(v) = c_1 v^{\alpha_1} K_n(c_2 v^{\alpha_2}), \quad (22)$$

with parameters c_i, α_j, n . On the other hand, exact distributions for their vertex-cell duals—the Voronoi tessellations—are presently unknown. We conjecture, however, that (22) should also govern the Voronoi areas V_i based on the following heuristic argument. Conditional independence of the volume of the *fundamental region* (a particular set containing the vertices of a Voronoi cell and the origin) and its shape (see [23]) suggests that one may assume a particular approximate shape for the typical Voronoi cell, say, a disk or regular hexagon. If its radius L_i is gamma-(inclusively, exponentially-)distributed, then V_i is proportional to L_i^2 , whose distribution is precisely of the form (22) (see [25]).

The universality of gamma distributions in random Voronoi tessellations—particularly beyond the Poisson-Voronoi case, and in geometric features beyond area—is remarkable. Concerning area, let us remark on this observation using a well-known characterization of the gamma distribution which seems to be analogous to the *independent scattering* property characterizing Poisson processes [9]. Let A be the typical partition size of a random tessellation and A_c be the size of the remainder. Then, the

total $T = A + A_c$ and fraction $F = A/(A + A_c)$ are independent if and only if A and A_c are independent gamma random variables of the same rate parameter (Lukacs’ theorem, [13]). This recapitulates the story of gamma laws describing entropy-maximizing configurations [8]; random tessellations which maximize the joint entropy $H(T, F)$ —where geometrically realizable, such as one-dimensional settings—have gamma-distributed partition sizes.

Even more remarkably, the emergence of these distributions in biological tissues, such as areas in *Volvox* and the volumes in “snowflake yeast” [5], as well as aspect ratios in a wide range of confluent tissues and inert matter [20] suggests that point processes are suitable idealized models for a quantitative description of the structure of a multitude of biological systems. In this sense, they play a role analogous to the idealized random walk models of polymers and the hard sphere model of fluids.

ACKNOWLEDGMENTS

We are grateful to T. Day, P. Yunker, and W. Ratcliff for numerous discussions. This work was supported in part by the Cambridge Trust (AS), The John Templeton Foundation and Wellcome Trust Investigator Grant 207510/Z/17/Z (REG).

- choanoflagellate *Salpingoeca rosetta*, *Dev. Biol.* **357**, 73–82 (2011).
- [2] S.R. Fairclough, M.J. Dayel and N. King, Multicellular development in a choanoflagellate, *Curr. Biol.* **20**, R875–R876 (2010).
 - [3] A. Hallmann, Morphogenesis in the Family Volvocaceae: Different Tactics for Turning an Embryo Right-side Out, *Protist* **157** 445–461 (2006).
 - [4] E.E. Ruppert, R.S. Fox, and R.D. Barnes, *Invertebrate Zoology*, 7th ed. (Thomson-Brooks/Cole, Belmont, CA, 2004).
 - [5] T. C. Day, S. S. Höhn, S. A. Zamani-Dahaj, D. Yanni, A. Burnetti, J. Pentz, A. R. Honerkamp-Smith, H. Wioland, H. R. Sleath, W. C. Ratcliff, *et al.*, Cellular organization in lab-evolved and extant multicellular species obeys a maximum entropy law, *eLife* **11**, e72707 (2022).
 - [6] W.C. Ratcliff, R.F. Denison, M. Borrello and M. Travisano, Experimental evolution of multicellularity, *Proc. Natl. Acad. Sci. USA* **109**, 1595–1600 (2012).
 - [7] S. Höhn, A. Honerkamp-Smith, P.A. Haas, P. Khuc Trong, and R.E. Goldstein, Dynamics of a *Volvox* Embryo Turning Itself Inside Out, *Phys. Rev. Lett.* **114**, 178101 (2015).
 - [8] T. Aste and T. Di Matteo, Emergence of Gamma distributions in granular materials and packing models, *Phys. Rev. E* **77**, 021309 (2008).
 - [9] See Supplemental Material at <http://link.aps.org/supplemental/xxx> for analytical and numerical details.
 - [10] R. Durrett, *Probability: Theory and Examples*, Vol. 49 (Cambridge University Press, 2019).
 - [11] D. J. Daley, D. Vere-Jones, *et al.*, *An Introduction to the Theory of Point Processes: Volume I: Elementary Theory and Methods* (Springer, 2003).
 - [12] R. Pyke, Spacings, *J. R. Stat. Soc. B* **27**, 395–436 (1965).
 - [13] E. Lukacs, A Characterization of the Gamma Distribution, *Ann. Math. Stat.* **26**, 319–324 (1955).
 - [14] D. Stoyan, W. S. Kendall, S. N. Chiu, and J. Mecke, *Stochastic Geometry and its Applications* (John Wiley & Sons, 2013).
 - [15] D. J. Grabiner, Brownian motion in a Weyl chamber, non-colliding particles, and random matrices, *Ann. IHP Prob. Stat.* **35**, 177–204 (1999).
 - [16] D. G. Hobson and W. Werner, Non-Colliding Brownian Motions on the Circle, *Bull. London Math. Soc.* **28**, 643–650 (1996).
 - [17] F. J. Dyson, A Brownian-Motion Model for the Eigenvalues of a Random Matrix, *J. Math. Phys.* **3**, 1191–1198 (1962).
 - [18] P. Diaconis and M. Shahshahani, On the eigenvalues of random matrices, *J. Appl. Prob.* **31**, 49–62 (1994).
 - [19] T. Kemp, Heat Kernel Empirical Laws on $\mathbb{U}(n)$ and $\mathbb{GL}(n)$, *J. Theor. Prob.* **30**, 397–451 (2017).
 - [20] L. Atia, *et al.*, Geometric constraints during epithelial jamming, *Nat. Phys.* **14**, 613–620 (2018).
 - [21] R. Li, *et al.*, E^2 and gamma distributions in polygonal networks, *Phys. Rev. Res.* **3**, L042001 (2021).
 - [22] P. Rathie, On the volume distribution of the typical Poisson–Delaunay cell, *J. App. Prob.* **29**, 740–744 (1992).
 - [23] J. Møller and S. Zuyev, Gamma-type results and other related properties of Poisson processes, *Adv. Appl. Prob.* **28**, 662–673 (1996).
 - [24] J. Møller, *Lectures on random Voronoi tessellations*, Vol. 87 (Springer Science & Business Media, 2012).
 - [25] H. J. Malik, Exact Distribution of the Product of Independent Generalized Gamma Variables with the Same Shape Parameter, *Ann. Math. Stat.* **39**, 1751–1752 (1968).
 - [26] E. Gilbert, Random Subdivisions of Space into Crystals, *Ann. Math. Stat.* **33**, 958–972 (1962).
 - [27] M. Tanemura, Statistical distributions of poisson voronoi cells in two and three dimensions, *FORMA-TOKYO* **18**, 221 (2003).
 - [28] J.-S. Ferenc and Z. Nédá, On the size distribution of Poisson Voronoi cells, *Physica A* **385**, 518–526 (2007).
 - [29] D. Weaire, J. Kermode, and J. Wejchert, On the distribution of cell areas in a Voronoi network, *Phil. Mag. B* **53**, L101–L105 (1986).
 - [30] J. van den Berg, R. Meester, and D. G. White, Dynamic Boolean models, *Stoch. Proc. Appl.* **69**, 247–257 (1997).
 - [31] R. Miles, Random points, sets and tessellations on the surface of a sphere, *Ind. J. Stat. A* **33**, 145–174 (1971).
 - [32] M. Van Den Berg and J. Lewis, Brownian motion on a hypersurface, *Bull. Lon. Math. Soc.* **17**, 144–150 (1985).
 - [33] E. Cépa and D. Lépingle, Brownian particles with electrostatic repulsion on the circle: Dyson’s model for unitary random matrices revisited, *ESAIM: Prob. Stat.* **5**, 203–224 (2001).
 - [34] M. L. Mehta, *Random Matrices* (Elsevier, 2004).
 - [35] G. Menon, Lesser known miracles of Burgers equation, *Acta Math. Sci.* **32**, 281–294 (2012).
 - [36] P. Findeisen, A simple proof of a classical theorem which characterizes the gamma distribution, *The Annals of Statistics*, 1165 (1978).
 - [37] N. Fusco, F. Maggi, and A. Pratelli, The sharp quantitative isoperimetric inequality, *Ann. Math.* **941**, (2008).
 - [38] S. Diamond and S. Boyd, CVXPY: A Python-embedded modeling language for convex optimization, *The Journal of Machine Learning Research* **17**, 2909 (2016).
 - [39] D. Kraft, A software package for sequential quadratic programming, *Forschungsbericht- Deutsche Forschungs- und Versuchsanstalt für Luft- und Raumfahrt* (1988).

SUPPLEMENTAL MATERIAL

This file contains analytical and computational details pertaining to the models discussed in the main body.

I. RANDOM VARIABLES

A. Transforms and convergence

Definition S1 (Pushforward measure). Let $g : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a diffeomorphism and $Y = g(X)$. The pushforward probability measure μ_Y is, for all measurable $U \subset \mathbb{R}^n$,

$$\mu_Y(U) = \mu_X(g^{-1}(U)). \quad (\text{S1})$$

a. Transforms of random variables. When μ_X has a Radon-Nikodym derivative f_X with respect to the Lebesgue measure —i.e. is expressible by the probability density $\mu_X(U) = \int_U f_X(x)dx$ —then by the rule for integration under a diffeomorphic change of coordinates $y = g(x)$,

$$\mu_X(U)(g^{-1}(U)) = \int_{g^{-1}(U)} f_X(x)dx = \int_U f_X(g^{-1}(y)) \left| \det \frac{\partial g^{-1}}{\partial y}(y) \right| dy. \quad (\text{S2})$$

As this is true for all U we conclude

$$f_Y(y) = f_X(g^{-1}(y)) \left| \det \frac{\partial g^{-1}}{\partial y}(y) \right|. \quad (\text{S3})$$

When preferable to work with g rather than g^{-1} , we may apply the chain rule to $g^{-1} \circ g = 1$ to convert (S3) to

$$f_Y(g(x)) = f_X(x) \left| \det \left(\frac{\partial g}{\partial x}(x) \right)^{-1} \right|. \quad (\text{S4})$$

Due to this fact, we will abbreviate the scaling factor as J^{-1} , denoting the inverse Jacobian determinant. For affine transforms $Y = cX + b$, (S3) becomes

$$f_Y(y) = \frac{1}{c} f_X \left(\frac{y - b}{c} \right). \quad (\text{S5})$$

b. Sums of random variables. Let X, Y be independent random variables taking values in $U_X, U_Y \subseteq \mathbb{R}$. Then their sum is distributed as the convolution

$$X + Y = Z \sim f_Z(z) = \int_{x \in U_X, x \leq z} f_X(x) f_Y(z - x) dx = f_X * f_Y. \quad (\text{S6})$$

This can alternatively be deduced by applying the transform rule (S3) to the map $(X, Y) \mapsto (X, X + Y)$.

Definition S2 (Convergence in distribution). A sequence of random variables $\{X_n\}$ taking values in an interval $U \subseteq \mathbb{R}$ is said to *converge in distribution* if, for all x at which the cumulative distribution function F_X is continuous, the c.d.f.s. F_{X_n} converge pointwise, i.e.

$$\lim_{n \rightarrow \infty} F_{X_n}(x) = F_X(x). \quad (\text{S7})$$

B. Basic properties of the exponential and gamma random variables

Let us recall with proof several basic properties of the gamma (and exponential, which is a special case) random variable.

Definition S3 (Gamma random variable.). The gamma random variable $Y_{k,\lambda}$ with shape parameter $k > 0$ and rate $\lambda > 0$ is the continuous random variable with probability density function

$$f_Y(y) = \frac{\lambda^k y^{k-1} \exp(-\lambda y)}{\Gamma(k)}. \quad (\text{S8})$$

Its namesake is the gamma function

$$\Gamma(k) = \int_0^\infty y^{k-1} e^{-y} dy. \quad (\text{S9})$$

Lemma S1 (Gamma random variables are closed under addition). *The sum of two independent gamma random variables Y_1, Y_2 with k_1, k_2 of common rate λ is $(k_1 + k_2)$ -gamma distributed.*

Proof. Since $Y_1, Y_2 \in [0, \infty)$, by (S6) we have

$$Y_1 + Y_2 \stackrel{\text{pdf}}{\sim} f_{Y_1} * f_{Y_2} = \frac{\lambda^{k_1+k_2}}{\Gamma(k_1)\Gamma(k_2)} \int_0^z y^{k_1-1} e^{-\lambda y} (z-y)^{k_2-1} e^{-\lambda(z-y)} dy. \quad (\text{S10})$$

Taking the Beta function

$$B(k_1, k_2) = \frac{\Gamma(k_1)\Gamma(k_2)}{\Gamma(k_1+k_2)} = \int_0^1 t^{k_1-1} (1-t)^{k_2-1} dt \quad (\text{S11})$$

with the change of variable $t = y/z$ yields

$$= \frac{1}{z^{k_1+k_2-1}} \int_0^z y^{k_1-1} (z-y)^{k_2-1} dx. \quad (\text{S12})$$

Substituting into (S10), we obtain

$$(\text{S10}) = \frac{\lambda^{k_1+k_2} z^{k_1+k_2-1} e^{-\lambda z}}{\Gamma(k_1+k_2)} \stackrel{\text{pdf}}{\sim} Y_{k_1+k_2, \lambda}. \quad (\text{S13})$$

□

Corollary. *The sum of n iid exponential random variables of rate λ are gamma-distributed with shape parameter n and rate λ .*

Lemma S2 (Beta-gamma convergence). *Let $Z_{2,m}$ be a beta random variable. Then $\frac{m}{\alpha} Z_{2,m}$ converges in distribution to the gamma random variable $Y_{2,\alpha}$ of rate α .*

Proof. By direct calculation,

$$\frac{m}{\alpha} Z_{2,m} \stackrel{\text{cdf}}{\sim} \text{P} \left[Z_{2,m} \leq \frac{z\alpha}{m} \right] = \frac{1}{B(2, m)} \int_0^{\frac{z\alpha}{m}} t(1-t)^{m-1} dt \quad z \in \left[0, \frac{m}{\alpha} \right], \quad (\text{S14})$$

and with the change of variables $s = 1 - t$,

$$= \frac{1}{B(2, m)} \int_{1-\frac{\alpha z}{m}}^1 (1-s)s^{m-1} ds \quad (\text{S15})$$

$$= m(m+1) \left[\frac{1}{m} s^m - \frac{1}{m+1} s^{m+1} \right]_{1-\frac{\alpha z}{m}}^1 \quad (\text{S16})$$

$$= 1 - \left((m+1) \left(1 - \frac{z\alpha}{m} \right)^m - m \left(1 - \frac{z\alpha}{m} \right)^{m+1} \right) \quad (\text{S17})$$

$$= 1 - \left(1 - \frac{z\alpha}{m} \right)^m (1 + z\alpha) \quad (\text{S18})$$

$$\xrightarrow{m \rightarrow \infty} 1 - e^{-z\alpha} (1 + z\alpha) \quad z \in [0, \infty). \quad (\text{S19})$$

Recalling the cumulative density function for $Y_{2,\alpha}$ (S8),

$$Y_{2,\alpha} \stackrel{\text{cdf}}{\sim} \int_0^z \alpha^2 y e^{-\alpha y} dy = -\alpha y e^{-\alpha y} \Big|_0^z + \int_0^z \alpha e^{-\alpha y} dy = 1 - e^{-z\alpha} (1 + z\alpha). \quad (\text{S20})$$

Hence, we have the distributional convergence $\frac{m}{\alpha} Z_{2,m} \xrightarrow{d} Y_{2,\alpha}$. □

Lemma S3 (Memoryless characterization of the exponential). *The only continuous random variable X which (i) possesses a cumulative distribution function F_X such that $F'_X(0)$ exists and (ii) satisfies the memoryless property*

$$P[X > x + y \mid X > x] = P[X > y] \quad (\text{S21})$$

is the exponential.

Proof. By Bayes' theorem, (S21) is equivalent to

$$P[X > x + y] = P[X > x]P[X > y] \Leftrightarrow 1 - F(x + y) = (1 - F(x))(1 - F(y)). \quad (\text{S22})$$

Let $G = 1 - F$; then (S22) is $G(x + y) = G(x)G(y)$. Then for all z ,

$$G'(z) = \lim_{h \rightarrow 0} \frac{G(z)G(h) - G(z)G(0)}{h} = G(z)G'(0). \quad (\text{S23})$$

Since by hypothesis (i) $F'(0)$ exists, this implies F' exists everywhere. Let u be such that $F(u) < 1$. Since F is nondecreasing, this implies $G > 0$ in the interval $(-\infty, u]$. Then for $w \in (-\infty, u]$, letting $G'(0) = -F'(0) = c$,

$$c = \frac{G'(w)}{G(w)} = \frac{d}{dw} \log G(w). \quad (\text{S24})$$

Integrating, we obtain $G(x) = b \exp(cx)$ for some b . By the conditions $F(0) = 0, \lim_{t \rightarrow \infty} F(x) = 1$, we have $b = 1, c < 0$. Letting $c = -\lambda$ for $\lambda > 0$ we obtain $F_X(x) = 1 - \exp(-\lambda x)$. Hence, $f_X(x) = F'_X(x) = \lambda \exp(-\lambda x)$, and X is an exponential random variable of rate λ . $\square$

Lemma S4 (Maximum-entropy characterization of the exponential). *The only nonnegative continuous random variable X with density f_X which maximizes the entropy with fixed mean $\mu > 0$ is the exponential.*

Proof. By hypothesis, f_X is a critical point of the functional

$$J[f] = \int_0^\infty L(f(x), \lambda_0, \lambda_1) dx, \quad (\text{S25})$$

for Lagrange multipliers λ_0, λ_1 constraining the 0th, 1st moments, and Lagrangian density

$$L(f(x), \lambda_0, \lambda_1) = f(x) \log f(x) + \lambda_1 f(x) + \lambda_1 x f(x). \quad (\text{S26})$$

Since for all test functions φ , the Fréchet derivative vanishes,

$$0 = \langle DJ[f], \varphi \rangle = \int_0^\infty \frac{\partial L}{\partial f(x)} \varphi(x) dx, \quad (\text{S27})$$

by the fundamental lemma of the calculus of variations,

$$0 = \frac{\partial L}{\partial f(x)} = 1 + \log f(x) + \lambda_0 + x \lambda_1, \quad (\text{S28})$$

we have $f(x) = \exp(-1 - \lambda_0 - x \lambda_1)$. Applying the total probability constraint,

$$1 = \int_0^\infty \exp(-1 - \lambda_0 - x \lambda_1) dx = \frac{1}{\lambda_1} \exp(-1 - \lambda_0), \quad (\text{S29})$$

where $\lambda_1 > 0$ necessarily. Then, $f_X(x) = \lambda_1 \exp(-\lambda_1 x)$ is the density of an exponential random variable with rate λ_1 . $\square$

C. Characterization theorem for gamma random variables

Theorem S5 (Lukacs 1955, [13]). *Let Y_1, Y_2 be independent random variables. Then*

$$A = Y_1 + Y_2, \quad B = \frac{Y_1}{Y_1 + Y_2} \quad (\text{S30})$$

are independent if and only if Y_1, Y_2 are gamma random variables of the same rate λ .

Proof ($\implies$). Consider the map

$$g(Y_1, Y_2) = \left(Y_1 + Y_2, \frac{Y_1}{Y_1 + Y_2} \right) =: (A, B). \quad (\text{S31})$$

Then for $a \neq 0$,

$$g^{-1}(a, b) = (ab, a - ab). \quad (\text{S32})$$

Therefore

$$\left| \det \frac{\partial g^{-1}}{\partial(a, b)} \right| = \left| \det \begin{bmatrix} b & 1-b \\ a & -a \end{bmatrix} \right| = a. \quad (\text{S33})$$

By (S4), the pushforward density is

$$f_{g(Y_1, Y_2)}(a, b) = a f_{Y_1}(ab) f_{Y_2}(a - ab) \quad (\text{S34})$$

$$= \frac{a(ab)^{k_1-1}(a-ab)^{k_2-1}}{\Gamma(k_1)\Gamma(k_2)} \lambda^{k_1+k_2} e^{-\lambda a} \quad (\text{S35})$$

$$= \frac{b^{k_1-1}(1-b)^{k_2-1}}{\Gamma(k_1)\Gamma(k_2)} a^{k_1+k_2-1} \lambda^{k_1+k_2} e^{-\lambda a}. \quad (\text{S36})$$

By inspection, the total A and fraction B are independent. Substituting the Beta function (S11),

$$(\text{S34}) = \frac{b^{k_1-1}(1-b)^{k_2-1}}{B(k_1, k_2)} \frac{a^{k_1+k_2-1} \lambda^{k_1+k_2} e^{-\lambda a}}{\Gamma(k_1 + k_2)} =: f_B(b) f_A(a). \quad (\text{S37})$$

Then $f_A(a)$ is a gamma distribution (as expected, S1) and $f_B(b)$ is a beta distribution. Therefore,

$$B = \frac{Y_{1;k_1,\lambda}}{Y_{1;k_1,\lambda} + Y_{2;k_2,\lambda}} = Z_{k_1, k_2} \quad (\text{S38})$$

is beta-distributed. $\square$

Note that (S38) is independent of the rate λ of Y_1, Y_2 . The converse ($\impliedby$), that gamma distributions are unique in possessing this independence property, is not proven here, but we refer the reader to a straightforward proof [36] relying on the fact that the gamma distribution is uniquely determined by its moments (Exercise 3.3.25, [10]).

II. POINT PROCESSES

A. Definitions

The usual Poisson process of rate λ on $[0, \infty)$ is realizable as the cumulative sum of i.i.d. exponential random variables $X_{i;\lambda}$. The construction of general point processes on a domain $K \subseteq \mathbb{R}^d$ is more technical, formalizing the notion of a “random almost-surely finite subset,” for which we recall several standard definitions [11]. Throughout, we will assume K is a closed set.

Definition S1 (Finite point process —2.2-2.4, [11]). A finite point process N on a complete separable metric space K is a family of random variables $N(E)$ for each Borel set $E \in \mathcal{F}_K$, such that, for every bounded E ,

$$\mathbb{P}[N(E) < \infty] = 1. \quad (\text{S39})$$

Formally, N is a *random measure* (see e.g. 5.1 [11] or Chapter 9 [11]). Without delving too deeply into this formalism, let us introduce the following definition based upon samples of N .

Definition S2 (Simple point process). A *simple* point process N is one whose points are non-overlapping. In other words, every sample of N can be written as the counting measure

$$\nu = \sum_{i \in I} \delta_{x_i} \quad (\text{S40})$$

where I is an index set and δ denotes the Dirac measure and $\mathbb{P}[x_i = x_j] = 0$ for all $i \neq j$.

Definition S3 (Non-atomic point process). A *nonatomic* point process N is one for whom the probability of realizing any particular point $x \in K$ is zero. That is,

$$\mathbb{P}[N(\{x\}) > 0] = 0. \quad (\text{S41})$$

These definitions are sufficient to define and analyze the Poisson point process.

Definition S4 (Poisson point process). A Poisson point process N on a c.s.m.s. K is defined by an *intensity measure* $\Lambda(E)$ such that, for all Borel sets $E \in \mathcal{F}_K$, $N(E)$ is a Poisson random variable, i.e.

$$N(E) \stackrel{\text{pdf}}{\sim} \frac{\Lambda(E)^k \exp(-\Lambda(E))}{k!}. \quad (\text{S42})$$

Example II.1 (Stationary Poisson process). A *stationary* Poisson process of rate λ is given by the intensity measure $\Lambda(E) = \lambda\mu(E)$, where μ is the Lebesgue measure.

Definition S5 (Independent scattering / complete independence). A point process N satisfies the *independent scattering* or *complete independence* property if, for all $n > 1$ and disjoint Borel sets $E_1, \dots, E_n \in \mathcal{F}_K$, the variables $N(E_1), \dots, N(E_n)$ are mutually independent.

B. Characterization theorem for Poisson point processes

Immediately, we see that Poisson point processes are simple and finite if the intensity measure $\Lambda(E)$ is given by

$$\Lambda(E) = \int_E \lambda(x) dx \quad (\text{S43})$$

for some function $\lambda : K \rightarrow \mathbb{R}_+$. Poisson point processes S4 satisfy the complete independence property S5, but remarkably, these properties are not logically independent.

Theorem S1 (Prekopa 1957, Theorem 2.4.V [11]). *A point process N is a non-atomic Poisson point process if and only if it is finite S1, simple S2, non-atomic S3, and completely independent S5.*

The characterization theorem S1 motivates and justifies the use of Poisson processes (i.e. Poisson-distributed count variables) in any scenario where points in any realization are non-interacting.

C. Sampling Poisson and Matérn point processes

Lemma S2. *The conditional distribution of a Poisson point process of intensity $\lambda(x)$ on a domain K is $\frac{1}{\Lambda(K)}\lambda(x)$.*

Lemma S2 allows one to sample a homogeneous Poisson point process of rate λ by first sampling a Poisson random variable $N_{\lambda\mu(K)} = n$, and scattering the n as points as i.i.d. uniform random variables in K . Point processes with a hard-core repulsion (such as the Matérn Type-II point process [14]) can often be realized as a *thinning* of a Poisson point process.

Definition S6 (Matérn type-II hard-core point process, [14]). Let N be a homogeneous Poisson point process of rate λ . To each point X_i of N , assign a *mark* M_i which are i.i.d. uniform random variables on $(0, 1)$. Then, construct the Matérn process N' with hard-core repulsion distance r as

$$N' = \{X_i \in N \mid M_i < M_j \ \forall M_j \in B(X_i, r), \ i \neq j\}. \quad (\text{S44})$$

The points of N' are situated at a minimum distance r from one another.

1. Poisson and Matérn processes on $\mathbb{S}^d$

Recall that a zero-mean multivariate Gaussian $W_{0,\Sigma}$ with Σ a $d \times d$ symmetric positive-definite covariance matrix has the ellipsoids $\{x \mid x^\top \Sigma^{-1} x = r^2\}$ as level sets of constant density. When $\Sigma = I$, this is the property of *spherical symmetry*. Combined with the fact that the map $x \mapsto x \|x\|^{-1}$ is surface-area-preserving up to a constant multiple,

this yields a computationally efficient and numerically stable method for generating iid uniform random variables U_i on $\mathbb{S}^d$: $U_i = W_{i;0,I} \|W_{i;0,I}\|^{-1}$.

Lemma S2 then allows the realization of homogeneous Poisson processes on d -spheres of radius r as $N_{\lambda A r^d} = n$, $\{U_i\}_{i=1}^n$, with A the surface area of the unit sphere $\mathbb{S}^d$ and λ the intensity per unit area. Conveniently, the parametrization of U_i in Cartesian coordinates allows the realization of Matérn processes (S6) using the geodesic (great-circle) distance $d(U_i, U_j) = r \arccos(U_i \cdot U_j)$.

III. GEOMETRY OF CLOSED CURVES

A. Moments of area

The area moments of simple closed curves on planar surfaces are useful for quantifying for geometric properties of internally connected objects such as cells. All tensorial quantities and identities below are for dimension 2.

Definition S1 (Planar n th area moment tensor). Let $x \in D$ be a two-dimensional domain bounded by a simple closed curve $\partial D = C$. The n th moment tensor of the area D with respect to a point v is defined as

$$I_{i_1 i_2 \dots i_n}^{(n)}(v) = \int_D (x_{i_1} - v_{i_1})(x_{i_2} - v_{i_2}) \dots (x_{i_n} - v_{i_n}) \quad (\text{S45})$$

for $i_1, \dots, i_n \in \{1, 2\}$. When the argument v is dropped, it is assumed $v = 0$.

We will assume D is a uniform mass-density domain in the following.

1. First moment, center of mass, and centrality

Applying the divergence theorem, $I^{(1)}$ becomes the surface integral

$$I^{(1)} = \int_D x = \frac{1}{3} \int_D \partial_i x_i x_j = \frac{1}{3} \int_C n_i x_i x_j \quad (\text{S46})$$

where n_i is the unit boundary normal.

Definition S2 (Center of mass of a closed curve). The center of mass of D is

$$\mu = \frac{I^{(1)}}{I^{(0)}}. \quad (\text{S47})$$

The notation μ is used for the probabilistic interpretation as the expected value of a uniform distribution supported on D .

Definition S3 (Variance-adjusted centrality of a test point within a polygon). For a test point y , we define the centrality metric

$$\text{centrality}(y) = \|W(y - \mu)\|^2 = (y - \mu)^\top \Sigma^{-1} (y - \mu) \quad (\text{S48})$$

with W a matrix defined as $W^\top W = \Sigma^{-1}$, and Σ the second central moment defined in the next part (S50).

This *whitening* procedure (using terminology from the probabilistic interpretation of Σ) enables comparison across curves C of varying moments—that is, centrality(y) is dimensionless. In probability, (S48) is the Mahalanobis distance of y and D .

2. Second moment, area moment of inertia, and isotropy

Again by divergence theorem,

$$I^{(2)} = \int_D x_i x_j = \frac{1}{4} \int_D \partial_i x_i x_j x_k = \frac{1}{4} \int_C n_i x_i x_j x_k. \quad (\text{S49})$$

Definition S4 (Second moment of area). The second central moment of area (i.e. the area moment of inertia about the center of mass) is

$$\Sigma = I^{(2)}(\mu) = I^{(2)} - (\mu \otimes I^{(1)} + I^{(1)} \otimes \mu) + I^{(0)}\mu \otimes \mu \quad (\text{S50})$$

where $x \otimes y = xy^\top$.

In connection to probability, Σ is precisely the covariance matrix of a mean-zero uniform distribution supported on the domain D .

Definition S5 (Principal axes of a polygon). Using (S50), we may define the principal axes and stretches of a polygon (represented by a closed curve C) via Hermitian eigendecomposition

$$\Sigma = PAP^{-1}$$

with v_i the columns of P being the principal axes and $\lambda_i > 0$ in ascending order (as Σ is symmetric positive-definite for non-degenerate curves).

We then define the “anisotropy” of the shape D as the distance of the second central moment Σ to its ℓ^2 -orthogonal projection onto the subspace $V \in \mathbb{R}^{d \times d}$ of isotropic matrices. There are two linearly independent isotropic matrices in dimension $d = 2$, the matrices $\Sigma = \delta_{ij}$ and $\Sigma = \epsilon_{ij}$. The latter identity cannot hold since $\Sigma_{ii} \neq 0$ for all i as defined by (S49), hence we take V to be the one-dimensional subspace αI with $\alpha \in \mathbb{R}$. Computing

$$0 = \frac{\partial}{\partial \alpha} \|\Sigma - \alpha I\|^2 = -2\langle \Sigma - \alpha I, I \rangle = -2\text{tr}(\Sigma) + 2d\alpha, \quad (\text{S51})$$

then the orthogonal projection of Σ onto V is simply $\frac{\text{tr}(\Sigma)}{d}I$ as expected. Thus in dimension d ($d = 2$ for planar curves), we define the anisotropy of D (in dimensionless units) as

$$\text{anisotropy}(D) = \left\| \frac{d}{(\text{tr} \Sigma)} \Sigma - I \right\|^2. \quad (\text{S52})$$

Clearly $\text{anisotropy}(D)$ is nonnegative. Furthermore, $\text{anisotropy}(D)$ vanishes if and only if D is a regular polygon. Furthermore, the “orientation” of an anisotropic domain D is given by the principal axes of Σ and “elongation” by the principal stretches. For an ellipse, $\Sigma^{-1} = M$ is precisely the ellipse matrix (S71).

3. Approximation by an ellipse.

Definition S6 (Aspect ratio of a polygon). Let λ_1, λ_2 and v_1, v_2 be the principal stretches and axes (in ascending order) of the second central moment Σ (S50). Define the aspect ratio:

$$R = \sqrt{\frac{\lambda_2}{\lambda_1}} \quad (\text{S53})$$

Then an ellipse with the same orientation, aspect ratio, and area (πab) as D (as defined previously by its n th moments) has the minor and major axes:

$$a = \sqrt{\frac{I^{(0)}}{\pi R}}, \quad b = Ra \quad (\text{S54})$$

Equation (S54) may be considered a superior approximation of the polygon D over e.g. a weighted ℓ^2 -minimization of vertex distance or a bounding ellipse approach.

4. Whitening a polygon.

Let a polygon C have second central moment Σ (S50) and centroid μ (S47). With C given in counterclockwise oriented coordinates as $\{x_i\}_{i=1}^n$, we define the *whitened* polygon C_w ,

$$C_w = \{W(x_i - \mu) \mid \forall i\}, \quad W = \Sigma^{-1/2}, \quad (\text{S55})$$

with the matrix square root W typically approximated by singular value decomposition (SVD) as

$$W = U(S + \varepsilon)^{-1/2}U^*, \quad W = USV^\top, \quad (\text{S56})$$

and W then referred to as the *ZCA whitening matrix* with a regularization constant $\varepsilon \ll 1$.

B. Isoperimetric inequality

We recall here several quantities which can be used as measures of the deviation of a closed plane curve C from a circle. The classical *isoperimetric inequality* is

$$4\pi A \leq L^2, \quad (\text{S57})$$

where A is the area bounded by C (denoted by the set E) and L its total arclength, which is an equality only for circles. Accordingly, one may define the *isoperimetric quotient*

$$Q = \frac{4\pi A}{L^2} \in [0, 1], \quad (\text{S58})$$

which is maximized for circles. A natural question, however, is whether (S57) can be used to define a set distance, in the sense of Hausdorff metric, of the area bounded by C to the “best” circle. This turns out [37] to be related to the problem of making (S57) *quantitative*, in the sense of a nonnegative quantity $\nu(E)$ such that

$$4\pi A + \nu(E) \leq L^2 \quad (\text{S59})$$

for all simple closed curves C , with $\nu(E) = 0$ iff C is a circle. Without proof, we cite [37] the result that the *isoperimetric deficit*, defined as the dimensionless quantity

$$D(E) = \frac{L}{\sqrt{4\pi A}} - 1 \quad (\text{S60})$$

upper-bounds any such quantity $\nu(E)$ via

$$\nu(E) \leq C_2 \sqrt{D(E)} \quad (\text{S61})$$

for some dimension-dependent constant C_2 . For convex E (applicable e.g. to the cells of a Voronoi tessellation of Euclidean space), (S60) upper-bounds the Hausdorff distance to the best-fit equal-volume ball as

$$\inf_{x \in \mathbb{R}^2} d_H(E, B + x) \leq C_2 D(E)^{\alpha_2} \quad (\text{S62})$$

for dimension-dependent constants C_2, α_2 [37]. Naturally this is sharp for E which is a ball.

C. Curves on surfaces of constant Gaussian curvature

Identity S1 (Area-angle formula for a polygon on $\mathbb{S}^2$ or $\mathbb{H}^2$). Let n denote the number of sides of a regular spherical or hyperbolic polygon with interior angles θ_i . Then by Gauss-Bonnet,

$$KA + \sum_{i=1}^n (\pi - \theta_i) = 2\pi \quad (\text{S63})$$

with K the Gaussian curvature.

IV. MODELS

Model IS.3 (variable N , fixed C)—As in Model IS.2, let us consider the k th order statistic of N uniform random variables (representing the fixed-circumference constraint), with N_λ now a Poisson-distributed random variable conditioned to be minimum k . The marginal distribution of this order statistic given λ is (a particular) *compound beta-Poisson distribution*, given by

$$\mathbb{P}(Z_{k,N-k+1} = x | \lambda, N \geq k) = \sum_{n=k}^{\infty} \mathbb{P}(Z_{k,n-k+1} = x | N = n) \mathbb{P}(N = n | \lambda, N \geq k). \quad (\text{S64})$$

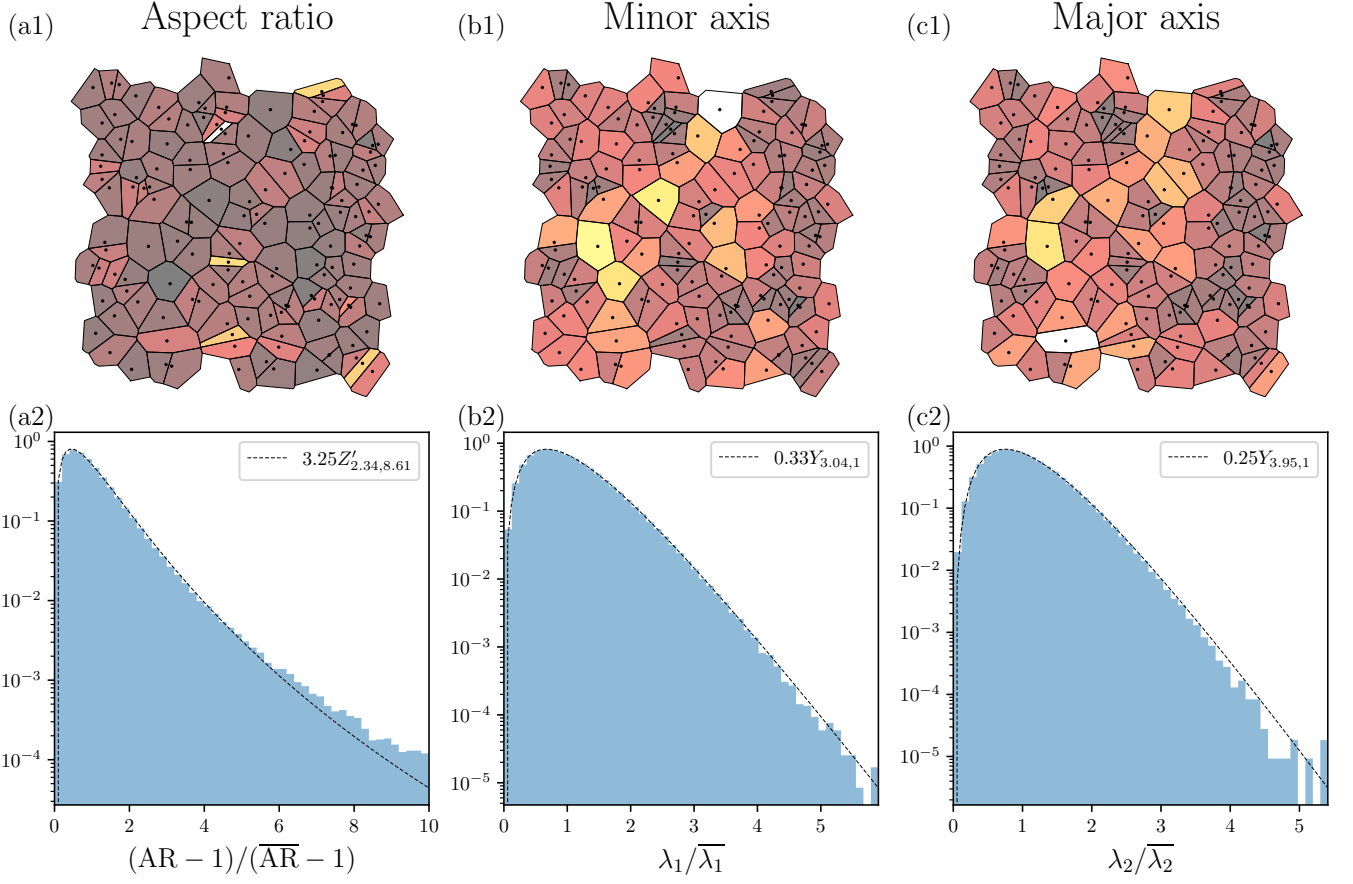


FIG. S1. Gamma-distributed stretches and approximately beta-prime distributed aspect ratio.

To see the parametrization more clearly, let $m = n - k + 1$; then,

$$(S64) = \frac{1}{1 - P(N < k)} \sum_{m=1}^{\infty} \frac{x^{k-1}(1-x)^{m-1}}{B(k, m)} \frac{\lambda^{m-1+k}e^{-\lambda}}{\Gamma(m+k)} \quad (S65)$$

$$= \frac{1}{1 - P(N < k)} \frac{\lambda^k x^{k-1} e^{-\lambda}}{\Gamma(k)} \sum_{m=1}^{\infty} \frac{(\lambda(1-x))^{m-1}}{\Gamma(m)} \quad (S66)$$

$$= \frac{1}{1 - P(N < k)} \frac{\lambda^k x^{k-1} e^{-\lambda x}}{\Gamma(k)} \quad (S67)$$

$$=: c_0 \frac{\lambda^k x^{k-1} e^{-\lambda x}}{\Gamma(k)}, \quad x \in [0, 1]. \quad (S68)$$

Recognizing the second factor as the gamma distribution, the first factor c_0 is simply a normalizing constant restricting the support to $[0, 1]$. Here we see the emergence of gamma distributions from the order statistics of a Poisson-distributed number of uniform random variables, which is in fact a roundabout way to construct the 1D Poisson process. As we take the support of this distribution (the circumference C) to infinity, we have $P(N_{C\lambda} < k) \rightarrow 0$ for any fixed k , hence $c_0 \rightarrow 1$ and we recover the true gamma distribution.

Model IIS.1 (periodic unit square)—Figure S1(b2-c2) shows that both major and minor axes of the Poisson-Voronoi tessellation, nondimensionalized to empirical mean 1, follow gamma distributions.

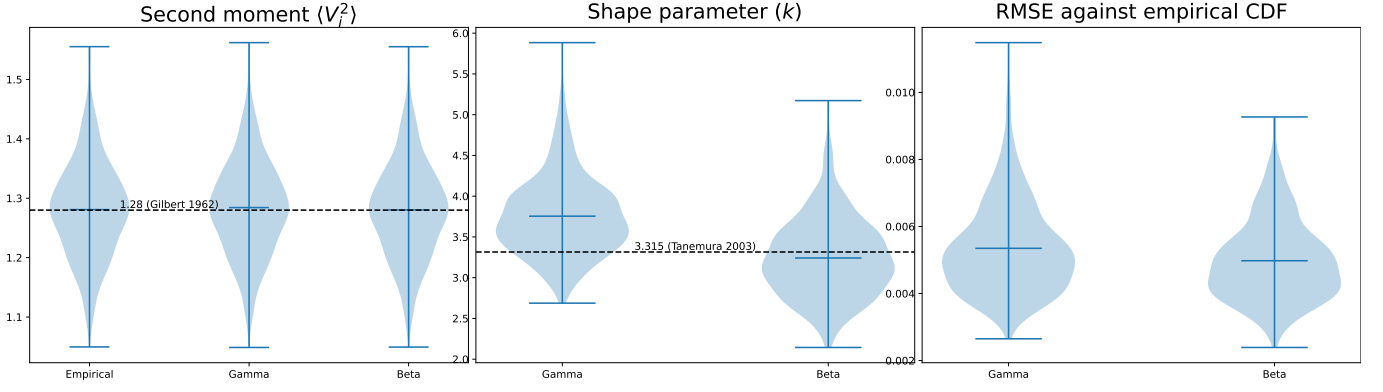


FIG. S2. Gamma- and Beta-maximum likelihood fits for Poisson-Voronoi tessellations of the flat torus $\mathbb{T}^2$. The numerically integrated value of the second moment $\mathbb{E}[V_i^2] = 1.280$ found by [26] is plotted as a dashed line in the left plot. The experimentally determined shape parameter $k = 3.315$ of the generalized-gamma fit found by [27] is plotted as a dashed line in the middle plot.

V. COMPUTATIONAL METHODS

A. Validation of the numerical method for Poisson-Voronoi tessellations

Let V_i be the measure (area, volume, etc.) of the typical Poisson-Voronoi cell. While the distribution of V_i is presently unknown, exact second moments of V_i in $\mathbb{R}^2$ and $\mathbb{R}^3$ are known [26], facilitating comparison with numerical study. Large simulations [27, 28] with $n > 10^6$ cells have found that gamma distributions, and in particular a 3-parameter generalization [27]

$$f_{Y;k,\lambda,a}(v) = \frac{a\lambda^{k/a}v^{k-1}e^{-\lambda v^a}}{\Gamma(k/a)} \quad (\text{S69})$$

achieve good maximum-likelihood fit to data with $< 1\%$ error relative to the analytical second moment. In Fig. S2, the estimated second moment $\langle V_i^2 \rangle$, shape parameter k , and CDF root mean square error are displayed for 500 trials of $N \stackrel{\text{pdf}}{\sim} \text{Poisson}(10^3)$ points. The average empirical, gamma, and beta second moments show good agreement with Gilbert's [26] numerically integrated value of 1.280 and are within 1% relative error, validating the numerical method. The estimated value of $k \approx 3.7$ for gamma on the torus $\mathbb{T}$ is consistent with prior results finding $k \approx 3.6$ [29] in the plane $\mathbb{R}^2$. On the other hand, the estimated value of $k \approx 3.2$ for beta is lower than gamma and closer to Tanemura's [27] generalized-gamma (S69) fit finding $k = 3.315$, suggesting that a beta hypothesis is a good substitute for the generalized gamma distribution. Lastly, we observe that the beta RMSE is slightly decreased compared to gamma.

B. Fitting ellipsoids in d dimensions

Let x, v be vectors in some d -dimensional basis. The equation of a $(d-1)$ -sphere S centered at v is

$$\|x - v\|_2^2 = 1 \quad \forall x \in S. \quad (\text{S70})$$

Applying a rotation P (an orthogonal matrix) of the sphere onto a set of principal axes and stretching along those axes by Λ (a positive diagonal matrix), one generalizes (S70) to an ellipsoid E via a symmetric positive-definite matrix $M = P\Lambda P^\top$ such that x satisfies

$$(x - v)^\top M (x - v) = 1 \quad \forall x \in E. \quad (\text{S71})$$

The elliptic radii r_i are then $r_i = \Lambda_{ii}^{-\frac{1}{2}}$ and the elliptic axes are the columns of P .

1. Minimum-volume bounding ellipsoid

Given the volume V

$$V \propto \prod_{i=1}^d r_i = \sqrt{\det(M^{-1})}, \quad (\text{S72})$$

it follows that maximizing $\log \det(M)$ minimizes V . Hence for a given dataset $\{x_i\}_{i=1}^n$, the following convex program computes the minimum-volume bounding ellipsoid:

$$\sup_{M,v} \log \det(M) \quad (\text{S73})$$

$$\text{subject to } (x_i - v)^\top M (x_i - v) \leq 1 \quad \forall i \quad (\text{S74})$$

$$M > 0. \quad (\text{S75})$$

Here, $M > 0$ is in the sense of linear matrix inequality (LMI), i.e. M is constrained to lie in the positive-definite cone. However, the offset $x_i - v$ produces a variable-product constraint which is not disciplined-convex. The following reparametrization uses the invertibility of M to eliminate the product from the problem:

$$\sup_{A,b} \log \det(A) \quad (\text{S76})$$

$$\text{subject to } \|Ax_i - b\|_2^2 \leq 1 \quad \forall i \quad (\text{S77})$$

$$A > 0 \quad (\text{S78})$$

$$M = A^2 = A^\top A \quad (\text{S79})$$

$$v = A^{-1}b. \quad (\text{S80})$$

DCP solvers such as CVXOPT [38] solve this problem. We reduce the problem size by considering the subset of X lying on the convex hull, computed in $O(n \log n)$.

2. Minimal ℓ^2 -distance projection to ellipsoid

Given a representation of an ellipsoid as (M, v) in the same basis as x , define the following convex program:

$$\inf_{\hat{Y}} \|\hat{Y} - \hat{X}\|_2^2 \quad (\text{S81})$$

$$\text{subject to } \|A\hat{x}_i\|_2^2 = 1 \quad \forall i \quad (\text{S82})$$

$$A^\top A = A^2 = M \quad (\text{S83})$$

$$\hat{X} = X - v \quad (\text{S84})$$

$$\hat{Y} = Y - v, \quad (\text{S85})$$

with A computed either by (Hermitian) eigendecomposition or Cholesky factorization of M . Then Y is the minimum- ℓ^2 -distance projection of X onto (M, v) . This problem is not DCP; however, since there are no matrix cone constraints, we can simply use non-DCP solvers compatible with nonlinear constraints, such as SLSQP [39]. The constraint Jacobian for (S82) is $2M\hat{x}_i$.

C. Fitting hyperplanes in d dimensions

The equation of a hyperplane H in d dimensions is

$$n \cdot (x - v) = 0 \quad \forall x \in H \quad (\text{S86})$$

for $n, v \in \mathbb{R}^d$. Without loss of generality, we may assume that $\|n\| = 1$, so that n is a unit normal to H ; expressing H in the form

$$n \cdot x - b = 0 \quad (\text{S87})$$

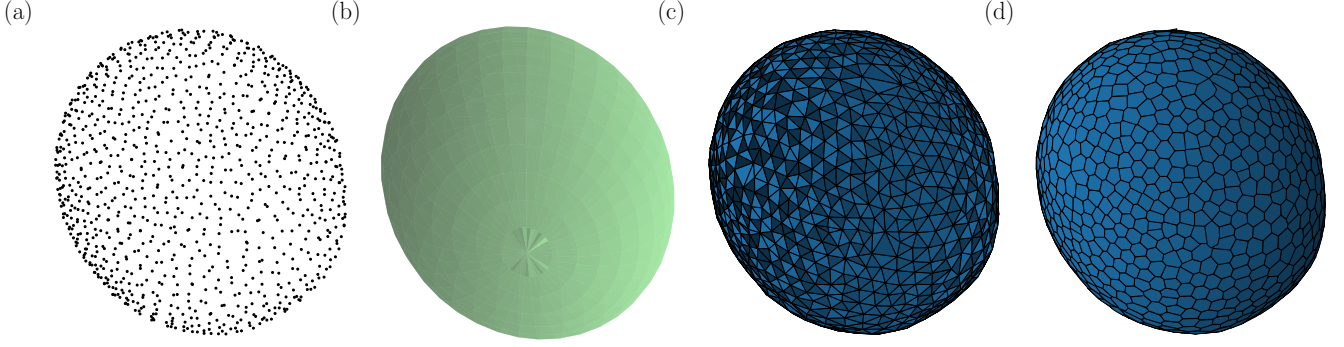


FIG. S3. Computational pipeline for extracting geometric features of *V. carteri*. (a) The somatic cell positions, available from [5], displayed in three-dimensional space. (b) The minimum-volume bounding ellipsoid containing the somatic cells is computed as in Section VB1. This establishes the approximate anterior-posterior axis of the spheroid. (c) The ℓ^2 -orthogonal projection of the somatic cell positions X to this ellipsoid are computed as in Section VB2, and the Delaunay tetrahedralization of this projected point cloud $\tilde{X}$ is computed. As $\tilde{X}$ is now its own convex hull, the index triplets corresponding to the triangular faces lying on the convex hull of the tetrahedral complex are taken to be an approximate Delaunay triangulation of the original point cloud X . (d) An approximate Voronoi tessellation of the surface is computed by sub-triangulating the Delaunay triangulation, connecting the circumcenters of adjacent triangular faces. The best-fit planes approximating these curved faces are computed as in Section VC, and the corresponding planar embeddings are treated as closed plane curves for downstream analysis as in Section III.

we see that $b = n \cdot v$ is the distance from the origin to H . It further follows that the distance from an arbitrary point $y \in \mathbb{R}^d$ to the plane is

$$d(y, H) = |n \cdot (y - v)| = |n \cdot y - b|. \quad (\text{S88})$$

Now, let $\{x_i\}_{i=1}^N = X \in \mathbb{R}^{N \times d}$ be a set of data points with $N \geq 3$. The best-fit hyperplane (in the ℓ^2 sense) is

$$\inf_{n, v} \|(X - v)n\|_2^2 \quad (\text{S89})$$

$$\text{subject to } \|n\|_2 = 1. \quad (\text{S90})$$

The cost is bi-convex in the parameters v and n . Calculating the gradient of the cost with respect to v , we find:

$$0 = \frac{\partial}{\partial v} \|(X - v)n\|_2^2 = \sum_{i=1}^N (x_i - v) \quad (\text{S91})$$

Thus $v = \frac{1}{N} \sum_{i=1}^N x_i$ is the centroid. Letting $\bar{X} = X - v$, the gradient of the cost with respect to n is:

$$0 = \frac{\partial}{\partial n} \|\bar{X}n\|_2^2 = 2\bar{X}^\top \bar{X}n \quad (\text{S92})$$

A common numerically stable method to estimate n is the singular vector of $\bar{X}$ corresponding to the smallest singular value, which is 0 if $\{x_i\}$ are coplanar. Moreover, by (S88) it follows that the ℓ^2 -orthogonal projection of X onto H is

$$Y = X - (X - v)nn^\top. \quad (\text{S93})$$

Let $u \in \mathbb{R}^d$ be a random vector such that $u \times n \neq 0$ (e.g. a Gaussian vector, for which this is almost surely true); then a planar embedding X_H of X is defined by a random orthonormal basis:

$$B = \frac{1}{\|v\|} [v \quad v \times n], \quad v = u \times n, \quad (\text{S94})$$

$$X_H = XB. \quad (\text{S95})$$

D. 3D reconstruction & analysis of *Volvox*

First, 3D meshes and an ellipsoidal approximation of the organism’s surface are constructed using the procedure detailed in Figure S3. Then, the anterior section—typically free of reproductive cells (gonidia) which interrupt the positions of somatic cells at the surface—is determined by separating cells into two subsets lying on either side of the plane normal to the ellipsoid’s major axis. The subset with smaller cell counts is used as a (generally good) approximation of the anterior side. Finally, the areas of the Voronoi faces are converted to solid-angles (4π times the area fraction of total) and are filtered by the cutoff $v_c = 0.007$ specified in [5]. The empirical area distribution for each organism is shifted by this cutoff and nondimensionalized to empirical mean 1, at which point they are combined across all 6 organisms.
