# Dissipation

We need first to incorporate dissipative contributions into the usual Lagrangian formulation of classical dynamics. Assume the Lagrangian is

$$\mathcal{L}(q, \dot{q}) = T - V$$

The minimum action principle is

$$\delta S = 0, \quad S = \int dt \mathcal{L} \qquad \Rightarrow \qquad \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}} - \frac{\partial \mathcal{L}}{\partial q} = 0$$

With a kinetic energy term $T = (1/2)m\dot{q}^2$ the Euler-Lagrange equation is

$$m\ddot{q} = -\frac{\partial V}{\partial q} \ .$$

In the limit of zero inertia (vanishing Reynolds number), simply dropping the kinetic energy term leaves us with no dynamics at all, so we need to introduce a *generalized force* associated with dissipation. If the viscous force is $\zeta\dot{q}$, the rate of dissipation is $\zeta\dot{q}^2$. So, we introduce the *Rayleigh dissipation function*

$$\boxed{\mathcal{R} = \frac{\zeta}{2}\dot{q}^2 \ ,}$$

proportional to the rate of dissipation.

# Dissipation - continued

The new variational principle is

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}} - \frac{\partial \mathcal{L}}{\partial q} = -\frac{\partial \mathcal{R}}{\partial \dot{q}} = -\zeta \dot{q}$$

In the overdamped limit we then have the Aristotelian law

$$\zeta \dot{q} = -\frac{\partial U}{\partial q}$$

The simplest generalization of this to a moving curve involves a local drag coefficient, call it $\Gamma$, such that

$$\mathcal{R} = \frac{\Gamma}{2}\int d\alpha \sqrt{g}\mathbf{r}_t^2 \quad \text{with} \quad -\frac{\delta \mathcal{R}}{\delta \mathbf{r}_t} = -\Gamma\sqrt{g}\mathbf{r}_t \ ,$$

and by recognizing that the potential energy function $V$ is really our generalized energy functional E for the curve we arrive at the equation of motion

$$\boxed{\Gamma \mathbf{r}_t = -\frac{1}{\sqrt{g}}\frac{\delta \mathcal{E}}{\delta \mathbf{r}} \ .}$$

# Some Examples

The simplest example of an energy functional is the length $L$ of a curve,

$$L = \int_0^1 d\alpha \sqrt{g}$$

This is a function of $\mathbf{r}_\alpha$ alone, so

$$-\frac{1}{\sqrt{g}}\frac{\delta L}{\delta \mathbf{r}} = \frac{1}{\sqrt{g}}\frac{\partial}{\partial \alpha}\left[\frac{1}{2}\frac{1}{\sqrt{g}}2\mathbf{r}_\alpha\right] = \frac{\partial}{\partial s}\hat{\mathbf{t}} = \boxed{-\kappa\hat{\mathbf{n}}\ ,}$$

so restoring force from line tension is proportional to the curvature.

A second calculation involves the area $A$ enclosed by a (closed) curve,

$$A = \frac{1}{2}\int_0^1 d\alpha \mathbf{r} \times \mathbf{r}_\alpha$$

where the cross product is here interpreted as a scalar: $(\mathbf{a} \times \mathbf{b})_i = \epsilon_{ij}\mathbf{a}_i\mathbf{b}_j$. A short calculation shows that

$$\boxed{-\frac{1}{\sqrt{g}}\frac{\delta A}{\delta \mathbf{r}} = \hat{\mathbf{n}}}$$

# Elastohydrodynamics

We recall that the viscous drag on a sphere of radius $a$ in low Reynolds number flow is given by the Stokes formula

$$\mathbf{F} = \zeta \mathbf{r}_t \qquad \zeta = 6\pi\eta a$$

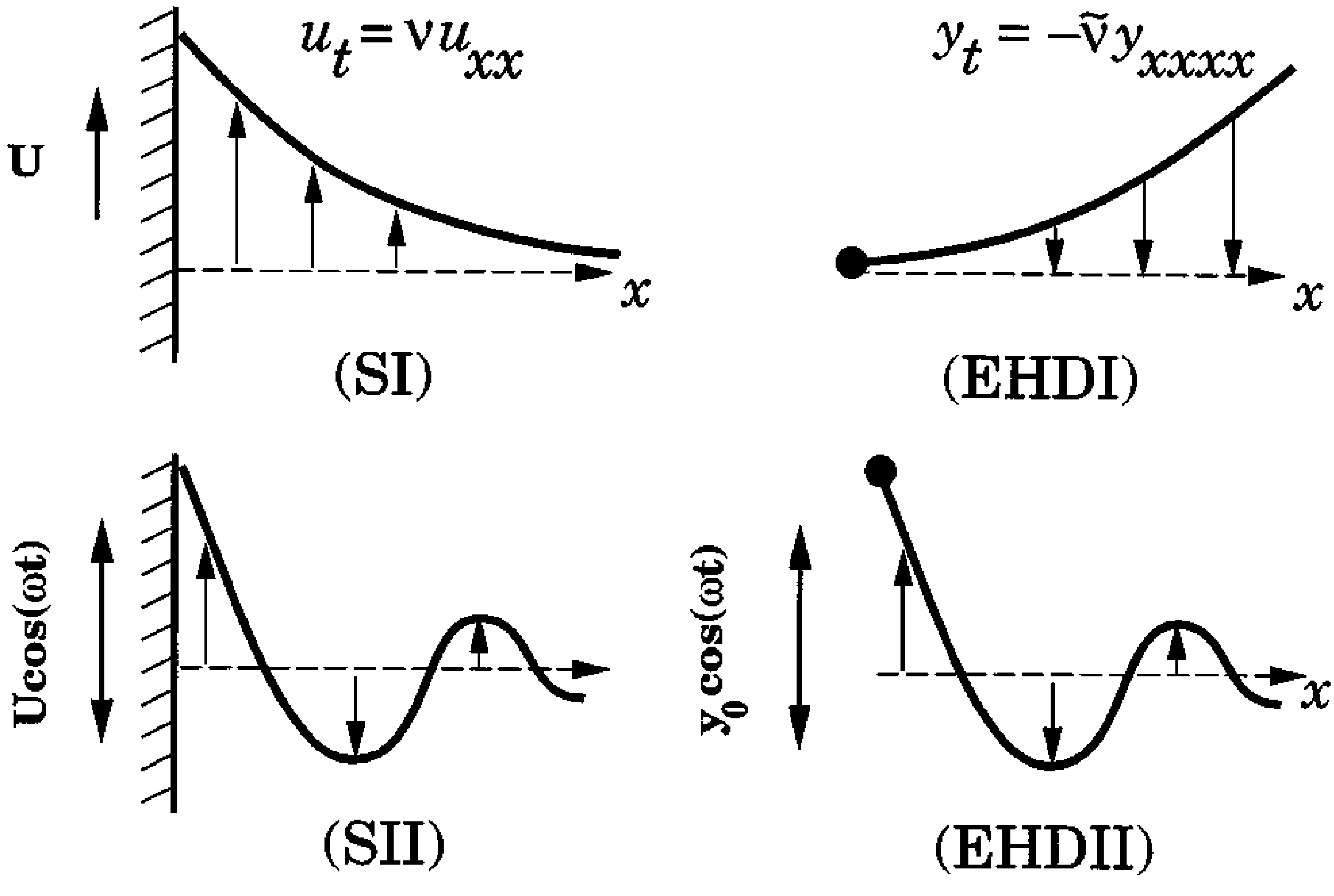
For a long, slender object of length $L$ and radius $a$, the calculation of the drag is a complicated nonlocal problem, but often the dominant behaviour is well-described by the introduction of local drag coefficients (so-called *Resistive Force Theory*) $\zeta_\perp$, $\zeta_{||}$, and $\zeta_r$ for motion perpendicular and parallel to the rod axis, and for rotational motion. These are

$$\zeta_\perp = 2\zeta_{||} = \frac{4\pi\eta}{\ln(L/2a)+c} \qquad \zeta_r = 4\pi\eta a^2$$

where for rotation motion there is a balance between the moment $m$ and angular frequency of the form $m = \zeta_r \omega$. With this anisotropic drag law, the equation of motion of the filament would be

$$\boxed{\left(\zeta_\perp \hat{\mathbf{n}}\hat{\mathbf{n}} + \zeta_{||}\hat{\mathbf{t}}\hat{\mathbf{t}}\right) \cdot \mathbf{r}_t = -\frac{1}{\sqrt{g}}\frac{\delta \mathcal{E}}{\delta \mathbf{r}}}$$

# Stokes and Elastohydrodynamic Problems



# Lessons From Stokes Problems

These are two pedagogical problems involving viscous fluids driven by the motion of a boundary parallel to itself. Starting with the Navier-Stokes equation

$$\rho\left(\mathbf{u}_t + \mathbf{u}\cdot\nabla\mathbf{u}\right) = -\nabla p + \eta\nabla^2\mathbf{u}$$

we recognize for wall-driven flow that the nonlinear term vanishes by symmetry, and we can ignore the pressure gradient. As the velocity will be of the form $\mathbf{u} = u(y)\hat{\mathbf{e}}_x$, the problem boils down to the solution of the diffusion equation

$$\boxed{u_t = \nu u_{yy}}$$

We will consider two situations:

1. Starting at $t = 0$, the wall begins to move with velocity $U$ in the $x$-direction. Only length scale is $\sqrt{\nu t}$.

2. The wall's velocity oscillates, as $U\mathrm{e}^{i\omega t}$. Only length scale is $\sqrt{\nu/\omega}$.

# The Similarity Solution (I)

Searching for a similarity solution of the form

$$u = Uf\left(\frac{y}{\sqrt{\nu t}}\right) \qquad u_{yy} = Uf_{\xi\xi}(\xi)\left(\frac{1}{\nu t}\right) \qquad u_t = Uf_{\xi}(\xi)\left(\frac{-1}{2t}\xi\right)$$

$u_t = \nu u_{yy}$ then becomes an ODE in $\xi$

$$\boxed{f_{\xi\xi} = -\frac{1}{2}\xi f_{\xi}}$$

Boundary conditions: $f(0) = 1$, $f(\infty) = 0$. Letting $g = f_{\xi}$

$$g_{\xi} = \frac{-1}{2}\xi g \qquad \Longrightarrow \qquad g = Ae^{-\xi^2/4} \, .$$

Invoking boundary conditions we obtain

$$f(\xi) = 1 - \frac{1}{\sqrt{\pi}}\int_0^{\xi} d\xi' e^{-\xi'^2/4}$$

and thus the similarity solution exists. The velocity is

$$\boxed{\frac{u}{U} = 1 - \mathrm{erf}\left(y/2\sqrt{\nu t}\right)}$$

# The Similarity Solution (II)

For the oscillating Stokes problem, we consider the situation post-transients, where we expect a similarity solution of the form

$$\boxed{u = U\mathrm{e}^{i\omega t} F\left(\frac{y}{\sqrt{\nu/\omega}}\right)}$$

and the PDE becomes an ODE in $\xi = y/\sqrt{\nu/\omega}$

$$F_{\xi\xi} = iF$$

Looking for exponential solutions, $F \sim \mathrm{e}^{\lambda\xi}$ gives

$$\lambda^2 = i \qquad \lambda_\pm = \pm\frac{(1+i)}{\sqrt{2}}$$

$$\text{So} \qquad F = \mathrm{e}^{-\xi/\sqrt{2}}\mathrm{e}^{-i\xi/\sqrt{2}} \; .$$

And we take the real part, giving the damped traveling-wave solution

$$\boxed{u = U\mathrm{e}^{-\xi/\sqrt{2}}\cos\left(\frac{\xi}{\sqrt{2}} - \omega t\right)} \; .$$

# EHD Problem I

For the oscillating EHD problem, the PDE governing small-amplitude deviations of the filament is

$$\boxed{\zeta_\perp h_t = -A h_{xxxx}}$$

and we will consider the situation post-transient in which the left-hand side is forced as $h(0,t) = h_0 \cos(\omega t)$, $h_{xx}(0,t) = 0$ and the distant end is free. Dimensional analysis shows there is a new elastohydrodynamic penetration length

$$\ell(\omega) = \left(\frac{A}{\zeta\omega}\right)^{1/4} .$$

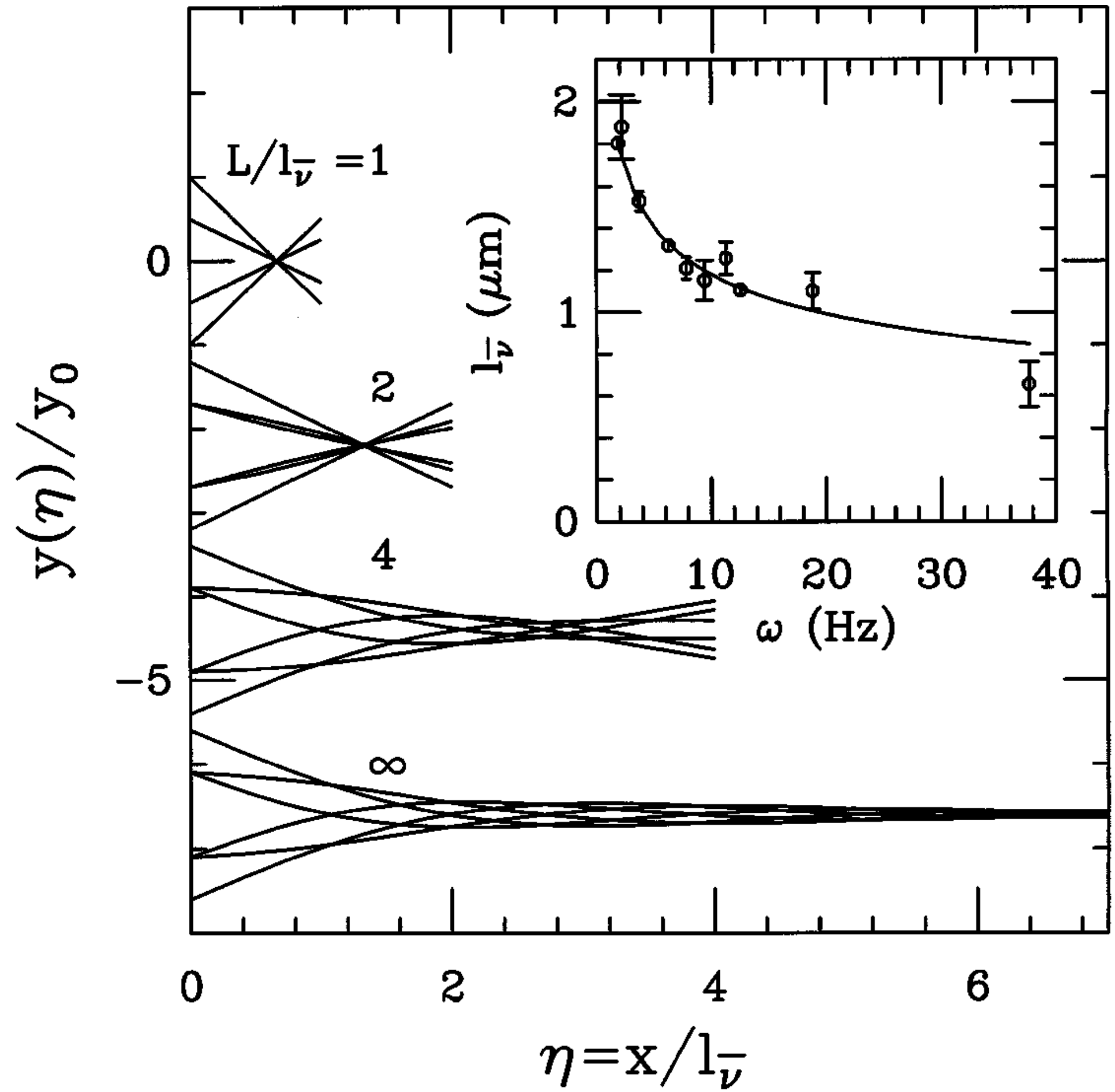
We expect a similarity solution of the form

$$h = h_0 \mathrm{Re}\left\{ \mathrm{e}^{i\omega t} F\left(\frac{x}{\ell(\omega)}\right)\right\} \quad \rightarrow \quad F_{\xi\xi\xi\xi} = -iF$$

Looking for exponential solutions, $F \sim \mathrm{e}^{\lambda\xi}$ gives $\lambda^4 = -i$, and when the dust settles we have ($C_8 = \cos(\pi/8)$, $S_8 = \sin(\pi/8)$),

$$\boxed{h(x,t) = \frac{1}{2} h_0 \left\{ \mathrm{e}^{-C_8\xi} \cos\left(\omega t + S_8\xi\right) + \mathrm{e}^{-S_8\xi} \cos\left(\omega t - C_8\xi\right)\right\}}$$

# The Undulating Shapes



# A Few Remarks

Let us estimate the EH penetration length. Writing $A = k_B T L_p$ and taking $L_p \sim 10\ \mu$m (actin), and measuring frequency in Hz, we find

$$\ell(\omega) \sim \left( \frac{4 \times 10^{-14} \cdot 10^{-3}}{4\pi \cdot 10^{-2} \cdot 1} \right)^{1/4} \boxed{\sim \frac{\text{a few microns}}{\omega^{1/4}}}$$

So, if we oscillate an actin filament at many Hz we will create undulations on a scale smaller than $L_p$. This can be done by optical trapping.
A second point (Machin, 1958), is that if we examine the shapes of undulating filaments that are end-actuated, the amplitudes of the subsequent peaks of the waveform are very small, totally unlike what is seen with e.g. sperm cells. This led Machin to the conclusion that eukaryotic flagella must not be actuated simply at their ends, but throughout their length. This is true!