Mapping of the classical kinetic balance equations onto the Pauli equation

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Abstract.
Here we find a mapping onto the Pauli equation of the first two balance equations derived from the classical Boltzmann equation. The essence of this mapping, which we previously used to obtain the particular case of the Sturm-Liouville operator known as Schrödinger's equation, consists of applying a Fourier transform to the momentum coordinate of the distribution function. This procedure introduces a natural parameter η with units of angular momentum. The main differences between the two cases are the conditions imposed on the probability distribution function, a difference most clearly understood at the level of the hydrodynamic equations generated in the first steps of the mapping. The case leading to the Sturm-Liouville operator corresponds to an irrotational flow, while here the ansatz leading to the Pauli equation corresponds to a fluid with non-zero vorticity. In the context of the fluid dynamics the magnitude of the angular momentum associated with the vorticity is $\eta/2$. To perform the mapping we follow the standard technique common in hydrodynamic problems, namely writing the Lagrangian for the Euler equations with the corresponding constraints expressed in terms of Clebsch variables.

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1. Introduction

One of the most intriguing aspects of quantum mechanics is the spin and its apparent lack of a classical equivalent. Through the years many attempts have been made to find a statistical interpretation of this physical quantity, in some cases within fluid mechanics [1, 2, 3, 4], in others using the dynamics of a dipole [5], and more recently within the context of many body physics [6]. As mentioned in a previous paper [7], the fact that fluid equations of motion are derivable from kinetic theory suggests that the same might be true of the hydrodynamic description of quantum mechanics. In that work, we showed that such an underlying kinetic approach exists for spinless quantum mechanics. Here we extend this work to show that it is also possible to find such a kinetic description for the Pauli equation. The derivation relies upon a Fourier transform on the momentum coordinate ($\mathbf{p}$) of the classical Boltzmann equation, and leads to the fluid Euler equations in $\mathbf{p}$-conjugate space. In the presence of vorticity they are more conveniently written in terms of a "new" set of variables that associate themselves with the spin in quite a natural way. The procedure we follow generates a unique free parameter in the theory that we call η. This single parameter has the units of angular momentum, the same as $\hbar$. Not without optimism we could view them as one and the same, given the striking fact that the transformed Euler equations can generate either a term identical to the quantum potential in the Madelung [8] representation of quantum mechanics or a term identical to the stress tensor found by Takabayasi [1] to correspond to the quantum potential accompanied by the presence of spin. If nothing else, the transformations we apply in this work, closely following the methods developed previously [7], lead to a classical way of thinking about the spin of a particle.

2. The Mapping

It is well known that the motion of an ensemble of N classical particles governed by Liouville's equation can be recast in a hierarchy of non-linear partial differential equations (PDEs) for the reduced probability functions defined as follows:

$$f_N(\mathbf{x}_1, \mathbf{p}_1, \ldots, \mathbf{x}_N, \mathbf{p}_N) = \frac{D}{\int_\Omega D d\Omega} \tag{1}$$

and for $1 \leq j < N$

$$f_j(\mathbf{x}^j, \mathbf{p}^j) = \int_\Omega f_N(\mathbf{x}^N, \mathbf{p}^N) \prod_{l=j+1}^{N} d\mathbf{x}_l d\mathbf{p}_l \tag{2}$$

where D represents the number density of points in phase space, Ω the volume in phase space and $(\mathbf{x}^N, \mathbf{p}^N) = (\mathbf{x}_1, \mathbf{p}_1, \ldots, \mathbf{x}_N, \mathbf{p}_N)$. These functions correspond to the probability of finding the subsystem of $j < N$ particles in the phase volume $\prod_{l=1}^{j} d\mathbf{x}_l d\mathbf{p}_l$ about the state $(\mathbf{x}_1, \mathbf{p}_1, \ldots, \mathbf{x}_j, \mathbf{p}_j)$. The N PDEs generated are known as the BBKGY hierarchy [9], the first two members of which (i.e. the equations for f_1 and f_2) determine the kinetic and potential energy of an aggregate of particles, and have a crucial role in fluid dynamics. One way to attempt a solution of these equations is to decouple them through an ansatz with regard to the properties of the functions f_j. When the Bogoliubov ansatz is imposed, the resulting equation for $f_1 = f_1(\mathbf{x}_1, \mathbf{p}_1, t)$ is the Boltzmann equation:

$$\begin{aligned} &\frac{\partial f_1}{\partial t} + \frac{\mathbf{p}_1}{m} \cdot \frac{\partial f_1}{\partial \mathbf{x}_1} + \mathbf{F}(\mathbf{x}_1, \mathbf{p}_1) \cdot \frac{\partial f_1}{\partial \mathbf{p}_1} \\ &\qquad = \int r_2 dr_2 g \int d\mathbf{p}_2 \left[f_1(\mathbf{p}_1') f_1(\mathbf{p}_2') - f_1(\mathbf{p}_1) f_1(\mathbf{p}_2) \right] \end{aligned} \tag{3}$$

where $\mathbf{F}$ is the external force averaged over all other coordinates, g is the magnitude of the relative velocity defined as $\mathbf{g} = (\mathbf{p}_2 - \mathbf{p}_1)/m$ and where we used $d\mathbf{x}_2 = r_2 dr_2 d\phi dz$. The Boltzmann equation has the property that when integrated over the momentum coordinate $\mathbf{p}_1$ it produces the conservation law for the number of particles, when multiplied by $\mathbf{p}_1$ and integrated over $\mathbf{p}_1$ it gives the momentum balance equation and when multiplied by $\mathbf{p}_1^2$ it gives the energy balance equation after integration over the momentum. Moreover, the right-hand-side cancels out in all three cases [9]. Thus the result for our first two balance equations reads:

$$\int_{-\infty}^{+\infty} d\mathbf{p} \left(\frac{\partial f_1}{\partial t} + \frac{\mathbf{p}}{m} \cdot \frac{\partial f_1}{\partial \mathbf{x}} \right) = 0 \tag{4}$$

and

$$\int_{-\infty}^{+\infty} \mathbf{p}\ d\mathbf{p} \left(\frac{\partial f_1}{\partial t} + \frac{\mathbf{p}}{m} \cdot \frac{\partial f_1}{\partial \mathbf{x}} + \mathbf{F}(\mathbf{x}, \mathbf{p}) \cdot \frac{\partial f_1}{\partial \mathbf{p}} \right) = 0 \tag{5}$$

where we have dropped the subindex 1 from $\mathbf{x}_1$ and $\mathbf{p}_1$ and also have assumed that any surface terms vanish due to the convergence properties of f_1. We will study the particular case for which $\mathbf{F}$ corresponds to the electromagnetic force on particles with charge q and magnetic moment $\mathbf{m}$,

$$\mathbf{F} = q\mathbf{E} + \frac{q}{c}\frac{\mathbf{p}}{m} \times \mathbf{B} + (\nabla \mathbf{B})\mathbf{m}\ . \tag{6}$$

We now introduce into (4) and (5) the following representation for f_1,

$$f_1(\mathbf{x}, \mathbf{p}, t) = \frac{1}{(2\pi\eta)^3} \int_{-\infty}^{+\infty} \exp\left(-i\frac{\mathbf{p}\cdot\mathbf{y}}{\eta} \right) \hat{f}(\mathbf{x}, \mathbf{y}, t) d\mathbf{y} \tag{7}$$

where $\hat{f}(\mathbf{x}, \mathbf{y}, t)$ is of course given by

$$\hat{f}(\mathbf{x}, \mathbf{y}, t) = \int_{-\infty}^{+\infty} \exp\left(i\frac{\mathbf{p}\cdot\mathbf{y}}{\eta} \right) f_1(\mathbf{x}, \mathbf{p}, t) d\mathbf{p}\ . \tag{8}$$

With these definitions and some straightforward algebra, equations 4 and 5 can be written as [7]

$$\lim_{\mathbf{y}\to 0} \left[\frac{\partial \hat{f}}{\partial t} + \frac{\eta}{im}\frac{\partial}{\partial \mathbf{x}} \cdot \frac{\partial \hat{f}}{\partial \mathbf{y}} \right] = 0 \tag{9}$$

$$\lim_{\mathbf{y}\to 0} \left[\frac{\partial}{\partial t}\left(\frac{\eta}{i}\frac{\partial \hat{f}}{\partial \mathbf{y}} \right) - \frac{\eta^2}{m}\frac{\partial}{\partial \mathbf{x}} \cdot \left(\frac{\overleftrightarrow{\partial^2 \hat{f}}}{\partial \mathbf{y}\partial \mathbf{y}} \right) - \mathbf{F}\cdot\hat{f} \right] = 0\ . \tag{10}$$

It is interesting to notice that two of the limits correspond to the following averages:

$$\begin{aligned} \lim_{\mathbf{y}\to 0} \hat{f} &= \lim_{\mathbf{y}\to 0} \int_{-\infty}^{+\infty} \exp\left(i\frac{\mathbf{p}\cdot\mathbf{y}}{\eta} \right) f_1(\mathbf{x}, \mathbf{p}, t) d\mathbf{p} \\ &= \int_{-\infty}^{+\infty} f_1(\mathbf{x}, \mathbf{p}, t)\ \ d\mathbf{p} = \frac{\rho(\mathbf{x}, t)}{m} \end{aligned} \tag{11}$$

$$\begin{aligned} \lim_{\mathbf{y}\to 0} \frac{\partial \hat{f}}{\partial \mathbf{y}} &= \lim_{\mathbf{y}\to 0} \frac{\partial}{\partial \mathbf{y}} \int_{-\infty}^{+\infty} \exp\left(i\frac{\mathbf{p}\cdot\mathbf{y}}{\eta} \right) f_1(\mathbf{x}, \mathbf{p}, t) d\mathbf{p} \\ &= \frac{i}{\eta} \int_{-\infty}^{+\infty} \mathbf{p}\ \ f_1(\mathbf{x}, \mathbf{p}, t) d\mathbf{p} \\ &= \frac{i}{\eta}\rho(\mathbf{x}, t)\mathbf{u}(\mathbf{x}, t) \end{aligned} \tag{12}$$

where we have defined the mean velocity $\mathbf{u}$ as the average, over the momentum $\mathbf{p}$ alone, of $\mathbf{p}/m$, with m the mass. We can see from these expressions that $\hat{f}$ is the generating function for the averages with respect to $\mathbf{p}$. Replacing these values into the balance equations we obtain the fluid equations

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho \mathbf{u})}{\partial \mathbf{x}} = 0 \tag{13}$$

and

$$\frac{1}{m}\frac{\partial(\rho \mathbf{u})}{\partial t} - \frac{\eta^2}{m}\lim_{\mathbf{y}\to 0}\frac{\partial}{\partial \mathbf{x}} \cdot \left(\frac{\overleftrightarrow{\partial^2 \hat{f}}}{\partial \mathbf{y} \partial \mathbf{y}} \right) - \frac{\rho}{m}\mathbf{F} = 0 \ . \tag{14}$$

To evaluate the tensor we introduce the canonical change of variables $\mathbf{y} = \mathbf{x}' - \mathbf{x}''$ and $\mathbf{x} = (\mathbf{x}' + \mathbf{x}'')/2$, which satisfies the following relationships:

$$\begin{aligned} \mathbf{x}' &= \mathbf{x} + \frac{\mathbf{y}}{2} \ , \qquad & \mathbf{x}'' &= \mathbf{x} - \frac{\mathbf{y}}{2} \\ \frac{\partial}{\partial \mathbf{y}} &= \frac{1}{2}\left(\frac{\partial}{\partial \mathbf{x}'} - \frac{\partial}{\partial \mathbf{x}''} \right) \ , & \frac{\partial}{\partial \mathbf{x}} &= \left(\frac{\partial}{\partial \mathbf{x}'} + \frac{\partial}{\partial \mathbf{x}''} \right) \ . \end{aligned} \tag{15}$$

Notice that the limit $\mathbf{y} \to 0$ corresponds to $\mathbf{x}' \to \mathbf{x}''$, $\mathbf{x}' = \mathbf{x}'' \equiv \mathbf{x}$. It has been pointed out previously [7] that a subclass of solutions of this equations corresponds to the case in which $\hat{f}$ is separable in the variables $\mathbf{x}'$ and $\mathbf{x}''$. Since $\hat{f}$ must be real in the limit $\mathbf{y} \to 0$ full separability, (i.e. $\hat{f}(\mathbf{x}', \mathbf{x}'', t) = h'(\mathbf{x}', t)h''(\mathbf{x}'', t)$) corresponds to h' and h'' being complex conjugate of each other, leaving only two independent real functions, and, as we will see below, this is not a sufficient number of independent functions to consider vortical flows [10]. With this in mind we propose that $\hat{f}$ is of the form

$$\hat{f}(\mathbf{x}', \mathbf{x}'', t) = h'(\mathbf{x}', t)h''(\mathbf{x}'', t) + g'(\mathbf{x}', t)g''(\mathbf{x}'', t) \tag{16}$$

where $h'(\mathbf{x}', t)$, $h''(\mathbf{x}'', t)$, $g'(\mathbf{x}', t)$ and $g''(\mathbf{x}'', t)$ are complex functions, which effectively correspond to eight real functions. This ansatz for $\hat{f}$ is not a very general one, but is less restrictive than considering full separability. The new ansatz for $\hat{f}$ on the other hand gives the minimum number of necessary variables to consider vortical flow still within the restrictive subclass of linear combinations of separable solutions. In this case, requiring that $\hat{f}$ be real in the limit $\mathbf{y} \to 0$ leads to h' and g' being the complex conjugate of h'' and g'', respectively, leaving four independent real functions to be found (corresponding to the three components of the velocity and the fluid density). It could be argued, rightfully so, that this subclass of solutions is a very particular one, since in most cases the initial and/or boundary conditions of the problem to be solved will not be separable and thus neither will be the solution. The relevance of these solutions lies in the fact that they are the ones that lead either to the Schrödinger equation, when full separability is invoked, or to the Pauli equation, when the extra freedom to consider vorticity is required. We are now ready to evaluate the tensor in the momentum balance equation. First, write its k, l-component as

$$\lim_{\mathbf{y}\to 0}\frac{\partial}{\partial y_k}\frac{\partial}{\partial y_l}\hat{f}(\mathbf{x}', \mathbf{x}'', t)$$
$$= \lim_{\mathbf{x}', \mathbf{x}''\to \mathbf{x}} \frac{1}{4}\left(\frac{\partial}{\partial x'_k} - \frac{\partial}{\partial x''_k} \right)\left(\frac{\partial}{\partial x'_l} - \frac{\partial}{\partial x''_l} \right) [h'h'' + g'g''] \ .$$

This yields the result

$$\lim_{\mathbf{y}\to 0}\frac{\partial}{\partial y_k}\frac{\partial}{\partial y_l}m\hat{f}(\mathbf{x}',\mathbf{x}'',t)$$
$$=\frac{1}{4}\Bigg[\psi_1^*\frac{\partial^2\psi_1}{\partial x_k\partial x_l}+\psi_2^*\frac{\partial^2\psi_2}{\partial x_k\partial x_l}-\frac{\partial\psi_1}{\partial x_k}\frac{\partial\psi_1^*}{\partial x_l}-\frac{\partial\psi_2}{\partial x_k}\frac{\partial\psi_2^*}{\partial x_l}$$
$$-\frac{\partial\psi_1^*}{\partial x_k}\frac{\partial\psi_1}{\partial x_l}-\frac{\partial\psi_2^*}{\partial x_k}\frac{\partial\psi_2}{\partial x_l}+\psi_1\frac{\partial^2\psi_1^*}{\partial x_k\partial x_l}+\psi_2\frac{\partial^2\psi_2^*}{\partial x_k\partial x_l}\Bigg]$$

where we have defined

$$\begin{aligned}\lim_{\mathbf{y}\to 0}\sqrt{m}\ h'(\mathbf{x}',t) &= \psi_1(\mathbf{x},t)\\ \lim_{\mathbf{y}\to 0}\sqrt{m}\ h''(\mathbf{x}'',t) &= \psi_1^*(\mathbf{x},t)\\ \lim_{\mathbf{y}\to 0}\sqrt{m}\ g'(\mathbf{x}',t) &= \psi_2(\mathbf{x},t)\\ \lim_{\mathbf{y}\to 0}\sqrt{m}\ g''(\mathbf{x}'',t) &= \psi_2^*(\mathbf{x},t)\ .\end{aligned} \qquad (17)$$

Introducing the notation

$$\Psi=\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}\qquad \Psi^\dagger=(\psi_1^*,\psi_2^*) \qquad (18)$$

the expressions for the density ρ and the mean velocity $\mathbf{u}$ as given by (11) and (12) can be rewritten as $\rho=\Psi^\dagger\Psi$ and $\mathbf{u}=-(\eta/2mi)(\Psi^\dagger\nabla\Psi-(\nabla\Psi^\dagger)\Psi)$. These definitions allows us also to rewrite (17) in a much more convenient and instructive way

$$\lim_{\mathbf{y}\to 0}\frac{\partial}{\partial y_k}\frac{\partial}{\partial y_l}m\hat{f}(\mathbf{x}',\mathbf{x}'',t)=\frac{1}{4}\left[\rho\frac{\partial^2\ln\rho}{\partial x_k\partial x_l}-4\rho\frac{m^2}{\eta^2}u_k u_l-\rho\frac{\partial\Sigma_i}{\partial x_k}\frac{\partial\Sigma_i}{\partial x_l}\right] \qquad (19)$$

where repeated indices indicate a summation, u_k and u_l are the k and l components of the average velocity $\mathbf{u}(\mathbf{x},t)$ and the quantities Σ_i have been defined as

$$\Sigma_i=\frac{\Psi^\dagger\sigma_i\Psi}{\Psi^\dagger\Psi} \qquad (20)$$

and σ_i are the Pauli matrices.

Finally, our two balance equations in Fourier space read

$$\frac{\partial\rho}{\partial t}+\boldsymbol{\nabla}\cdot(\rho\mathbf{u})=0 \qquad (21)$$

and

$$\frac{\partial}{\partial t}(m\rho u_k)=-\partial_j\left[\rho\left(mu_k u_j-\frac{\eta^2}{4m}\frac{\partial^2\ln\rho}{\partial x_k\partial x_j}+\frac{\eta^2}{4m}\frac{\partial\Sigma_i}{\partial x_k}\frac{\partial\Sigma_i}{\partial x_j}\right)\right]-\rho F_k \qquad (22)$$

where $\boldsymbol{\nabla}\equiv\partial/\partial\mathbf{x}$. It is interesting to notice that the only difference between this pair of balance equations and the ones obtained with the usual method of averaging without Fourier transforming is the presence of terms proportional to η^2 instead of an arbitrary pressure tensor. The tensor in equation 22 can be expressed in a more transparent and meaningful manner if we pay careful attention to the term involving the magnitudes Σ_i. As they are written, they can be thought of as the components of a dimensionless unit vector. We can absorb the constant $\eta^2/4$ into the vector product and define a new vector $\mathbf{s}$ with magnitude $\eta/2$ as

$$\mathbf{s}=\frac{\eta}{2}\boldsymbol{\Sigma}=\frac{\eta}{2}\frac{\Psi^\dagger\hat{\sigma}\Psi}{\Psi^\dagger\Psi} \qquad (23)$$

where $\hat{\sigma}$ is the vector whose components are the Pauli matrices. Given the units of η, this choice allows us to associate $\mathbf{s}$ with an angular momentum. Recalling that $\mathbf{m} = (e/mc)\mathbf{L}$, where $\mathbf{L}$ is a characteristic angular momentum, it is natural to identify $\mathbf{L} \equiv \mathbf{s}$. Then, when (22) is transformed into the standard form of Euler's equations by using the continuity relation to eliminate the time derivative of ρ, it becomes

$$m(\partial_t u_k + u_j \partial_j u_k) - \frac{1}{m\rho}\partial_j \left[\rho\left(\frac{\eta^2}{4}\frac{\partial^2 \ln\rho}{\partial x_k \partial x_j} - \partial_k s_i \partial_j s_i\right)\right] + qE_k + \frac{q}{c}(\mathbf{u}\times\mathbf{B})_k + \frac{q}{mc}s_j \partial_k B_j = 0 \qquad (24)$$

where we have used the notation $\partial_i \equiv \partial/\partial x_i$. This is identical to the hydrodynamic equation Takabayasi [1, 2] found by making a suitable change of variables on the Pauli equation.

If $\mathbf{s} = (\eta/2)(\Psi^* \hat{\sigma}\Psi/\Psi^*\Psi)$ is considered to be a new variable, then it is necessary to find its equation of motion. This is most easily done by observing that the term containing $\mathbf{s}$ can be expressed as a gradient plus a contribution of the same form as the last term in (24),

$$m(\partial_t u_k + u_j \partial_j u_k) - \partial_k\left[\frac{\eta^2}{2m}\frac{\partial_j \partial_j \rho^{1/2}}{\rho^{1/2}} - \frac{1}{2m}\partial_j s_i \partial_j s_i\right] + \frac{1}{m}\partial_k\left[\frac{1}{\rho}\partial_j(\rho\partial_j s_i) + \frac{q}{c}B_i\right]s_i + qE_k + \frac{q}{c}(\mathbf{u}\times\mathbf{B})_k = 0\ . \qquad (25)$$

This rewriting of the momentum balance equation is the same as the one presented by Takabayasi [2], and is very enlightening with regard to the role of $\mathbf{s}$. As he observed, the extra term plays the role of an "effective" space dependent magnetic field, and thus the equation of motion for the variable $\mathbf{s}$ is given by

$$\begin{aligned}\frac{d\mathbf{s}}{dt} &= \frac{1}{m}\mathbf{s}\times\mathbf{\Omega} \\ &= \frac{q}{mc}\mathbf{s}\times\mathbf{B} + \frac{1}{m\rho}\mathbf{s}\times\partial_k(\rho\partial_k\mathbf{s}) \\ &= \frac{q}{mc}\mathbf{s}\times\mathbf{B}_{eff}\end{aligned} \qquad (26)$$

where $\mathbf{\Omega} = \mathbf{\nabla}\times\mathbf{u}$ is the vorticity. The presence of a non-zero vorticity makes it harder to find the Lagrangian of the system and its associated Hamilton-Jacobi equation. As noted elsewhere [7], when $\hat{f}$ is fully separable, and thus the vorticity $\mathbf{\Omega} = 0$, it is possible to use the classical definition of the action S in terms of the momentum ($\mathbf{p} = \mathbf{\nabla}S$) and reduce the equations of motion to a Hamilton-Jacobi equation. In the present case, this is not possible anymore in so simple a fashion. Not even with the introduction of the canonical momentum $\mathbf{p} = \mathbf{\nabla}S - (q/c)\mathbf{A}$, where $\mathbf{A}$ is the vector potential, is it possible to rewrite (25) as a total gradient. In the presence of the non-zero vorticity it is necessary to use a more general expression for the true canonical momentum. In fact, the most general way to write the canonical momentum in three dimensions when there is no helicity [11] is

$$\mathbf{p} = \mathbf{\nabla}S - \frac{q}{c}\mathbf{A} + \zeta\mathbf{\nabla}\omega\ . \qquad (27)$$

The new variables ζ and ω are known as Clebsch potentials [12]. In our particular case, they have a very specific physical meaning as a consequence of representing $\mathbf{s}$ with the Pauli matrices: ζ corresponds the z-component of the vector $\mathbf{s}$ and ω corresponds to the azimuthal angle, i.e, the canonical conjugate variable of s_z. This correspondence can be seen when the curl of (27) is taken

$$\begin{aligned}\boldsymbol{\Omega} &= \frac{1}{m}\boldsymbol{\nabla}\times\mathbf{p}\\ &= \frac{q}{mc}\mathbf{B}+\frac{1}{m}\boldsymbol{\nabla}\zeta\times\boldsymbol{\nabla}\omega\\ &= \frac{q}{mc}\mathbf{B}+\frac{1}{m\rho}\partial_k(\rho\partial_k\mathbf{s})\ .\end{aligned} \tag{28}$$

Then, the expressions for ζ and ω as a function $\mathbf{s}$ are

$$\begin{aligned}\zeta &= s_z\\ \omega &= \tan^{-1}\left(\frac{s_x}{s_y}\right)\ .\end{aligned} \tag{29}$$

Since the equivalence between equations 13, 24 and 26 and the Pauli equation for a non-relativistic particle with spin $\eta/2$ has already been proven by different authors [1, 4] in what follows we will only summarize the procedure. As a first step we will write the action that will give rise to the equations of motion when its variation is performed. Then, we will introduce a change of variables into that action. This change of variables is such that its variation respect of the new coordinates yields Pauli's equation. This is not a conceptually complicated procedure, however it involves lengthy calculations. The action that corresponds to our equations of motion is

$$\begin{aligned}\mathcal{A} = \int d^3\mathbf{x}\int dt\Big[&\frac{1}{2}m\rho\mathbf{u}^2 - q\rho\Phi + \frac{q}{c}\rho\mathbf{A}\cdot\mathbf{u} - \frac{q}{mc}\rho\mathbf{B}\cdot\mathbf{s}\\ &-\frac{\eta^2}{8m}\frac{(\boldsymbol{\nabla}\rho)^2}{\rho} - \frac{\rho}{2m}\left[(\boldsymbol{\nabla}s_x)^2+(\boldsymbol{\nabla}s_y)^2+(\boldsymbol{\nabla}s_z)^2\right]\Big]\end{aligned} \tag{30}$$

with the Lin constraints $\partial\omega/\partial t+(\mathbf{u}\cdot\boldsymbol{\nabla})\omega = 0$ and $\partial S/\partial t+(\mathbf{u}\cdot\boldsymbol{\nabla})S = 0$, and Lagrange multipliers $\rho\zeta$ and ρ respectively. The first constraint is not conserved and this will lead to its modification, by fixing the gauge of the Clebsch potentials [1]. The second constrain corresponds to the continuity equation which is indeed conserved. This action can be fully expressed as a function of the Clebsch potentials making use of the following identities [5]:

$$\mathbf{B}\cdot\mathbf{s} = \left(\frac{\eta^2}{4}-\zeta^2\right)^{1/2}\left[B_x\sin\omega + B_y\cos\omega\right] + \zeta B_z \tag{31}$$

and

$$(\boldsymbol{\nabla}s_x)^2 + (\boldsymbol{\nabla}s_y)^2 + (\boldsymbol{\nabla}s_z)^2 \ = \frac{\eta^2}{4}\left[\frac{(\boldsymbol{\nabla}\zeta)^2}{(\eta^2/4-\zeta^2)} + \frac{4}{\eta^2}\left(\frac{\eta^2}{4}-\zeta^2\right)(\boldsymbol{\nabla}\omega)^2\right]\ . \tag{32}$$

To obtain the equations of motion we need to calculate the variation of $\mathcal{A}$ respect of

$\mathbf{u}$, ρ, S, ζ and ω. Then,

$$
\begin{aligned}
\frac{\delta\mathcal{A}}{\delta\mathbf{u}} &: m\mathbf{u} + \frac{q}{c}\mathbf{A} - \zeta\boldsymbol{\nabla}\omega - \boldsymbol{\nabla}S = 0 \\
\frac{\delta\mathcal{A}}{\delta\rho} &: \frac{\partial S}{\partial t} + \zeta\frac{\partial\omega}{\partial t} + \frac{1}{2m}\left(\boldsymbol{\nabla}S + \zeta\boldsymbol{\nabla}\omega - \frac{q}{c}\mathbf{A}\right)^2 \\
&\quad - \frac{q}{mc}\mathbf{B}\cdot\mathbf{s} - q\Phi - \frac{\eta^2}{2m}\frac{\boldsymbol{\nabla}^2\rho^{1/2}}{\rho^{1/2}} \\
&\quad + \frac{\eta^2}{8m}\left[\frac{(\boldsymbol{\nabla}\zeta)^2}{(\eta^2/4-\zeta^2)} + \frac{4}{\eta^2}\left(\frac{\eta^2}{4}-\zeta^2\right)(\boldsymbol{\nabla}\omega)^2\right] = 0 \\
\frac{\delta\mathcal{A}}{\delta S} &: \frac{\partial\rho}{\partial t} + \boldsymbol{\nabla}\cdot(\rho\mathbf{u}) = 0 \\
\frac{\delta\mathcal{A}}{\delta\zeta} &: \frac{\partial\omega}{\partial t} + (\mathbf{u}\cdot\boldsymbol{\nabla})\omega \\
&\quad - \frac{q}{mc}\left[\frac{\zeta}{(\eta^2/4-\zeta^2)}(B_x\sin\omega + B_y\cos\omega) - B_z\right] \\
&\quad - \zeta\frac{\eta^2}{4m}\left[-\frac{(\boldsymbol{\nabla}\zeta)^2}{(\eta^2/4-\zeta^2)^2} + \frac{4}{\eta^2}(\boldsymbol{\nabla}\omega)^2\right] \\
&\quad - \frac{\eta^2}{2m}\frac{1}{\rho}\boldsymbol{\nabla}\left(\rho\frac{(\boldsymbol{\nabla}\zeta)^2}{(\eta^2/4-\zeta^2)}\right) = 0 \\
\frac{\delta\mathcal{A}}{\delta\omega} &: \frac{\partial\zeta}{\partial t} + (\mathbf{u}\cdot\boldsymbol{\nabla})\zeta \\
&\quad - \frac{q}{mc}\left(\frac{\eta^2}{4}-\zeta^2\right)^{1/2}[B_x\cos\omega - B_y\sin\omega] \\
&\quad + \frac{1}{m\rho}\boldsymbol{\nabla}\left[\rho\left(\frac{\eta^2}{4}-\zeta^2\right)\boldsymbol{\nabla}\omega\right] = 0\ .
\end{aligned}
\tag{33}
$$

The first three equations correspond to the canonical momentum, the equation of motion for $\mathbf{u}$ and continuity and the last two correspond to the equation of motion for $\mathbf{s}$. The equation of motion for $\mathbf{u}$ can be recast in the form (24) by taking its gradient and then replacing into it the expressions for $d\omega/dt$ and $d\zeta/dt$ obtained from the last two variations, where $d/dt \equiv \partial/\partial t + (\mathbf{u}\cdot\boldsymbol{\nabla})$.

Replacing the expression for the generalized momentum and the Lin constraints into the action $\mathcal{A}$, we find

$$
\begin{aligned}
\mathcal{A} = &- \int d^3\mathbf{x}\int dt\ \rho\left[\frac{1}{2m}(\boldsymbol{\nabla}S + \zeta\boldsymbol{\nabla}\omega - \frac{q}{c}\mathbf{A})^2\right. \\
&+ \frac{\partial S}{\partial t} + \zeta\frac{\partial\omega}{\partial t} + \frac{q}{mc}\mathbf{B}\cdot\mathbf{s} - q\Phi + \frac{\eta^2}{8m}\frac{(\boldsymbol{\nabla}\rho)^2}{\rho^2} \\
&\left. + \frac{1}{2m}\left[(\boldsymbol{\nabla}s_x)^2 + (\boldsymbol{\nabla}s_y)^2 + (\boldsymbol{\nabla}s_z)^2\right]\right]
\end{aligned}
\tag{34}
$$

where, in order to keep the equations more compact, we have not substituted the expressions (31) and (32). Motivated by the fact that $\mathbf{s}$ is a vector with constant magnitude $\eta/2$, it can be expressed as

$$
\mathbf{s} = \frac{\eta}{2}\left[\sin\theta\sin\omega\hat{\mathbf{x}} + \sin\theta\cos\omega\hat{\mathbf{y}} + \cos\theta\hat{\mathbf{z}}\right]\ . \tag{35}
$$

Now introducing the following change of variables into $\mathcal{A}$

$$\begin{aligned} \zeta &= \frac{\eta}{2}\cos\theta \\ \Psi &= R\mathrm{e}^{(i/\eta)S}\begin{pmatrix} \cos\frac{\theta}{2}\mathrm{e}^{i\omega/2} \\ i\sin\frac{\theta}{2}\mathrm{e}^{-i\omega/2} \end{pmatrix} \end{aligned} \tag{36}$$

we obtain

$$\begin{aligned} \mathcal{A} = \int d^3\mathbf{x}\int dt \Bigg[& \frac{i\eta}{2}\left(\Psi^\dagger\frac{\partial\Psi}{\partial t} - \frac{\partial\Psi^\dagger}{\partial t}\Psi\right) \\ & - \frac{1}{2m}\left(i\eta\boldsymbol{\nabla}\Psi^\dagger - \frac{q}{c}\Psi^\dagger\mathbf{A}\right)\left(-i\eta\boldsymbol{\nabla}\Psi - \frac{q}{c}\mathbf{A}\Psi\right) \\ & - q\Phi\Psi^\dagger\Psi + \frac{q\eta}{2mc}\Psi^\dagger\hat{\sigma}\cdot\mathbf{B}\Psi\Bigg] \end{aligned} \tag{37}$$

Finally when the variation of $\mathcal{A}$ respect of $\Psi^\dagger$ is performed we obtain:

$$i\eta\frac{\partial\Psi}{\partial t} = \frac{1}{2m}\left(-i\eta\boldsymbol{\nabla} - \frac{e}{c}\mathbf{A}\right)^2\Psi + e\Phi\Psi - \frac{e\eta}{2mc}\hat{\sigma}\cdot\mathbf{B}\Psi \tag{38}$$

which is Pauli's equation for a particle with spin $\eta/2$ and charge $q = e$. As shown by Takabayasi [1], the relation (36) along with the Kelvin-Helmholtz theorem for the hydrodynamic equations leads to the usual quantization of $\mathbf{s}$.

3. Conclusions

This work has been based on the premise of our previous study in which a Fourier transform on the $\mathbf{p}$ (momentum) variable of the classical Boltzmann equation leads to a mapping onto the particular subclass of Sturm-Liouville operators known as the Schrödinger equation. As has been mentioned above, the mapping onto the Schrödinger equation is done by imposing a separability condition on the Fourier transform of the one particle distribution function. Also in our previous work, we showed that the rules to calculate the averages of physical quantities in the $\mathbf{p}$-conjugate space read like the postulates of quantum mechanics, and that the hydrodynamic equations that are generated as an intermediate step of the mapping present a term identical to the quantum potential when our parameter η is replaced by $\hbar$. Here we have shown that changing the ansatz for $\hat{f}$ leads to a new term in the hydrodynamic equations that coincides, upon replacement of the parameter η by $\hbar$, with the equations that map onto the Pauli equation for a particle with spin $\hbar/2 \equiv \eta/2$. Moreover, the value of the magnitude of the spin is naturally fixed to be $\eta/2$ and it is also a natural consequence of the rewriting of the equations that the spin is an angular momentum. It is also apparent from the derivation that the "spin" we find is an internal degree of freedom, introduced because of the need for more variables to account for vorticity. It is an interesting question whether one might infer properties of the underlying classical system associated with either one of the ansatzes for the distribution function. An answer to this question might give insight into how the role of the measurement apparatus in quantum mechanics might be understood.

Finally, we find very appealing the connection between Clebsch variables and $\mathbf{s}$ since the method of choice in quantum mechanics to work efficiently with angular momentum is a variant of the method of Clebsch, whose original intent was the representation of non-exact differential forms and which was widely used in the late 19th century to study vortical Eulerian flows.

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