ASYMPTOTIC METHODS

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Introduction

In this course, we will be interested in finding so-called asymptotic series, leading to approximations to the values of integrals depending on some parameter, or to the solutions of differential equations. For instance, consider *Stirling's Approximation*

$$n! \sim (2\pi n)^{1/2}\left(\frac{n}{\mathrm{e}}\right)^n \quad \text{as } n \to \infty \qquad \text{or} \qquad \Gamma(n) = (n-1)! \sim \left(\frac{2\pi}{n}\right)^{1/2}\left(\frac{n}{\mathrm{e}}\right)^n$$

We will also be interested in the *Stieltjes Integral*: let $\rho(t)$ be a well-behaved non-negative function that decays sufficiently fast as $t \to \infty$. Then

$$\int_0^\infty \frac{\rho(t)}{1+xt}\,\mathrm{d}t \sim \int_0^\infty \rho(t)\,\mathrm{d}t \quad \text{for small } x > 0$$

The *modified Bessel function* of order zero, I_0, solves the differential equation

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + \frac{1}{x}\frac{\mathrm{d}y}{\mathrm{d}x} - y = 0$$

and has the following expansion with infinite radius of convergence:

$$\mathrm{I}_0(x) = \sum_{n=0}^{\infty} \frac{x^{2n}}{2^{2n}(n!)^2} \sim (2\pi x)^{-1/2}\mathrm{e}^x \quad \text{as } x \to \infty$$

Asymptotic expansions are widely used in applied mathematics and theoretical physics, e.g. when deriving the leading-order contribution to the electrostatic potential corresponding to a charge distribution of limited spatial extent when viewed from far away. In number theory, the prime number theorem states that the number $\pi(n)$ of primes at most n satisfies $\pi(n) \sim \frac{n}{\log n}$ as $n \to \infty$.

In the above examples, the symbol $\sim$ means that the ratio of the two sides tends to unity in the appropriate limit. If we seek better approximations, we can derive *asymptotic series*, such as the Stirling series

$$\Gamma(n) \sim \left(\frac{2\pi}{n}\right)^{1/2}\left(\frac{n}{\mathrm{e}}\right)^n\left[1 + \frac{1}{12n} + \frac{1}{288n^2} + \cdots\right]$$

where the coefficients are known, but not elementary. The coefficients get smaller for a while, but eventually grow rapidly. The series does not converge for any n. Upon truncating, this series gives a sequence of approximations:

$$\frac{\Gamma(n)}{\left(\frac{2\pi}{n}\right)^{1/2}\left(\frac{n}{\mathrm{e}}\right)^n} - 1 = o(1), \quad \frac{\Gamma(n)}{\left(\frac{2\pi}{n}\right)^{1/2}\left(\frac{n}{\mathrm{e}}\right)^n} - 1 - \frac{1}{12n} = o\left(\frac{1}{n}\right), \quad \ldots \quad \text{for large } n$$

For $n = 4$, these approximations give $3! \approx 5.877$ and $3! \approx 5.999$. For $n = 4$, the approximation using six terms gives the best answer; with more terms, the error increases, since the series does not converge.

For the Stieltjes integral, we will show that, using the Taylor expansion $(1+xt)^{-1} = 1 - xt + x^2t^2 + \cdots$ valid for $t < 1/x$,

$$\int_0^\infty \frac{\rho(t)}{1+xt}\,\mathrm{d}t \sim \sum_{n=0}^{\infty}(-1)^n c_n x^n \quad \text{where } c_n = \int_0^\infty t^n\rho(t)\,\mathrm{d}t$$

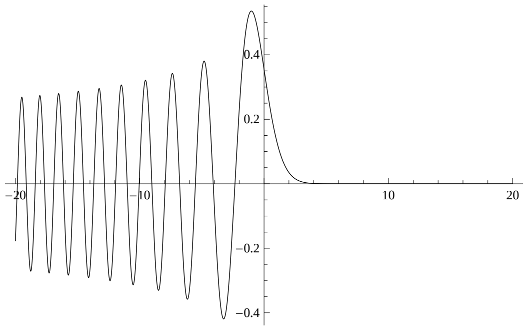
for small $x > 0$, provided that $\rho(t) \to 0$ as $t \to \infty$ fast enough for the moments in the series to exist. For the modified Bessel function, it is possible to show that

$$\mathrm{I}_0(x) \sim (2\pi x)^{-1/2} \mathrm{e}^x \left[1 + \frac{1^2}{1!} \frac{1}{8x} + \frac{1^2 3^2}{2!} \frac{1}{(8x)^2} + \frac{1^2 3^2 5^2}{3!} \frac{1}{(8x)^3} + \cdots \right]$$

For instance, the five term approximation for $x = 20$ has an error of about 0.0009%. As before, the series does not converge, since the coefficients increase as $n!$.

The Rule of Thumb is to sum up the series as far as, but not including the smallest term. This term gives an estimate of the error. The place to stop depends on the value at which one wants to evaluate the series.

Asymptotic series used to be important for precise evaluation of special functions, but, in that area, they have now been superseded. Often, they give insight because they replace some sophisticated mathematical function by elementary functions, such as exponential, trigonometric functions or powers. They play a central role in computation techniques such as shooting methods. The behaviour of these functions can be rich, such as the behaviour of the *Airy function* shown below, which exhibits both trigonometric oscillations and exponential decay.



Another feature of asymptotic series that we will discuss in this course is *Stokes' phenomenon*. Many classical and special functions, e.g. those satisfying second-order ordinary differential equations, are entire functions on the complex plane, so, unless they are constant, which is kind of duff, they are unbounded. We therefore look for asymptotic series of the form

$$f(z) \sim c_1 \mathrm{e}^{s_1(z)} + c_2 \mathrm{e}^{s_2(z)} \quad \text{as } z \to \infty$$

The functions s_1, s_2 are *Liouville–Green Functions*, and we will discuss them later on. Interestingly, one finds that these expansions are only valid for a restricted range of the complex argument; the constants c_1, c_2 have different values in different sectors of this kind. The sector boundaries are called *Stokes Lines*. In fact, s_1, s_2 are often rational functions, i.e. not entire, in which case Stokes lines are inevitable. The analytic continuation of the asymptotic approximation fails to be an asymptotic approximation outside the sector.

Asymptotic Series

A real-valued function $f(x)$ has an *asymptotic series* or *asymptotic expansion* around x_0, written

$$f(x) \sim \sum_{n=0}^{\infty} a_n (x - x_0)^n \quad \text{as } x \to x_0$$

where the series should be understood as a formal power series, if

$$f(x) - \sum_{n=0}^{N} a_n (x - x_0)^n \ll (x - x_0)^N \quad \text{for each } N \geqslant 0 \text{ and for } x - x_0 \text{ small}$$

or, more formally, if

$$\frac{f(x) - \sum\limits_{n=0}^{N} a_n (x - x_0)^n}{(x - x_0)^N} \to 0 \quad \text{as } x \to x_0, \text{ for each } N \geqslant 0.$$

For a truncated asymptotic series, there is just a finite number of inequalities or limits. Note that the definition only involves partial sums: the series itself need not converge. Also, we consider N to be fixed in the limit $x \to x_0$; this is the other way round for convergent power series. Similarly, a function $f(x)$ has an asymptotic series about infinity, written

$$f(x) \sim \sum_{n=0}^{\infty} a_n x^{-n} \quad \text{as } x \to \infty$$

if it holds that

$$f(x) - \sum_{n=0}^{N} a_n x^{-n} \ll x^{-N} \quad \text{for all } N \geqslant 0, \text{ as } x \to \infty.$$

These definitions extend to complex functions, but it turns out that one needs to be more careful about regions of validity. One often needs to take out a prefactor to be able to use the definition; for instance,

$$\frac{\mathrm{I}_0(x)}{(2\pi x)^{-1/2}\mathrm{e}^x} \sim 1 + \frac{1}{8x} + \frac{9}{128x^2} + \cdots \quad \text{as } x \to \infty$$

This definition involves a function and its expansion. The expansion has little meaning by itself, and the function must be defined in its own right first, by contrast with Taylor series.

Lemma. A function has a most one asymptotic expansion about some given point, or about infinity.

Suppose that a function $f(x)$ has two asymptotic expansions about some point x_0, the argument for expansions about infinity being analogous, viz

$$f(x) \sim \sum_{n=0}^{\infty} a_n (x-x_0)^n \quad \text{and} \quad f(x) \sim \sum_{n=0}^{\infty} b_n (x-x_0)^n$$

Suppose that, for some $m \geqslant 0$, $a_m \neq b_m$, but $a_n = b_n$ for $0 \leqslant n < m$. Then

$$\frac{f(x) - \sum\limits_{n=0}^{m} a_n (x-x_0)^n}{(x-x_0)^m} \to 0, \qquad \frac{f(x) - \sum\limits_{n=0}^{m} b_n (x-x_0)^n}{(x-x_0)^m} \to 0 \quad \text{as } x \to x_0.$$

Upon subtracting these two limits, $a_m - b_m \to 0$, and thus $a_m = b_m$, a contradiction. Hence the two asymptotic expansions are equal.

Note that there are plenty of functions with no asymptotic series, or with only a finite asymptotic series. For instance, it is easy to see that e^{-x} does not have an asymptotic series around infinity, for all coefficients in the asymptotic expansion vanish. One says that e^{-x} is 'small beyond all orders' in inverse powers of x. This also shows that two different functions can have the same asymptotic series: if $f(x)$ has an asymptotic series about infinity, then $f(x) + \mathrm{e}^{-x}$ has the same asymptotic series about infinity.

Elementary Properties

Consider two functions $f(x)$ and $g(x)$ with respective asymptotic expansions

$$f(x) \sim \sum_{n=0}^{\infty} a_n (x-x_0)^n \quad \text{and} \quad g(x) \sim \sum_{n=0}^{\infty} b_n (x-x_0)^n$$

about x_0. Then $f(x) + g(x)$ and $f(x)g(x)$ both have asymptotic expansions,

$$f(x) + g(x) \sim \sum_{n=0}^{\infty} (a_n + b_n)(x-x_0)^n \quad \text{and} \quad f(x)g(x) \sim \sum_{n=0}^{\infty} c_n (x-x_0)^n \quad \text{where } c_n = \sum_{k=0}^{n} a_k b_{n-k}$$

Assume furthermore that $f(x)$ is integrable. Then

$$\int_{x_0}^{x} f(\xi)\ \mathrm{d}\xi \sim \sum_{n=0}^{\infty} \frac{a_n}{n+1}(x-x_0)^{n+1} \quad \text{as } x \to x_0$$

Indeed, for any $N > 0$ and for any $\varepsilon > 0$, there exists $\eta > 0$ such that, whenever $|x - x_0| < \varepsilon$,

$$\int_{x_0}^{x} \left| f(\xi) - \sum_{n=0}^{N} a_n(\xi - x_0)^n \right|\ \mathrm{d}\xi \leqslant \eta \int_{x_0}^{x} |\xi - x_0|^N\ \mathrm{d}\xi = \frac{\eta}{N+1}|x-x_0|^{N+1}$$

The claim then follows from the observation that

$$\begin{aligned}\left| \int_{x_0}^{x} f(\xi)\ \mathrm{d}\xi - \sum_{n=0}^{N} \frac{a_n}{n+1}(x-x_0)^{n+1} \right| &= \left| \int_{x_0}^{x} \left(f(\xi) - \sum_{n=0}^{N} a_n(\xi - x_0)^n \right) \mathrm{d}\xi \right| \\ &\leqslant \int_{x_0}^{x} \left| f(\xi) - \sum_{n=0}^{N} a_n(\xi - x_0)^n \right| \mathrm{d}\xi\end{aligned}$$

Asymptotics of Integrals

As an example, we consider the *Stieltjes Integral* more rigorously. Let $\rho(t)$ be a non-negative function that does not vanish identically, and such that the moments

$$c_n = \int_0^{\infty} t^n \rho(t)\ \mathrm{d}t \quad \text{exist for all } n \geqslant 0.$$

We are interested in deriving an asymptotic series for the integral

$$I(x) = \int_0^{\infty} \frac{\rho(t)}{1+xt}\ \mathrm{d}t \quad \text{where } x > 0.$$

Note that there is an exact expansion

$$\frac{1}{1+xt} = 1 - xt + x^2t^2 + \cdots + (-1)^N x^N t^N + \frac{(-1)^{N+1}x^{N+1}t^{N+1}}{1+xt}$$

It follows that

$$I(x) = \sum_{n=0}^{N} (-1)^n c_n x^n + (-1)^{N+1} x^{N+1} \int_0^{\infty} \frac{t^{N+1}\rho(t)}{1+xt}\ \mathrm{d}t$$

But $1 + xt \geqslant 1$ since $x \geqslant 0$, and thus

$$x^{-N} \left| (-1)^{N+1} x^{N+1} \int_0^{\infty} \frac{t^{N+1}\rho(t)}{1+xt}\ \mathrm{d}t \right| \leqslant x c_{N+1} \to 0 \quad \text{as } x \to 0.$$

We have thus derived the asymptotic expansion

$$I(x) = \int_0^{\infty} \frac{\rho(t)}{1+xt}\ \mathrm{d}t \sim \sum_{n=0}^{\infty} (-1)^n c_n x^n \quad \text{as } x \to 0.$$

One way to see that this is not a convergent Taylor expansion is to note that the left-hand side is ill-defined for negative x, due to the pole of the integrand at $t = -1/x$. Alternatively, one can evaluate the moments explicitly for simple examples, such as $\rho(t) = \mathrm{e}^{-t}$.

Integration by Parts Formulae

Integration by parts is useful for determining asymptotic expansions of integrals where the end point dominates. Let $f^{(n)}(t)$ denote the n-th derivative of a function $f(t)$. (We shall always, in this course, assume that functions are sufficiently smooth.) Integrating by parts n times gives

$$\int_a^b f(t)g^{(n)}(t)\,\mathrm{d}t = \sum_{k=1}^{n}(-1)^{k-1}\Big[f^{(k-1)}(t)g^{(n-k)}(t)\Big]_a^b + (-1)^n\int_a^b f^{(n)}(t)g(t)\,\mathrm{d}t$$

Example of an Asymptotic Expansion of an Integral

Let $a < b$ be fixed, and let $f(t)$ be a smooth function such that $f(b) \neq 0$. We are interested in obtaining an asymptotic expansion, as $x \to \infty$, for the integral

$$I(x) = \int_a^b f(t)\mathrm{e}^{xt}\,\mathrm{d}t$$

Observe that the endpoint $t = b$ completely dominates in this integral. Integrating by parts gives

$$\begin{aligned} I(x) &= \sum_{k=1}^{n}(-1)^{k-1}\Big[f^{(k-1)}(t)\mathrm{e}^{xt}x^{-k}\Big]_a^b + \frac{(-1)^n}{x^n}\int_a^b f^{(n)}(t)\mathrm{e}^{xt}\,\mathrm{d}t \\ &= \mathrm{e}^{xb}\left[\sum_{k=1}^{n}(-1)^{k-1}f^{(k-1)}(b)x^{-k} + \frac{(-1)^n}{x^n}\int_a^b f^{(n)}(t)\mathrm{e}^{x(t-b)}\,\mathrm{d}t - \mathrm{e}^{-x(b-a)}\left(\sum_{k=1}^{n}(-1)^{k-1}f^{(k-1)}(a)x^{-k}\right)\right] \end{aligned}$$

Since $b - a > 0$, the term in $\mathrm{e}^{-x(b-a)}$ is small beyond all orders. What is more,

$$\left|\int_a^b f^{(n)}(t)\mathrm{e}^{x(t-b)}\,\mathrm{d}t\right| \leqslant \left|\max_{t\in[a,b]} f^{(n)}(t)\right|\left(\int_a^b \mathrm{e}^{x(t-b)}\,\mathrm{d}t\right) = \left|\max_{t\in[a,b]} f^{(n)}(t)\right|\frac{1-\mathrm{e}^{-x(b-a)}}{x} \to 0 \quad \text{as } x \to \infty.$$

We thus have, provided that $f(b) \neq 0$, the asymptotic expansion

$$I(x) = \int_a^b f(t)\mathrm{e}^{xt}\,\mathrm{d}t \sim \mathrm{e}^{xb}\left[\sum_{k=1}^{n}(-1)^{k-1}f^{(k-1)}(b)x^{-k}\right] \quad \text{as } x \to \infty.$$

Again, this cannot be an exact, or convergent, series, as the result is independent of a. The leading term in this approximation is

$$\int_a^b f(t)\mathrm{e}^{xt}\,\mathrm{d}t \sim \mathrm{e}^{xb}f(b)x^{-1} \quad \text{which arises from} \quad \int_{-\infty}^b \mathrm{e}^{xt}\,\mathrm{d}t = \frac{\mathrm{e}^{xb}}{x}$$

More generally, we get any number of terms in this asymptotic series using a Taylor expansion of $f(t)$ around $t = b$, and integrating from $-\infty$ to b, making an error that is small beyond all orders.

Laplace's Integral

The ideas of the above example are generalised by the *Laplace Integral*. Let $f(t)$ be a smooth function with $f(b) \neq 0$. We are interested in expansions of

$$I(x) = \int_a^b f(t)\mathrm{e}^{x\phi(t)}\,\mathrm{d}t$$

If $\phi'(t)$ does not change sign on $[a, b]$, the Laplace integral just reduces to the integral in our above example: assume without loss of generality that $\phi'(t) > 0$ in $[a, b]$. Change variables to $u = \phi(t)$, so that

$$I(x) = \int_{\phi(a)}^{\phi(b)} \frac{f(t)}{\phi'(t)}\mathrm{e}^{xu}\,\mathrm{d}u \quad \text{where } t = \phi^{-1}(u)$$

Having inverted this change of variables (which is pretty straightforward in examples), we obtain an expansion for the integral, which is rather messy in general. The leading-order approximation is

$$I(x) \sim \mathrm{e}^{x\phi(b)}\frac{f(b)}{\phi'(b)}\frac{1}{x}$$

If $\phi(t)$ attains its maximum value in $[a,b]$, at $t=b$, with $\phi'(b)>0$ and $f(b)\neq 0$ as usual, this leading-order approximation is still valid: to leading order, the only relevant contribution to the integral comes from a neighbourhood of $t=b$.
A variant of the simple Laplace integral we have been considering above is the following: suppose that $f(t)$ is defined by a convergent power series in some neighbourhood of $t=0$, viz

$$f(t)=\sum_{n=0}^{\infty} a_n t^n$$

Note that $f^{(n)}(0)=n!a_n$. Hence, adapting our previous results, for $b>0$, we have the asymptotic series

$$\int_0^b f(t)\mathrm{e}^{-xt}\ \mathrm{d}t \sim \sum_{n=0}^{\infty} n!a_n x^{-(n+1)} \quad \text{as } n\to\infty$$

Watson's Lemma

Before generalising the previous results, we need two preliminaries. First, note that, so far, we have only considered asymptotic expansions with integral powers. Some functions have asymptotic expansions in fractional powers of their argument, of the form

$$f(x)\sim\sum_{n=0}^{\infty} c_n x^{\alpha+\beta n} \quad \text{as } x\to 0_+$$

where $\beta>0$. Taking out the prefactor x^α, and changing variables to $\xi=x^\beta$, we recover the standard definition of an asymptotic series.

Let us now make a few remarks about the *Gamma function*, which is defined for all complex numbers with $\operatorname{Re} z>0$ by the integral

$$\Gamma(z)=\int_0^\infty t^{z-1}\mathrm{e}^{-t}\ \mathrm{d}t$$

Integrating by parts, one obtains

$$\Gamma(z+1)=z\Gamma(z)$$

Thus, it is possible to analytically continue the Gamma function to a meromorphic function on the entire complex plane, with poles at $z=0,-1,-2,\ldots$.
Clearly, $\Gamma(1)=1$, so $\Gamma(n+1)=n!$ by induction. Another special value that is sometimes useful is $\Gamma\left(\frac{1}{2}\right)=\sqrt{\pi}$; it is obtained by evaluating a Gaussian integral.

Theorem (Watson's Lemma). Let $f(t)$ be a function such that $f(t)$ is bounded on $[t,1]$, for any $0<t<1$, and that $\left|f(t)\right|<\mathrm{e}^{ct}$ at large t, for some constant c. Let $f(t)$ have the asymptotic series

$$f(t)\sim\sum_{n=0}^{\infty} c_n t^{\alpha+\beta n} \quad \text{as } t\to 0_+$$

where $\alpha>-1$ and $\beta>0$. Then

$$\int_0^b f(t)\mathrm{e}^{-xt}\ \mathrm{d}t\sim\sum_{n=0}^{\infty}\Gamma(\alpha+\beta n+1)c_n x^{-(\alpha+\beta n+1)} \quad \text{as } x\to\infty$$

Observe that the conditions imposed on $f(t)$ are sufficient for the integral to exist. Also notice that the result is independent of b, and that the case $\alpha=0$, $\beta=1$ corresponds to our previous result. Fix $N\geqslant 0$, and take $\varepsilon>0$. Then

$$I(x)\sim\int_0^\varepsilon f(t)\mathrm{e}^{-xt}\ \mathrm{d}t \quad \text{with an error } \left|\int_\varepsilon^b f(t)\mathrm{e}^{-xt}\ \mathrm{d}t\right|\leqslant C\int_\varepsilon^b \mathrm{e}^{-xt}\ \mathrm{d}t=\mathcal{O}\left(\frac{\mathrm{e}^{-x\varepsilon}}{x}\right)$$

which is small beyond all orders as $x \to \infty$. Taking ε to be small enough,

$$I(x) \sim \int_0^\varepsilon \left(\sum_{n=0}^N c_n t^{\alpha+\beta n} \right) \mathrm{e}^{-xt} \mathrm{d}t + \mathcal{O}\left(\int_0^\varepsilon t^{\alpha+\beta(N+1)} \mathrm{e}^{-xt} \; \mathrm{d}t \right)$$

Watson's Lemma then follows from the fact that, up to an error that is exponentially small,

$$\int_0^\varepsilon t^{\alpha+\beta n} \mathrm{e}^{-xt} \; \mathrm{d}t = x^{-(\alpha+\beta n+1)} \int_0^{\varepsilon t} \underbrace{u^{\alpha+\beta n} \mathrm{e}^{-u}}_{g(u)} \; \mathrm{d}u \sim x^{-(\alpha+\beta n+1)} \int_0^\infty g(u) \; \mathrm{d}u = \Gamma(\alpha+\beta n+1) x^{-(\alpha+\beta n+1)}$$

Example. $I(x) = \displaystyle\int_0^5 \left(t^2+2t\right)^{-1/2} \mathrm{e}^{-xt} \; \mathrm{d}t \sim \frac{\Gamma(\frac{1}{2})}{(2x)^{1/2}} - \frac{\Gamma(\frac{3}{2})}{2(2x)^{3/2}}$.

Laplace's Integral with Interior Maximum

We now consider the Laplace integral

$$I(x) = \int_a^b f(t) \mathrm{e}^{x\phi(t)} \; \mathrm{d}t \quad \text{as } x \to \infty$$

where $\phi(t)$ has a smooth non-zero quadratic maximum at some $c \in (a,b)$, i.e. $\phi'(c) = 0$ and $\phi''(c) < 0$, and where $f(c) \neq 0$. The integral is dominated by a neighbourhood of c, in which the integral becomes Gaussian upon writing $f(t) \approx f(c)$ and $\phi(t) \approx \phi(c) + \frac{1}{2}\phi''(c)(t-c)^2$. The leading-order asymptotics are

$$I(x) \sim \int_{c-\varepsilon}^{c+\varepsilon} f(c) \exp\left[x\left(\phi(c) + \tfrac{1}{2}\phi''(c)(t-c)^2 \right) \right] \; \mathrm{d}t = f(c)\mathrm{e}^{x\phi(c)} \int_{c-\varepsilon}^{c+\varepsilon} \mathrm{e}^{\frac{1}{2}x\phi''(c)(t-c)^2} \; \mathrm{d}t$$

where the error is exponentially small. Change variables to $s = \left(-\frac{1}{2}\phi''(c)\right)^{1/2}(t-c)$, noting that $\phi''(c) < 0$. It follows that

$$I(x) \sim \frac{f(c)\mathrm{e}^{x\phi(c)}}{\left(-\frac{1}{2}x\phi''(c)\right)^{1/2}} \int_{-(-\frac{1}{2}x\phi''(c))^{1/2}\varepsilon}^{(-\frac{1}{2}x\phi''(c))^{1/2}\varepsilon} \mathrm{e}^{-s^2} \; \mathrm{d}s \sim \frac{f(c)\mathrm{e}^{x\phi(c)}}{\left(-\frac{1}{2}x\phi''(c)\right)^{1/2}} \int_{-\infty}^{\infty} \mathrm{e}^{-s^2} \; \mathrm{d}s$$

where, again, the error is exponentially small. Thus, finally,

$$I(x) \sim \left(\frac{2\pi}{-\phi''(c)} \right)^{1/2} f(x) \mathrm{e}^{x\phi(c)} x^{-1/2} \quad \text{as } x \to \infty$$

One can similarly deal with the case where $\phi(t)$ has a quadratic maximum at one of a, b. Note that, in that case, the above expansion has an extra factor of one half, since we only pick up half a Gaussian integral. That case can also be dealt with by changing variables and using Watson's Lemma, but that approach is rather fiddly.

Example: The Modified Bessel Function. $\mathrm{I}_0(x) = \displaystyle\frac{1}{\pi}\int_0^\pi \mathrm{e}^{x\cos t} \; \mathrm{d}t \sim \frac{\mathrm{e}^x}{\sqrt{2\pi x}}$.

We can similarly obtain higher-order terms in the asymptotic expansions using Taylor series. We illustrate the technique by example. Consider the integral

$$I(x) = \int_0^{\frac{\pi}{2}} \mathrm{e}^{-x\sin^2 t} \; \mathrm{d}t \quad \text{as } x \to \infty$$

Note that $\phi(t) = -\sin^2 t$ has a maximum at $t = 0$, and $\sin^2 t \approx t^2 - \frac{1}{3}t^4$ near $t = 0$. The integral is exponentially dominated by a region near $t = 0$, so, for some $\varepsilon > 0$,

$$I(x) \sim \int_0^\varepsilon \mathrm{e}^{-x\sin^2 t} \; \mathrm{d}t \sim \int_0^\varepsilon \mathrm{e}^{-xt^2} \left(1 + \tfrac{1}{3}xt^4\right) \mathrm{d}t$$

Changing variables to $s = \sqrt{x}t$ takes the upper limit of the integral to infinity in the limit $x \to \infty$, and we thus obtain

$$I(x) \sim \frac{\sqrt{\pi}}{2} x^{-1/2} + \frac{\sqrt{\pi}}{8} x^{-3/2} \quad \text{as } x \to \infty$$

Notice that, as before, the leading term arises from half a Gaussian integral. Observe that it is important not to Taylor expand the leading term in the exponential. If the integral had a prefactor, the latter would have to be Taylor expanded, too.

Fourier–Type Integrals and the Stationary Phase Method

In this subsection, we study the asymptotics of integrals of the form

$$I(\omega) = \int_a^b f(t) \mathrm{e}^{\mathrm{i}\omega t}\ \mathrm{d}t \quad \text{or, more generally,} \quad I(\omega) = \int_a^b f(t) \mathrm{e}^{\mathrm{i}\omega \phi(t)}\ \mathrm{d}t \quad \text{as } |\omega| \to \infty$$

Here, ω is a real variable, and the function $\phi(t)$ is real. Note that the first case corresponds to the Fourier transform of a localised function. Let us remark that the real and imaginary parts of $\mathrm{e}^{\mathrm{i}\omega t}$ are rapidly oscillating for large $|\omega|$. Heuristically, we therefore expect the negative and positive parts to cancel approximately, so that $I(\omega)$ is small.

In fact, we can make this remark more precise by recalling that a well-behaved function $f\colon [0, 2\pi] \to \mathbb{C}$ has a Fourier series given by

$$f(t) = \sum_{n=-\infty}^{\infty} c_n \mathrm{e}^{\mathrm{i}\omega t} \quad \text{where } c_n = \frac{1}{2\pi} \int_0^{2\pi} f(t) \mathrm{e}^{\mathrm{i}\omega t}\ \mathrm{d}t$$

Parseval's Theorem for Fourier series, as introduced in the *Methods* course, states that

$$\int_0^{2\pi} \left|f(t)\right|^2\ \mathrm{d}t = 2\pi \sum_{n=-\infty}^{\infty} |c_n|^2$$

Now, the series on the right-hand side only converges if $c_n \to 0$ as $n \to \pm\infty$. This property of the Fourier coefficients is generalised by the *Riemann–Lebesgue Lemma*, which will form the basis for our study of the asymptotics of $I(\omega)$:

Theorem (Riemann–Lebesgue Lemma). Let $f\colon [a, b] \to \mathbb{C}$ be Riemann integrable on a closed interval $[a, b]$. Then it holds that

$$\lim_{|\omega| \to \infty} \left(\int_a^b f(t) \mathrm{e}^{\mathrm{i}\omega t}\ \mathrm{d}t \right) = 0$$

Let us first remark that this result is rather obvious under the stronger assumption of f being continuously differentiable. In that case, we may integrate by parts to obtain

$$I(\omega) = \left[\frac{f(t)\mathrm{e}^{\mathrm{i}\omega t}}{\mathrm{i}\omega} \right]_a^b - \frac{1}{\mathrm{i}\omega} \int_a^b f'(t) \mathrm{e}^{\mathrm{i}\omega t}\ \mathrm{d}t \tag{$*$}$$

Since continuous functions on a closed bounded interval are bounded, this implies that $I(\omega) \to 0$ as $|\omega| \to \infty$. Similarly, the result holds for piecewise constant functions. The result follows in general, because, by definition of Riemann integrability, any Riemann integrable function can be bounded above and below by piecewise constant functions, the integrals of which differ by no more than ε, for any $\varepsilon > 0$, i.e. there is a partition of $[a, b]$ with associated piecewise constant functions $m(t)$ and $M(t)$ such that

$$m(t) \leqslant f(t) \leqslant M(t) \quad \text{and} \quad \left| \int_a^b \big(M(t) - m(t)\big) \mathrm{e}^{\mathrm{i}\omega t}\ \mathrm{d}t \right| \leqslant \int_a^b \big(M(t) - m(t)\big)\ \mathrm{d}t \leqslant \varepsilon.$$

We now consider the asymptotic expansion of integrals of the form in the Riemann-Lebesgue lemma, assuming f to be a smooth function. By $(*)$ above, since f' is smooth, the remainder integral decays as $o(1/\omega)$. Hence we actually have an asymptotic expansion

$$I(\omega) \sim \frac{f(b)\mathrm{e}^{\mathrm{i}\omega b} - f(a)\mathrm{e}^{\mathrm{i}\omega a}}{\mathrm{i}\omega} \quad \text{as } |\omega| \to \infty$$

to leading order. Notice that, unlike the case of Laplace integrals, both endpoints contribute to the asymptotic expansion. Since f is smooth, we can integrate by parts repeatedly to obtain the infinite series

$$I(\omega) \sim \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{(\mathrm{i}\omega)^k} \left[f^{(k-1)}(b)\mathrm{e}^{\mathrm{i}\omega b} - f^{(k-1)}(a)\mathrm{e}^{\mathrm{i}\omega a} \right] \quad \text{as } |\omega| \to \infty$$

Example. $I(\omega) = \displaystyle\int_0^1 \frac{\mathrm{e}^{\mathrm{i}\omega t}}{1+t}\,\mathrm{d}t \sim \sum_{k=0}^{\infty} \frac{(k-1)!}{(\mathrm{i}\omega)^k}\left(\frac{\mathrm{e}^{\mathrm{i}\omega}}{2^k} - 1\right)$ as $|\omega| \to \infty$.

Integrals on Infinite Intervals and Discontinuities

Suppose that the function f and all its derivatives exist, and vanish as $t \to \infty$. Then, letting the limits in the integral above tend to $\pm\infty$, we find that the integral

$$I(\omega) = \int_{-\infty}^{\infty} f(t)\mathrm{e}^{\mathrm{i}\omega t}\,\mathrm{d}t$$

vanishes beyond all orders as $|\omega| \to \infty$. This is saying that the Fourier transform of f vanishes faster than any inverse power of ω. This is a familiar result: for example, the Fourier transform of a Gaussian is a Gaussian (and thus vanishes exponentially fast). Similarly, if f has a finite number of discontinuities, we may split up the integral, and thus $I(\omega) = \mathcal{O}(1/\omega)$. If f is continuous, with f' having finitely many discontinuities, then $I(\omega) \sim \mathcal{O}(1/\omega^2)$ and so on.

Generalised Fourier Integrals: The Stationary Phase Method

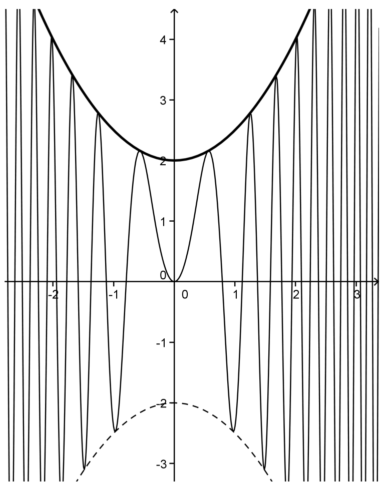
We now consider generalised Fourier integrals of the form

$$I(\omega) = \int_a^b f(t)\mathrm{e}^{\mathrm{i}\omega\phi(t)}\,\mathrm{d}t \quad \text{as } |\omega| \to \infty$$

A generalised version of the Riemann–Lebesgue lemma shows that this integral vanishes as $|\omega| \to \pm\infty$, provided that $\phi'(t)$ is continuous, and that $\phi(t)$ is not constant on any interval of positive length. We are interested in finding the leading asymptotics for smooth f.

The simplest case has $\phi'(t) > 0$ for all $t \in [a,b]$. The case where $\phi'(t) < 0$ for all $t \in [a,b]$ is of course completely analogous. Upon changing variables, our previous analysis applies, and we obtain that, to leading order,

$$I(\omega) \sim \frac{1}{\mathrm{i}\omega}\left[\frac{f(b)}{\phi'(b)}\mathrm{e}^{\mathrm{i}\omega\phi(b)} - \frac{f(a)}{\phi'(a)}\mathrm{e}^{\mathrm{i}\omega\phi(a)}\right] \quad \text{as } |\omega| \to \infty$$



The case where $\phi'(c) = 0$ for some $c \in (a,b)$ is more interesting. In that case, the integral is said to have a *stationary phase* at c. Intuitively, the condition $\phi'(c) = 0$ means that cancellation is slower in a neighbourhood of c, as illustrated on the right. We estimate

$$I(\omega) = \int_{c-\varepsilon}^{c+\varepsilon} f(t)\mathrm{e}^{\mathrm{i}\omega\phi(t)}\,\mathrm{d}t + \mathcal{O}\left(\frac{1}{\omega}\right)$$

where $\varepsilon > 0$. Of course, this only makes sense if we show that the leading order term in the first integral vanishes more slowly that $1/\omega$. Taylor expand, writing $\phi(t) = \phi(c) + \frac{1}{2}\phi''(c)(t-c)^2$, where we assume that

$\phi''(c) \neq 0$. For convenience, suppose that $\phi''(c) > 0$. Also estimate $f(t) \approx f(c)$, assuming that $f(c) \neq 0$. We then obtain the leading approximation

$$I(\omega) \sim f(c)\mathrm{e}^{\mathrm{i}\omega\phi(c)} \int_{c-\varepsilon}^{c+\varepsilon} \mathrm{e}^{\frac{1}{2}\mathrm{i}\omega\phi''(c)(t-c)^2}\ \mathrm{d}t \sim f(c)\mathrm{e}^{\mathrm{i}\omega\phi(c)} \left(\frac{2}{\omega\phi''(c)}\right)^{\frac{1}{2}} \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}s^2}\ \mathrm{d}s$$

upon changing variables, and taking the limits of integration to infinity. The so-called *Fresnel integral* above actually converges, and it can be evaluated by contour integration. More generally, one can show that

$$\int_0^\infty \mathrm{e}^{\mathrm{i}xs^2}\ \mathrm{d}s = \frac{1}{2}\sqrt{\frac{\pi}{x}}\mathrm{e}^{\mathrm{i}\pi/4} \quad \text{provided that } x > 0$$

Hence, we obtain the leading-order asymptotic expansion

$$I(\omega) \sim f(c)\mathrm{e}^{\mathrm{i}\omega\phi(c)} \left(\frac{2\pi}{\omega\phi''(c)}\right)^{\frac{1}{2}} \mathrm{e}^{\mathrm{i}\pi/4} \quad \text{or} \quad I(\omega) \sim f(c)\mathrm{e}^{\mathrm{i}\omega\phi(c)} \left(\frac{2\pi}{\omega|\phi''(c)|}\right)^{\frac{1}{2}} \mathrm{e}^{(\mathrm{i}\pi/4)\mathrm{sign}(\phi''(c))}$$

if we only suppose that $\phi''(c) \neq 0$. Note that these leading asymptotics are $\mathcal{O}\big(\omega^{-1/2}\big)$, so they dominate the effects of the endpoints at a and b. There are higher-order corrections from both the endpoints and from the neighbourhood of c. Obtaining these higher-order terms is quite tricky.

Example. $\displaystyle\int_{-\infty}^{\infty} \frac{\mathrm{e}^{\mathrm{i}\omega t}}{1+t^2}\ \mathrm{d}t \sim \left(\frac{\pi}{\omega}\right)^{\frac{1}{2}} \mathrm{e}^{\mathrm{i}\pi/4}.$

Of course, we can deal with similar integrals involving cosines and sines by taking real parts in the above results. The method of stationary phase can be generalised to the case where ϕ has higher-order stationary points at some $c \in (a, b)$. In that case, one needs to evaluate higher-order Fresnel integrals, by choosing appropriate contours.

The Method of Steepest Descent

In this section, we take the idea of Laplace-type integrals and the stationary phase method one step further, by extending the results to integrals in the complex plane of the form

$$I(x) = \int_{\mathscr{C}} f(z)\mathrm{e}^{x\phi(z)}\ \mathrm{d}z \quad \text{as } x \to \infty$$

where x is a real parameter. Here, $\mathscr{C}$ is some curve in the complex plane, and the functions $f(z)$ and $\phi(z)$ are analytic in a domain containing $\mathscr{C}$.

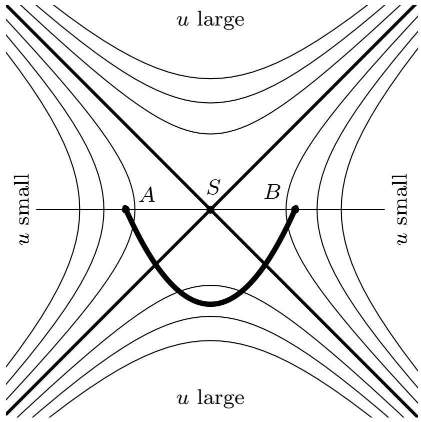
Write $z = p + \mathrm{i}q$, with $p = \operatorname{Re} z$, $q = \operatorname{Im} z$, and let $\phi(z) = u(p, q) + \mathrm{i}v(p, q)$, where the functions u, v are real-valued. Since ϕ is analytic, these functions satisfy the Cauchy–Riemann equations

$$\frac{\partial u}{\partial p} = \frac{\partial v}{\partial q} \quad \text{and} \quad \frac{\partial u}{\partial q} = -\frac{\partial v}{\partial p}$$

which are saying that the gradients of u and v have the same magnitude, but are orthogonal, i.e. the curves of constant u and v are at right angles. Note that the gradient of u is perpendicular to the curves of constant v, and vice versa.

The 'optimal' contour is one of 'steepest descent' of u: in that case, the integral becomes Laplace-like, dominated by the part where u has its maximum. Such a path of steepest descent is parallel, but in opposite direction, to $\boldsymbol{\nabla} u$, i.e. parallel to the lines of constant v. Thus the phase is constant on such a path.

By Cauchy's theorem, we can deform the contour of integration. The idea of the method of steepest descent is to deform the original contour to some other contour on which the phase is constant as far as possible. Paths of steepest descent typically start at saddle points of u, or at infinity. Note that, since $\nabla^2 u = 0$, the function u has no local maxima or minima.



Note that paths of steepest descent are orthogonal to the contours of u. In the figure above, we have chosen endpoints with the same phase at the saddle. We shall see that it is possible to deal with situations where the endpoints have different phases. Note that one might think that, in the plot above, the region where u is large dominates the integral, but there, the integrand oscillates.

To obtain the leading asymptotics, it suffices to note that the integral is Gaussian near the saddle point, where $\phi'(z) = 0$. Watson's lemma allows calculating the complete asymptotic expansion using local data at the saddle point, but we will not go into that.

If the phases at the endpoints of the integral differ, or if they are different from the phase at the saddle point, we need two or three paths of steepest descent, joined up at infinity.

Let us now determine the basic contribution from a simple saddle. Let $\mathscr{C}$ be a steepest descent path from a saddle at z_0. Consider the integral

$$I_0 = \int_{\mathscr{C}} \mathrm{e}^{x\phi(z)}\ \mathrm{d}z \quad \text{where } \phi(z) = \alpha \mathrm{e}^{\mathrm{i}\beta}(z - z_0)^2$$

where $\alpha > 0$ and $-\pi < \beta \leqslant \pi$. Changing variables to $y = -\mathrm{i}\mathrm{e}^{\mathrm{i}\beta/2}(z - z_0)$ moves the contour onto the positive real axis. Then

$$I_0 = \mathrm{i}\mathrm{e}^{-\mathrm{i}\beta/2} \int_{\mathscr{C}'} \mathrm{e}^{-x\alpha y^2}\ \mathrm{d}y = \frac{\mathrm{i}}{2}\sqrt{\frac{\pi}{x\alpha}}\mathrm{e}^{-\mathrm{i}\beta/2} = \mathrm{i}\sqrt{\frac{\pi}{2\phi''(z_0)x}}$$

More generally, by Taylor expanding around a general saddle point, we obtain the leading asymptotics for a path of steepest descent starting at a saddle point,

$$I(x) = \int_{\mathscr{C}} f(z)\mathrm{e}^{x\phi(z)}\ \mathrm{d}z \sim \mathrm{i}f(z_0)\mathrm{e}^{\phi(z_0)x}\sqrt{\frac{\pi}{2\phi''(z_0)x}} \quad \text{as } x \to \infty$$

Example: Stirling's Formula for the Gamma Function. Applying these techniques to the Gamma function is a bit of an overkill. Let us do it nonetheless. We have

$$\Gamma(x+1) = \int_0^\infty t^x \mathrm{e}^{-t}\ \mathrm{d}t = \int_0^\infty \mathrm{e}^{x\log t - t}\ \mathrm{d}t = x^{x+1}\int_0^\infty \mathrm{e}^{x(\log t - t)}\ \mathrm{d}t$$

where we have changed variables to bring the integral into the general form of integrals that we have been considering above. The meromorphic function $\phi(t) = \log t - t$ has a saddle on the real axis, at $t = 1$, where $\phi(t)$ viewed as a real function has its maximum.
The method of steepest descent and the Laplace method are the same in this case. Noting that there are two contributions to the integral, one from each side of the saddle, we obtain the expansion

$$\Gamma(x+1) \sim \sqrt{2\pi}x^{x+\frac{1}{2}}\mathrm{e}^{-x} \quad \text{as } x \to \infty$$

In fact, in this case, we can obtain higher-order terms by using $\Gamma(x+1) = x\Gamma(x)$, and Taylor expanding the expansion. We thus obtain, after ploughing through a certain amount of algebra,

$$\Gamma(x+1) \sim \sqrt{2\pi} x^{x+\frac{1}{2}} \mathrm{e}^{-x} \left(1 + \frac{1}{12x} + \frac{1}{288x^2} + \cdots\right) \quad \text{as } x \to \infty$$

Example: A Finite-Range Fresnel Integral. Consider the finite-range Fresnel integral

$$I(x) = \int_0^1 \mathrm{e}^{\mathrm{i}xt^2} \, \mathrm{d}t \quad \text{as } x \to \infty$$

Set $\phi(t) = \mathrm{i}t^2$. Note that $\operatorname{Im}\phi(0) \neq \operatorname{Im}\phi(1)$. It is easy to see that the paths of steepest descent or ascent are hyperbolae with axes $y^2 = x^2$. We can thus join up any such paths by a small segment at infinity, in the first quadrant, where the integrand decays. We thus obtain the asymptotic expansion

$$I(x) \sim \sqrt{\frac{\pi}{4x}} \mathrm{e}^{\mathrm{i}\pi/4} - \frac{\mathrm{i}}{2x} \mathrm{e}^{\mathrm{i}x} \quad \text{as } x \to \infty$$

The first term comes from the saddle at the origin, while the second term comes from a path of steepest descent through 1; we obtain the second term using the usual Laplace method. Note that, to be rigorous, we would need to check that there is no contribution of order $\mathcal{O}(1/x)$ from the first path.

Dispersive Waves and Group Velocity

Let us now turn to an application of the method of steepest descent. Consider a wave $\mathrm{e}^{\mathrm{i}(kx-\omega t)}$ with phase velocity ω/k. The dispersion relation that relates the frequency to the wave number is determined by the wave equation, e.g. the scaled Schrödinger equation

$$\mathrm{i}\frac{\partial \psi}{\partial t} = -\frac{\partial^2 \psi}{\partial x^2} \quad \text{with dispersion relation} \quad \omega = k^2$$

Consider a smooth wavepacket having a narrow maximum at $k = k_0$, defined by the integral

$$\psi(x, t) = \int_0^\infty b(k) \mathrm{e}^{\mathrm{i}(kx - \omega(k)t)} \, \mathrm{d}k$$

so that $b(k)$ is just the Fourier transform of $\psi(x, 0)$. Consider the asymptotics of $\psi(x, t)$ in the large-time limit $t \to \infty$, with $x = Vt$ for some constant V. Use the stationary phase approximation with $\phi(t) = kV - \omega(k)$. Note that the phase is stationary when $\omega'(k) = V$. Let k' be a solution of this equation, and assume, for convenience, that it is the only solution. For the Schrödinger equation, $k' = \frac{1}{2}V$. Now, by the stationary phase approximation,

$$\psi(x, t) \sim b(k') \mathrm{e}^{\mathrm{i}(k'x - \omega(k')t)} \frac{A}{t^{1/2}} \quad \text{as } t \to \infty, \text{ with } x = Vt$$

where A is a constant. Hence the wave amplitude decays as $t^{-1/2}$, by dispersion (note that there is no mechanism for energy loss) along the line $x = Vt$, for each V. Note that the wave disturbance is maximal when $k' = k_0$, i.e. along the line $x = \omega'(k_0)t$. The *group velocity* $c_{\mathrm{g}} = \omega'(k)$ is a function of k.
Hence, by assumption, the wavepacket moves with speed $c_{\mathrm{g}}(k_0)$, and the width of the wavepacket increases as $t \to \infty$. Note that, in the case of the free Schrödinger equation, the group velocity is related to the phase velocity $c = \omega/k$ by $c_{\mathrm{g}} = 2c$. These approximate results are to compared to the Gaussian wavepacket considered in the *Quantum Mechanics* course.

The Airy Function and the Airy Integral

In the section, we first introduce the *Airy function*, to which we will apply the various techniques introduced in this course: conveniently, it can be defined in terms of an integral, but also, it satisfies a differential equation (and we shall come back to asymptotics of solutions of differential equations). The Airy function is important for finding approximate solutions of the Schrödinger equation with non-zero potential. It is defined by

$$\mathrm{Ai}(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}(xs+\frac{1}{3}s^3)}\ \mathrm{d}s$$

where, for the time being, x is a real variable. Note that the Airy function is real-valued for real x. The integral converges for reasons similar to those explaining why the Fresnel integral converges. Note that we can deform the contour, initially the real axis, to any contour such that $\mathrm{Im}\, s > 0$ and $\mathrm{Im}\, s^3 > 0$ far away from the origin, i.e. such that $0 < \theta < \frac{\pi}{3}$ or $\frac{2\pi}{3} < \theta < \pi$ for large s, where $\theta = \arg s$. We need to impose this restriction lest the integrand blow up.

Let us thus deform the initial contour to a new contour $\mathscr{C}'$ contained in the strict upper half-plane and obeying the above restrictions. Observe that

$$\frac{\mathrm{d}^2\mathrm{Ai}}{\mathrm{d}x^2} - x\mathrm{Ai}(x) = \frac{1}{2\pi}\int_{\mathscr{C}'} -\left(s^2+x\right)\mathrm{e}^{\mathrm{i}(xs+\frac{1}{3}s^3)}\ \mathrm{d}s = \frac{\mathrm{i}}{2\pi}\int_{\mathscr{C}} \frac{\mathrm{d}}{\mathrm{d}s}\left(\mathrm{e}^{\mathrm{i}(xs+\frac{1}{3}s^3)}\right)\ \mathrm{d}s = 0$$

Note that we needed to deform the contour, because taking the derivative inside the integral is not well-defined when we integrate along the real axis. Hence the Airy function is a solution of the simple differential equation

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = xy$$

This is sometimes called the *Airy equation*. It has two linearly independent solutions, the Airy function and the 'Bairy function' $\mathrm{Bi}(x)$. By the standard method for series solutions, we find that the general solution of the above has the Taylor series about $x = 0$,

$$y(x) = \sum_{n=0}^{\infty} a_n x^n \quad \text{where } a_0, a_1 \text{ are arbitrary, } a_2 = 0, \text{ and } a_n = \frac{a_{n-3}}{n(n-1)} \text{ for } n \geqslant 3,$$

so the general solution is

$$y(x) = a_0\left(1 + \frac{x^3}{6} + \frac{x^6}{180} + \cdots\right) + a_1\left(x + \frac{x^4}{12} + \frac{x^7}{504} + \cdots\right)$$

This series has an infinite radius of convergence. The values of the constants a_0 and a_1 for the Airy function are determined by $\mathrm{Ai}(0)$ and $\mathrm{Ai}'(0)$. One shows that

$$a_0 = \frac{3^{-1/6}}{2\pi}\Gamma\left(\tfrac{1}{3}\right) \quad \text{and} \quad a_1 = -\frac{3^{1/6}}{2\pi}\Gamma\left(\tfrac{2}{3}\right)$$

Asymptotics of the Airy Function

Let us first consider asymptotics of the Airy function for large positive x. Shift the contour to a contour in the strict upper half-plane as before. Reparametrise the contour by setting $s = x^{1/2}t$, and letting $y = x^{3/2}$. Then

$$\mathrm{Ai}(x) = \frac{1}{2\pi}\int_{\mathscr{C}'} \mathrm{e}^{\mathrm{i}(xs+\frac{1}{3}s^3)}\ \mathrm{d}s \quad \Longrightarrow \quad \mathrm{Ai}\big(y^{2/3}\big) = \frac{y^{1/3}}{2\pi}\int_{\mathscr{C}'} \mathrm{e}^{\mathrm{i}y(t+\frac{1}{3}t^3)}\ \mathrm{d}t$$

Use the method of steepest descent, using the saddle point at i. Steepest descent contours through this saddle have $\mathrm{Im}\,\phi = 0$, and hence, if $t = p + \mathrm{i}q$, we have $p = 0$ or $q^2 = 1 + \frac{1}{3}p^2$ along paths of steepest descent. The second path is a hyperbola, the asymptotes of which are at angles of magnitude $\frac{\pi}{6}$ to the real axis, and therefore lie in the appropriate sector. We thus obtain

$$\mathrm{Ai}\big(y^{3/2}\big) \sim \frac{y^{-1/6}}{2\sqrt{\pi}}\mathrm{e}^{-\frac{2}{3}y} \quad \text{as } y \to \infty$$

and thus, upon reintroducing the original variable,

$$\mathrm{Ai}(x) \sim \frac{x^{-1/4}}{2\sqrt{\pi}} \mathrm{e}^{-\frac{2}{3}x^{3/2}} \quad \text{as } x \to \infty$$

Hence the Airy function decays exponentially as $x \to \infty$. This is as expected, because the phase of the integrand has no points of stationary phase along the real axis.

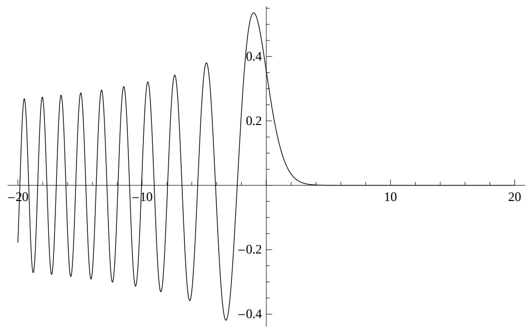
Let us now seek the asymptotics of $\mathrm{Ai}(-x)$ as $x \to \infty$. This is in fact much easier, because the integrand now has points of stationary phase on the real axis. Making the same change of variables as before,

$$\mathrm{Ai}\big(-y^{2/3}\big) = \frac{y^{1/3}}{2\pi} \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}y(-t+\frac{1}{3}t^3)} \, \mathrm{d}t \sim \frac{y^{1/3}}{2\pi} \left(\frac{\pi}{y}\right)^{\frac{1}{2}} \left[\mathrm{e}^{-\mathrm{i}(\frac{2}{3}y-\frac{\pi}{4})} + \mathrm{e}^{\mathrm{i}(\frac{2}{3}y-\frac{\pi}{4})}\right] \quad \text{as } y \to \infty$$

It follows that, collecting the complex exponentials into cosines

$$\mathrm{Ai}(-x) \sim \frac{x^{-1/4}}{\sqrt{\pi}} \cos\left(\frac{2}{3}x^{3/2} - \frac{\pi}{4}\right) \quad \text{as } x \to \infty$$

As expected, this is a real result. The function now exhibits oscillatory decay; the decay is however no longer exponential. Note that the frequency of the oscillations increases as $x \to \infty$. This result is in fact not strictly right, for the cosine on the right-hand side may vanish, in which case we would need to go to higher-order terms in the series. Notice that this series can be used to find approximations to the zeros of the Airy function; the effect of the higher-order terms is to slightly change the position of these zeros. Using these asymptotics, we can sketch the Airy function:



Note that the Airy equation looks like a (scaled) Schrödinger equation with linear potential $V(x) = x$. Hence the Airy function is the stationary state wavefunction for a particle coming in from the left and bouncing off the potential $V(x) = x$, being totally reflected in the process.

Asymptotics of Solutions of Differential Equations

In this section, we consider the asymptotics of solutions of ordinary differential equations. Consider a general second-order differential equation

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + p(x)\frac{\mathrm{d}y}{\mathrm{d}x} + q(x)y = 0$$

Use dashes to denote differentiation. Recall the Frobenius series in the neighbourhood of an ordinary point or a regular singular point x_0, of the form

$$y(x) = (x - x_0)^{\alpha}\big(a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + \cdots\big)$$

There is at least one solution of this type; possibly, there is a logarithmic solution, too. The expansion near an irregular singular point at x_0 is more difficult. This occurs when $p(x)$ is more singular than $\frac{1}{x-x_0}$, i.e. the singularity is more complicated than a simple pole, and/or $q(x)$ is more singular than $\frac{1}{(x-x_0)^2}$.

The Liouville–Green Method

The *Liouville–Green Method* is a method of obtaining approximate solutions in the neighbourhood of an irregular singular point. The basic idea is to try a solution of the form $y(x) = \mathrm{e}^{S(x)}$. Substituting into the differential equation, we obtain the exact result

$$S'' + S'^2 + p(x)S' + q(x) = 0$$

We assume that $S'' \ll S'^2$. The idea is to make this approximation, and to check later on that the solution one obtains indeed satisfies this inequality. Note that, if $S(x) = a(x - x_0)^{-b}$, for some $b > 0$, then we have $S'(x)^2 = a^2b^2(x - x_0)^{-2b-2}$, while $S''(x) = ab(b + 1)(x - x_0)^{-b-2}$, so $S'' \ll S'^2$ near the singularity, as required for our approximation. Making this approximation, we have

$$S'^2 + p(x)S' + q(x) = 0$$

This is a quadratic equation for S', which we can solve in principle. We can then integrate to get an approximate solution of the differential equation.

Example. Let us illustrate this result for $y'' = y/x^3$. This has an irregular singular point at the origin. We obtain approximate solutions $y \approx A\mathrm{e}^{2x^{-1/2}}$ and $y \approx B\mathrm{e}^{-2x^{-1/2}}$. The first solution is exponentially large, while the second one is exponentially suppressed. It is therefore not very useful to consider a linear combination of these solutions, for the error in the first solution is very much larger than the second solution. However, if we require y to be regular at $x = 0$, the second solution gives a useful approximation.

Improved Approximations

Let us now try to get a better approximation, and in fact get the leading-order asymptotic behaviour. We will illustrate the method using the above example. Look for corrections of the form

$$S(x) = \frac{2}{x^{1/2}} + C(x) \quad \text{where } C(x) \ll x^{-1/2}$$

in the limit $x \to 0^+$. Note that we have chosen one of the two possible solutions. Substituting into the exact equation gives

$$\frac{3}{2}x^{-5/2} + C'' - 2x^{-3/2}C' + C'^2 = 0 \quad \Longrightarrow \quad \frac{3}{2}x^{-5/2} - 2x^{-3/2}C' \approx 0$$

where we make the approximations $C' \ll x^{-3/2}$, so that $C'x^{-3/2} \gg C'^2$, and $C'' \ll x^{-5/2}$. In general, one cannot of course differentiate inequalities in this sense, but we aim to check retrospectively whether our approximations make sense. Rearranging the above and integrating yields $C' = \log x^{4/3}$ up to a constant of integration, and thus, we have obtained that

$$y(x) \sim Ax^{3/4}\mathrm{e}^{2x^{-1/2}} \quad \text{as } x \to 0^+$$

where A is a constant. This is in fact an asymptotic approximation, and one can obtain further terms in the asymptotic series by setting $C(x) = \frac{3}{4}\log x + D(x)$, where $D(x) \ll \log x$. It turns out, that, in fact, $D(x)$ is finite as $x \to 0^+$, and all further corrections decay as $x \to 0^+$, and so we get a proper asymptotic series with the above as prefactor.

This Liouville–Green approach outlined above has wide validity, but justifying these results rigorously is rather difficult.

The Liouville–Green Method for Irregular Singular Points at Infinity

Changing variables to $t = 1/x$, the general second-order ordinary differential equation considered above becomes

$$\frac{\mathrm{d}^2y}{\mathrm{d}t^2} + \left(\frac{2}{t} - \frac{P(t)}{t^2}\right)\frac{\mathrm{d}y}{\mathrm{d}t} + \frac{Q(t)}{t^4}\,y = 0 \quad \text{where } P(t) = p\big(\tfrac{1}{t}\big),\ Q(t) = q\big(\tfrac{1}{t}\big)$$

Typically, i.e. unless $p(x)$ and $q(x)$ decay rapidly as $x \to \infty$, this equation has an irregular singular point at $t = 0$, i.e. at infinity in x-space.

Example: The Airy Equation. The Airy equation $y''-xy=0$ has an irregular singular point at infinity. Rather than expanding about the origin after changing variables, we shall develop the Liouville–Green method for expansions about infinity.
Try a solution of the form $y(x)=\mathrm{e}^{S(x)}$. Then $S''+S'^2-x=0$. Assuming that $S''\ll S'^2$ gives $S'(x)=\pm x^{1/2}$ (which enables us to verify our assumption), so $S(x)=\pm\frac{2}{3}x^{3/2}$. Choose the exponentially decaying solution. To get an asymptotic approximation, set $S(x)=-\frac{2}{3}x^{3/2}+C(x)$, with $C(x)\ll x^{3/2}$. We thus obtain, similarly to before,

$$y(x)\sim A\mathrm{e}^{-\frac{2}{3}x^{3/2}}x^{-1/4}$$

However, there is now way of finding A for the Airy function by this method. Our integral asymptotics however led to $A=\frac{1}{2}\pi^{-1/2}$.

Observe that, in these examples, we were able to solve the equation for S' and hence find S 'exactly' once we had assumed that $S''\ll S'^2$. In other examples, this can be messy or impossible. Instead, it is often easier to expand S in suitable powers of x.

Equations with Small Leading Parameter: The WKB Method

Let $0<\varepsilon\ll 1$ be a small positive parameter. In this subsection, we consider second-order ordinary differential equations of the form

$$\varepsilon^2\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=q(x)y$$

which do not have singular points. We can still use the ideas of the Liouville–Green method developed above; in this context, they are referred to as the *WKB approximation*, after Wenzel, Kramer, Brillouin and sometimes Jeffreys, or *semiclassical method.* Note that, often, $q(x)$ has a definite length scale, so we cannot just rescale x and ignore ε.
As before, we try a solution of the form $y(x)=\mathrm{e}^{S(x)}$, so that $\varepsilon^2\left(S''+S'^2\right)-q(x)=0$. Supposing as before that $S''\ll S'^2$, we get approximate solutions

$$S(x)=\pm\frac{1}{\varepsilon}\int_?^x q(\xi)^{1/2}\ \mathrm{d}\xi$$

This shows that there is a major difference between $q(x)>0$ and $q(x)<0$: in the first case, $S(x)$ is real, so the solution grows or decays; in the second case, the solution oscillates. Solutions of $q(x)=0$ are referred to as *turning points.* Seek an improved approximation,

$$S(x)=\pm\frac{1}{\varepsilon}\int_?^x q(\xi)^{1/2}\ \mathrm{d}\xi+C(x)\quad\text{so that}\quad \pm\tfrac{1}{2}\varepsilon q^{-1/2}(x)q'(x)\pm 2\varepsilon q^{1/2}(x)C'(x)+\varepsilon^2C'^2=0$$

Balancing the terms that are $\mathcal{O}(\varepsilon)$,

$$C'(x)=-\frac{q'(x)}{4q(x)}\quad\Longrightarrow\quad C(x)=\log q(x)^{-1/4}\quad\text{assuming that } q(x)>0$$

Again, this solution breaks down close to turning points. It is remarkable that we can integrate this term. Hence we obtain a solution

$$y(x)\sim A|q(x)|^{-1/4}\exp\left(\pm\frac{1}{\varepsilon}\int_a^x|q(\xi)|^{1/2}\ \mathrm{d}\xi\right)$$

Note that this expansion is not however valid across or too close to a turning point, where the logarithm behaves badly.

Connection Across a Turning Point

The question becomes: how can we match asymptotic expansions on either side of a turning point? Assume that there is a turning point at $x=a$, with $\mu=q'(a)>0$ for convenience. Choose the solution which is exponentially decaying for large positive x, so that

$$y(x) \sim \begin{cases} \dfrac{A}{q(x)^{1/4}} \exp\left(-\dfrac{1}{\varepsilon}\displaystyle\int_a^x q(\xi)^{1/2}\,\mathrm{d}\xi\right) & \text{if } x > a \\ \dfrac{B}{|q(x)|^{1/4}} \cos\left(\dfrac{1}{\varepsilon}\displaystyle\int_x^a |q(\xi)|^{1/2}\,\mathrm{d}\xi - \gamma\right) & \text{if } x < a \end{cases}$$

where γ is an unknown phase. The above approximations are only valid if $|x-\alpha| \gg \varepsilon$. The idea is to note that the slope close to $x = a$ is approximately constant, so we can approximate the solution using the Airy function: near $x = a$, $q(x) \approx \mu(x-a)$, so

$$\varepsilon^2 \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = \mu(x-a)y \quad\implies\quad \frac{\mathrm{d}^2 y}{\mathrm{d}t^2} = ty \quad \text{where } t = \left(\frac{\mu}{\varepsilon^2}\right)^{1/3}(x-a)$$

Comparing the above asymptotics with those of the Airy function as $t \to \pm\infty$ implies that we can have the expansions match up provided that $B = 2A$ and $\gamma = \frac{\pi}{4}$.

Bounded Solutions with Two Turning Points

Now assume that $q(x)$ has two turning points at a, b, where $a > b$. We assume that the solution of the differential equation decays exponentially in $x > a$ and $x < b$. The resulting problem is essentially an eigenvalue problem. Approximating the solutions for $x < a$ and $x > b$ by their asymptotic series,

$$y(x) \sim \frac{D}{|q(x)|^{1/4}} \cos\left(\frac{1}{\varepsilon}\int_x^a \left|q(\xi)\right|^{1/2}\,\mathrm{d}\xi - \frac{\pi}{4}\right) \quad\text{and}\quad y(x) \sim \frac{D'}{|q(x)|^{1/4}} \cos\left(\frac{1}{\varepsilon}\int_b^x \left|q(\xi)\right|^{1/2}\,\mathrm{d}\xi - \frac{\pi}{4}\right)$$

where D, D' are constants. The expansions overlap and must thus agree. Note that requiring the arguments to be equal does not work, for the arguments have oppositive derivatives with respect to x. However, we may require

$$\left[\frac{1}{\varepsilon}\int_x^a |q(\xi)|^{1/2}\,\mathrm{d}\xi - \frac{\pi}{4}\right] = n\pi - \left[\frac{1}{\varepsilon}\int_b^x |q(\xi)|^{1/2}\,\mathrm{d}\xi - \frac{\pi}{4}\right] \quad\implies\quad \frac{1}{\varepsilon}\int_b^a |q(\xi)|^{1/2}\,\mathrm{d}\xi = \left(n+\tfrac{1}{2}\right)\pi$$

for some $n \in \{0, 1, 2, \dots\}$, provided that $D' = (-1)^n D$. If $q(x)$ satisfies this integral relationship, there is a bounded solution $y(x)$. Note that the amplitude of the oscillations is largest close to the turning points. This result is often useful for all natural values of n, but really, to get proper asymptotics, n needs to be suitably large, because of the $\frac{1}{\varepsilon}$ factor on the left-hand side above.

Application: Bound States in Quantum Mechanics

Consider the stationary Schrödinger equation in one-dimensional quantum mechanics, for a particle of mass m and energy E moving in a potential well $V(x)$,

$$-\frac{\hbar^2}{2m}\frac{\mathrm{d}^2\psi}{\mathrm{d}x^2} + V(x)\psi = E\psi$$

Taking $\hbar$ as our small parameter, and writing $q(x) = 2m\big(V(x) - E\big)$, and requiring the bound states to be normalisable, i.e. exponentially decaying in the region where $q(x) > 0$, the eigenvalues are, in the WKB approximation, given by

$$\frac{\sqrt{2m}}{\hbar}\int_b^a \big(E_n - V(x)\big)^{1/2}\,\mathrm{d}x = \left(n+\tfrac{1}{2}\right)\pi \quad\implies\quad \int_a^b \big(E_n - V(x)\big)^{1/2}\,\mathrm{d}x = \frac{\hbar\pi}{\sqrt{2m}}\left(n+\tfrac{1}{2}\right)$$

This is the *Bohr–Sommerfeld Quantisation Condition*. Note that a, b are functions of E. The WKB approximation is generally particularly good for large n, in the so-called *semi-classical regime*. Note that, classically, $|q(x)|^{1/2}$ is the momentum of the particle, so, in the WKB approximation,

$$\oint p\,\mathrm{d}x = (2n+1)\hbar\pi$$

where the integral is the integral over one classical orbit of the particle, i.e. the area of phase space corresponding to one orbit. The local wavelength satisfies $\frac{2\pi}{\lambda} = \frac{p}{\hbar}$, which rearranges to the well-known de Broglie relation $\lambda = 2\pi\hbar/p$. Roughly, there is one quantum state per area $2\pi\hbar$ of phase space.
Finally, the probability density averaged over one wavelength is proportional to $1/p$, as expected. This is proportional to the classical time spent in a spatial interval. Note that, as expected, the probability density is larger close to the turning points, as expected from the classical limit.

Example: The Quantum Harmonic Oscillator. The classical Hamiltonian for a harmonic oscillator is given by $\mathcal{H} = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2$. Hence the orbits in phase space are the curves

$$\frac{1}{2m}p^2 + \frac{m\omega^2}{2}x^2 = E$$

These are ellipses of semi-axes $(2mE)^{1/2}$ and $\left(2E/m\omega^2\right)^{1/2}$, and therefore of area $2\pi E/\omega$. Plugging this into the Bohr-Sommerfeld conditions gives

$$E_n = \left(n + \tfrac{1}{2}\right)\hbar\omega$$

Hence the WKB approximations gives the exact result for the energy levels of the quantum harmonic oscillator. This is why the WKB method gives a good approximation to all the energy levels, and not only the high energy levels, for potentials that look roughly like the potential of a harmonic oscillator. By contrast, the approximations for the energy levels in an infinite square well are not great, because the potential is infinite at the walls.

Stokes' Phenomenon

Suppose that $f(z) \sim g(z)$ as $|z| \to \infty$. Such an asymptotic expansion is generally only valid in a limited *sector* or range of $\arg z$, because an exponentially small error, or difference may become exponentially large when crossing a sector boundary.
For instance, e^{-z} is exponentially small for $-\frac{\pi}{2} \leqslant \arg z \leqslant \frac{\pi}{2}$, but exponentially large for $\frac{\pi}{2} \leqslant \arg z \leqslant \frac{3\pi}{2}$. Such a boundary is called a *Stokes line*.

Stokes Lines for Liouville–Green Solutions

Consider a Liouville–Green solution $f(z) \sim A\mathrm{e}^{S_1(z)} + B\mathrm{e}^{S_2(z)}$ as $|z| \to \infty$, for a second-order linear ordinary differential equation. Suppose that, in some sector, $\operatorname{Re} S_2(z) \ll \operatorname{Re} S_1(z)$, where, for once, signs matter in the comparison.
In that case, we say that the first term in the asymptotic solution is *dominant*, while the second term is *recessive* and is absent from the asymptotics.
In this setup, we define a *Stokes line* to be where $\operatorname{Re} S_1(z) = \operatorname{Re} S_2(z)$. The dominant and recessive terms may change across a Stokes line. Even keeping A, B, which is good near a Stokes line, the asymptotics expansion above is still not valid for all values of $\arg z$, because $S_1(z)$ and $S_2(z)$ are not usually entire functions, even if f is.
The coefficient of the recessive exponential therefore needs to jump in places. The best place for such a jump is where the real parts of $S_1(z)$ and $S_2(z)$ are maximally different. This occurs on the *anti-Stokes lines*. They are often where the imaginary parts of $S_1(z)$ and $S_2(z)$ are equal.

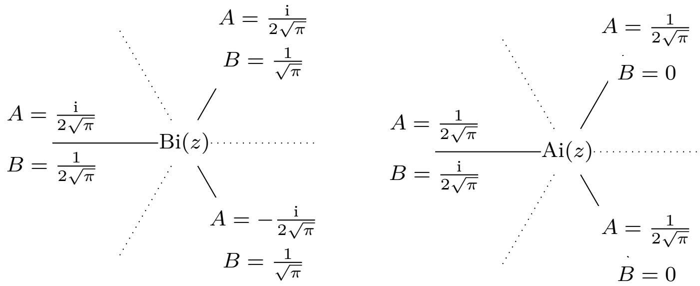
Example: The Airy and Bairy Functions
We have seen previously that the asymptotics of the solutions of $y'' = zy$ are dominated by $\mathrm{e}^{\frac{2}{3}z^{3/2}}$ and $\mathrm{e}^{-\frac{2}{3}z^{3/2}}$ for large $|z|$. Stokes lines are therefore $\arg z = \frac{\pi}{3}, \pi, \frac{5\pi}{3}$, and anti-Stokes lines are $\arg z = 0, \frac{2\pi}{3}, \frac{4\pi}{3}$. This requires some care, because the square root functions are not defined on the entire complex plane. The general solution of the Airy equation can be shown to have the asymptotic expansion

$$y(z) \sim Az^{-1/4}\mathrm{e}^{-\frac{2}{3}z^{3/2}}\left(1 - \frac{5}{48}z^{-3/2} + \cdots\right) + Bz^{-1/4}\mathrm{e}^{\frac{2}{3}z^{3/2}}\left(1 + \frac{5}{48}z^{-3/2} + \cdots\right)$$

The Airy function and the Bairy function (the former of which is exponentially decaying, while the latter is exponentially increasing as $z \to \infty$) have the expansions

$$\left.\begin{array}{l}\mathrm{Ai}(z)\\ \mathrm{Bi}(z)\end{array}\right\} \sim \pi^{-1/2}|z|^{-1/4}\cos\left(\frac{2}{3}|z|^{3/2} \mp \frac{\pi}{4}\right) \quad \text{as } z \to -\infty$$

The values of the constants A, B for the Airy function and the Bairy function are as follows in the different sectors. (Again, because the square root function is not defined on the entire complex plane, this requires some care.) The expansions shown below are valid between the (dashed) anti-Stokes lines (or slightly beyond them), and overlap in the sectors defined by the Stokes lines. Notice that the Airy function is special: since it decays exponentially as $z \to \infty$, we must have $B = 0$ along the positive real axis, so neither A or B can jump across this anti-Stokes line, and there are only two regimes, rather than three.



The Airy function also has applications in optics. In ray optics, rays can form a *caustic*: rays that are refracted by a lens can be tangent to a curve, the caustic, in such a way that on one side of the ray, there are two rays through any point, whereas there are no rays on the other side of the caustic. An example of a caustic in everyday life is a rainbow; note that light is refracted at different angles for different colours, thence the colouring of the rain bow. The intensity of the light is strongest at the caustic, where it has a inverse square-root singularity. In wave optics, one ends up with a wave amplitude

$$\varPsi(\xi) = 2\pi\mathrm{Ai}(\xi) = \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}(\xi s + \frac{1}{3}s^3)}\, \mathrm{d}s$$

where ξ is some coordinate. The rays correspond to stationary phase points with $\xi + s^2 = 0$. Hence there are no real solutions for $\xi > 0$, and two real solutions for $\xi < 0$. Interestingly, there are additional peaks, or supernumerary bows, apart from the caustic. By contrast to the caustic predicted by ray theory, the Airy function smoothes out the caustic.

Sometimes, it is possible for two caustics to meet at a cusp, diving the plane into a region where there is one ray through each point, and a region where there are three rays through each point. The corresponding amplitude is represented by a *Pearcey function*

$$\varPsi(\xi, \eta) = \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}(\eta t + \frac{1}{2}\xi t^2 + \frac{1}{4}t^4)}\, \mathrm{d}t$$

where ξ and η are coordinates. The phase of the integral is stationary when $\eta + \xi t + t^3 = 0$. For real ξ, η, it follows that there are three real solutions if $|\eta| \leqslant \left(4\xi^3/27\right)^{1/2}$, and one otherwise. This leads to a cusp.

Example: The Modified Bessel Function

Consider the modified Bessel function $\mathrm{I}_0(z)$, which is an entire solution of the second-order ordinary differential equation $zy'' + y'z - y = 0$.

Liouville–Green solutions to this differential equations are asymptotic to $(2\pi z)^{-1/2}\mathrm{e}^{z}$ and $(2\pi z)^{1/2}\mathrm{e}^{-z}$ as $|z| \to \infty$. The asymptotic solution for I_0 turns out to be

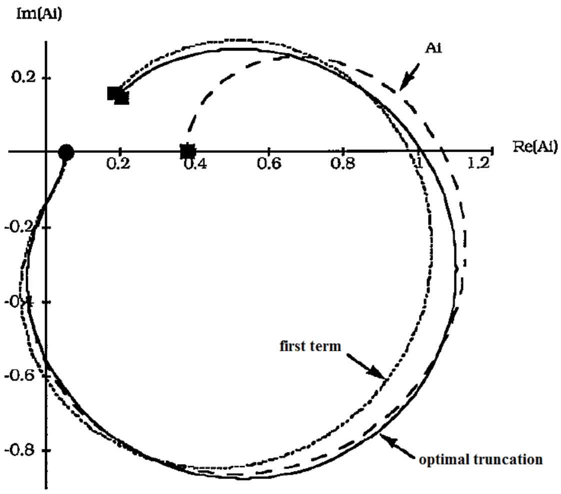
$$\mathrm{I}_0(z) \sim (2\pi z)^{-1/2}\mathrm{e}^{z}\left(1 + \frac{1}{8z} + \cdots\right) + \mathrm{i}(2\pi z)^{-1/2}\mathrm{e}^{-z}\left(1 - \frac{1}{8z} + \cdots\right) \quad \text{as } |z| \to \infty,\ \text{for } -\tfrac{\pi}{2} < \arg z < \tfrac{3\pi}{2}$$

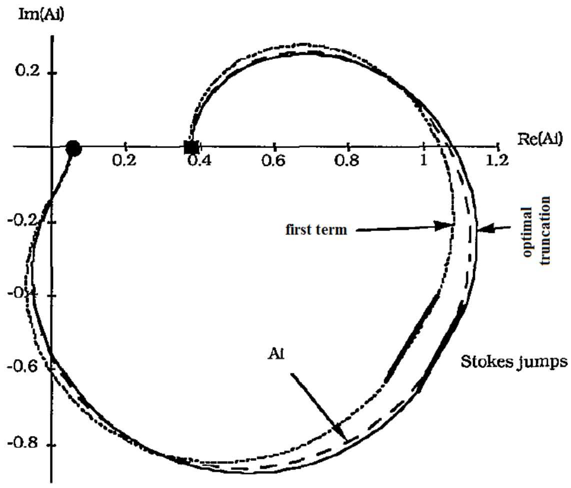
This is essentially J_0 on the imaginary axis. Notice that we need both terms close to the Stokes line $\arg z = \frac{\pi}{2}$. The asymptotic result is valid except in a small neighbourhood of $\arg z = -\frac{\pi}{2}$, but the asymptotic approximations get poorer once we cross the anti-Stokes lines $\arg z = 0$ and $\arg z = \pi$.

Asymptotics Beyond all Orders

In the previous sections, we have derived results that are valid as $|z| \to \infty$. What about actually using these expansions to get approximations results for small finite values of z?

Let us illustrate this for the Airy function, in the range $0 \leqslant \arg z \leqslant \pi$. The graphs below suggest that we need to include the recessive term for accuracy at finite $|z|$, despite this being completely negligible in the limit $|z| \to \infty$. Notice that the asymptotic expansion in the sector $0 \leqslant \arg z \leqslant \frac{2\pi}{3}$ used for the first graph becomes increasingly bad as we approach $\arg z = \pi$. Also notice that optimal truncation gives are large gain in accuracy.





Let us make some remarks on the magnitude of the erros in truncated asymptotic series. Divergent series often have 'late' terms of the form $n!X^{-n}$. Optimal truncation is at the smallest term, where $n/X \approx 1$, i.e. $n \approx X$. Hence we estimate the truncation error as $X!/X^X \approx X^{1/2}e^{-X}$ using Stirling's approximation.

Note that, for the Airy function, $X = \frac{2}{3}z^{3/2}$, so the error in the dominant series is exponentially small, and comparable with the recessive term.

This suggests as above that one can improve the asymptotic approximation by optimal truncation and inclusion of the recessive term. However, the coefficient of the recessive term jumps. It was only recently discovered that there is a something better than Stokes jumps: use the optimally truncated dominating and recessive series with coefficients A and B, and replace the discontinuous $|B|$ by an error function, i.e. an integral of Gaussian, with width depending on $|z|$: for large $|z|$, the Gaussian is narrow, for smaller $|z|$, it is wide. This greatly increases the accuracy of the 'hyperasymptotic' approximation.