Example 4 *Leading behavior of* $\int_0^\infty \cos(xt^2 - t)\,dt$ *as* $x \to +\infty$. To use the method of stationary phase, we write this integral as $\int_0^\infty \cos(xt^2 - t)\,dt = \text{Re}\int_0^\infty e^{i(xt^2-t)}\,dt$. The function $\psi(t) = t^2$ has a stationary point at $t = 0$. Since $\psi''(0) = 2$, (6.5.12) with $p = 2$ gives $\int_0^\infty \cos(xt^2 - t)\,dt \sim \text{Re}\,\frac{1}{2}\sqrt{\pi/x}\,e^{i\pi/4} = \frac{1}{2}\sqrt{\pi/2x}$ $(x \to +\infty)$.

Example 5 *Leading behavior of* $J_n(n)$ *as* $n \to \infty$. When n is an integer, the Bessel function $J_n(x)$ has the integral representation

$$J_n(x) = \frac{1}{\pi}\int_0^\pi \cos(x \sin t - nt)\,dt \tag{6.5.13}$$

(see Prob. 6.54). Therefore, $J_n(n) = \text{Re}\int_0^\pi e^{in(\sin t - t)}\,dt/\pi$. The function $\psi(t) = \sin t - t$ has a stationary point at $t = 0$. Since $\psi''(0) = 0$, $\psi'''(0) = -1$, (6.5.12) with $p = 3$ gives

$$\begin{aligned} J_n(n) &\sim \frac{1}{\pi}\text{Re}\left[\frac{1}{3}e^{-i\pi/6}\left(\frac{6}{n}\right)^{1/3}\Gamma\left(\frac{1}{3}\right)\right], & x \to +\infty, \\ &= \frac{1}{\pi}2^{-2/3}3^{-1/6}\Gamma\left(\frac{1}{3}\right)n^{-1/3}, & n \to \infty. \end{aligned} \tag{6.5.14}$$

Observe that because $\psi''(0) = 0$, $J_n(n)$ vanishes less rapidly than $n^{-1/2}$ as $n \to \infty$.
If n is not an integer, (6.5.14) still holds (see Prob. 6.55).

In this section we have obtained only the leading behavior of generalized Fourier integrals. Higher-order approximations can be complicated because nonstationary points may also contribute to the large-x behavior of the integral. Specifically, the second integral on the right in (6.5.8) must be taken into account when computing higher-order terms because the error incurred in neglecting this integral is usually algebraically small. By contrast, recall that the approximation in (6.4.2) for Laplace's method is valid to all orders because the errors are exponentially, rather than algebraically, small. To obtain the higher-order corrections to (6.5.12), one can either use the method of asymptotic matching (see Sec. 7.4) or the method of steepest descents (see Sec. 6.6).

6.6 METHOD OF STEEPEST DESCENTS

The method of steepest descents is a technique for finding the asymptotic behavior of integrals of the form

$$I(x) = \int_C h(t)e^{x\rho(t)}\,dt \tag{6.6.1}$$

as $x \to +\infty$, where C is an integration contour in the complex-t plane and $h(t)$ and $\rho(t)$ are analytic functions of t. The idea of the method is to use the analyticity of the integrand to justify deforming the contour C to a new contour C' on which $\rho(t)$ has a constant imaginary part. Once this has been done, $I(x)$ may be evaluated asymptotically as $x \to +\infty$ using Laplace's method. To see why, observe that on the contour C' we may write $\rho(t) = \phi(t) + i\psi$, where ψ is a real constant and $\phi(t)$ is a real function. Thus, $I(x)$ in (6.6.1) takes the form

$$I(x) = e^{ix\psi}\int_{C'} h(t)e^{x\phi(t)}\,dt. \tag{6.6.2}$$

Although t is complex, (6.6.2) can be treated by Laplace's method as $x \to +\infty$ because $\phi(t)$ is real.

Our motivation for deforming C into a path C' on which Im $\rho(t)$ is a constant is to eliminate rapid oscillations of the integrand when x is large. Of course, one could also deform C into a path on which Re $\rho(t)$ is a constant and then apply the method of stationary phase. However, we have seen that Laplace's method is a much better approximation scheme than the method of stationary phase because the full asymptotic expansion of a generalized Laplace integral is determined by the integrand in an arbitrarily small neighborhood of the point where Re $\rho(t)$ is a maximum on the contour. By contrast, the full asymptotic expansion of a generalized Fourier integral typically depends on the behavior of the integrand along the entire contour. As a consequence, it is usually easier to obtain the full asymptotic expansion of a generalized Laplace integral than of a generalized Fourier integral.

Before giving a formal exposition of the method of steepest descents, we consider three preliminary examples which illustrate how shifting complex contours can greatly simplify asymptotic analysis. In the first example we consider a Fourier integral whose asymptotic expansion is difficult to find by the methods used in Sec. 6.5. However, deforming the contour reduces the integral to a pair of integrals that are easy to evaluate by Laplace's method.

Example 1 *Conversion of a Fourier integral into a Laplace integral by deforming the contour.* The behavior of the integral

$$I(x) = \int_0^1 \ln t \, e^{ixt} \, dt \tag{6.6.3}$$

as $x \to +\infty$ cannot be found directly by the methods of Sec. 6.5 because there is no stationary point. Also, integration by parts is useless because $\ln 0 = -\infty$. Integration by parts is doomed to fail because, as we will see, the leading asymptotic behavior of $I(x)$ contains the factor $\ln x$ which is not a power of $1/x$.

To approximate $I(x)$ we deform the integration contour C, which runs from 0 to 1 along the real-t axis, to one which consists of three line segments: C_1, which runs up the imaginary-t axis from 0 to iT; C_2, which runs parallel to the real-t axis from iT to $1 + iT$; and C_3, which runs down from $1 + iT$ to 1 along a straight line parallel to the imaginary-t axis (see Fig. 6.5). By Cauchy's theorem, $I(x) = \int_{C_1+C_2+C_3} \ln t \, e^{ixt} \, dt$. Next we let $T \to +\infty$. In this limit the contribution from C_2 approaches 0. (Why?) In the integral along C_1 we set $t = is$, and in the integral along C_3 we set $t = 1 + is$, where s is real in both integrals. This gives

$$I(x) = i \int_0^\infty \ln (is) \, e^{-xs} \, ds - i \int_0^\infty \ln (1 + is) \, e^{ix(1+is)} \, ds. \tag{6.6.4}$$

The sign of the second integral on the right is negative because C_3 is traversed downward.

Observe that both integrals in (6.6.4) are Laplace integrals. The first integral can be done exactly. We substitute $u = xs$ and use $\ln (is) = \ln s + i\pi/2$ and the identity $\int_0^\infty e^{-u} \ln u \, du = -\gamma$, where $\gamma = 0.5772\ldots$ is Euler's constant, and obtain

$$i \int_0^\infty \ln (is) \, e^{-xs} \, ds = -i(\ln x)/x - (i\gamma + \pi/2)/x.$$

We apply Watson's lemma to the second integral on the right in (6.6.4) using the Taylor expansion $\ln (1 + is) = -\sum_{n=1}^\infty (-is)^n/n$, and obtain

$$-i \int_0^\infty \ln (1 + is) \, e^{ix(1+is)} \, ds \sim ie^{ix} \sum_{n=1}^\infty \frac{(-i)^n (n-1)!}{x^{n+1}}, \qquad x \to +\infty.$$

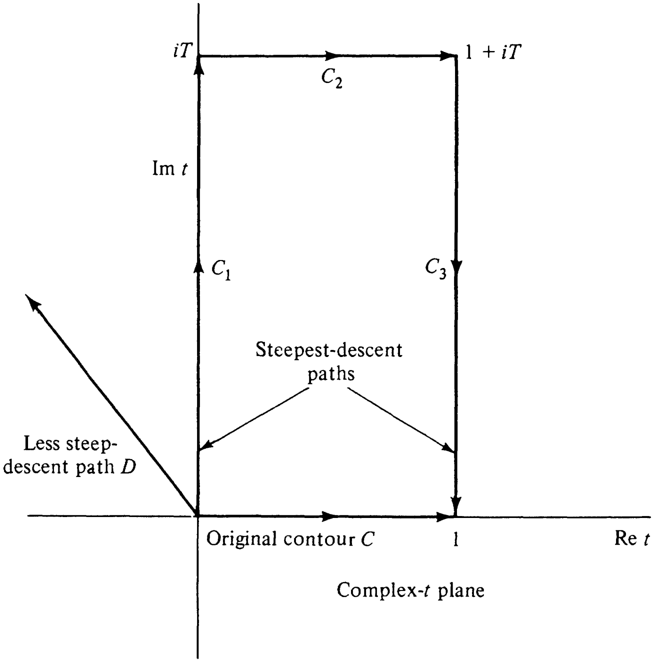

Figure 6.5 It is possible to convert the Fourier integral $I(x)$ in (6.6.3) into a Laplace integral merely by deforming the original contour C into $C_1 + C_2 + C_3$ as shown above and then allowing $T \to \infty$. C_1 and C_3 are called steepest-descent paths because $|\exp[x\rho(t)]|$ decreases most rapidly along these paths as t moves up from the real-t axis; $|\exp[x\rho(t)]|$ also decreases along D, but less rapidly per unit length than along C_1.

Combining the above two expansions gives the final result:

$$I(x) \sim -\frac{i \ln x}{x} - \frac{i\gamma + \pi/2}{x} + ie^{ix} \sum_{n=1}^{\infty} \frac{(-i)^n (n-1)!}{x^{n+1}}, \qquad x \to +\infty.$$

Let us review the calculation in the preceding example. For the integral (6.6.3), $\rho(t) = it$. For this function, paths of constant Im $\rho(t)$ are straight lines parallel to the imaginary-t axis. On the particular contours C_1 and C_3, Im $\rho(t) = 0$ and 1, respectively. Note that Im $\rho(t)$ is not the same constant on C_1 and C_3, but this does not matter; we have applied Laplace's method separately to each of the integrals on the right side of (6.6.4). Since Im $\rho(t = 0) \neq$ Im $\rho(t = 1)$, it is clear that there is no continuous contour joining $t = 0$ and $t = 1$ on which Im $\rho(t)$ is constant. This is why it is necessary to deform the original contour C into C_1 and C_3 which are joined at ∞ by C_2 along which the integrand vanishes. In general, we expect that if Im $\rho(t)$ is not the same at the endpoints of the original integration contour C, then we cannot deform C into a continuous contour on which Im $\rho(t)$ is constant; the best one can hope for is to be able to deform C into distinct constant-phase contours which are joined by a contour on which the integrand vanishes.

Now we can explain why the procedure used in Example 1 is called the method of steepest descents. The contours C_1 and C_3 are called contours of constant phase because the phase of the complex number $e^{x\rho(t)}$ is constant. At the same time, C_1 and C_3 are also called steepest-descent paths because $|e^{x\rho(t)}|$ decreases most rapidly along these paths as t ranges from the endpoints 0 and 1 toward ∞. Any path originating at the endpoints 0 and 1 and moving upward in the complex-t plane is a path on which $|e^{x\rho(t)}|$ decreases (see Fig. 6.5). However, after traversing any given length of arc, $|e^{x\rho(t)}|$ decreases more along the vertical paths C_1 and C_3 than along any other path leaving the endpoints 0 and 1, respectively. We will explain this feature of steepest-descent paths later in this section.

Example 2 *Full asymptotic behavior of* $\int_0^1 e^{ixt^2}\,dt$ *as* $x \to +\infty$. The method of stationary phase can be used to find the leading behavior of the integral $I(x) = \int_0^1 e^{ixt^2}\,dt$. Here $\psi(t) = t^2$, so the stationary point lies at $t = 0$ and, using (6.5.12), $I(x) \sim \frac{1}{2}\sqrt{\pi/x}\, e^{i\pi/4}$ $(x \to +\infty)$. The method of steepest descents gives an easy way to determine the full asymptotic behavior of $I(x)$. [The method of integration by parts also works (see Prob. 6.57).]

As in Example 1, we try to deform the contour C: $0 \le t \le 1$ into contours along which Im $\rho(t)$ is constant, where $\rho(t) = it^2$. We begin by finding a contour which passes through $t = 0$ and on which Im $\rho(t)$ is constant. Writing $t = u + iv$ with u and v real, we obtain Im $\rho(t) = u^2 - v^2$. At $t = 0$, Im $\rho = 0$. Therefore, constant-phase contours passing through $t = 0$ must satisfy $u = v$ or $u = -v$ everywhere along the contour (see Fig. 6.6). On the contour $u = -v$, Re $\rho(t) = 2v^2$, so $|e^{x\rho(t)}| = e^{2xv^2}$ increases as $t = (i - 1)v \to \infty$. This is called a steepest-ascent contour; since there is no maximum of $|e^{x\rho(t)}|$ on this contour, Laplace's method cannot be applied. On the other hand, the contour $u = v$ is a steepest-descent contour because Re $\rho(t) = -2v^2$, so $|e^{x\rho(t)}| = e^{-2xv^2}$ decreases as $t = (1 + i)v \to \infty$. The contour C_1: $t = (1 + i)v$ $(0 \le v < \infty)$ is comparable to the contour C_1 employed in Example 1.

Next, we must find a steepest-descent contour passing through $t = 1$ along which Im $\rho(t)$ is constant. At $t = 1$, the value of Im $\rho(t)$ is 1. Therefore, the constant-phase contour passing through $u = 1$, $v = 0$ is given by $u = \sqrt{v^2 + 1}$. Since Re $\rho(t) = -2uv$ decreases as $t = u + iv \to \infty$ along the portion of this constant-phase contour with $0 \le v < \infty$, the steepest-descent contour passing through $t = 1$ is given by C_3: $t = \sqrt{v^2 + 1} + iv$, $0 \le v < \infty$. Note that C_1 and C_3 become tangent as $v \to +\infty$ (see Fig. 6.6).

The next step is to deform the original contour C: $0 \le t \le 1$ into $C_1 + C_3$, in which C_3 is traversed from $t = \infty$ to $t = 1$. Along C_1, Im $\rho(t) = 0$, while along C_3, Im $\rho(t) = 1$. Since the value of Im $\rho(t)$ is different on C_1 and C_3, it is clear that the original contour cannot be continuously deformed into $C_1 + C_3$. Rather, we must include a third contour C_2 which bridges the gap between C_1 and C_3. We take C_2 to be the straight line connecting the points $(1 + i)V$ on C_1 and $\sqrt{V^2 + 1} + iV$ on C_3 (see Fig. 6.6). C can be continuously deformed into C_2 together with the portions of C_1 and C_3 satisfying $0 \le v \le V$. Now, as $V \to \infty$, the contribution from the contour C_2 vanishes. (Why?) Thus,

$$I(x) = \int_{C_1} e^{ixt^2}\,dt - \int_{C_3} e^{ixt^2}\,dt. \tag{6.6.5}$$

The integral along C_1 can be evaluated exactly. Setting $t = (1 + i)v$, we obtain

$$\int_{C_1} e^{ixt^2}\,dt = (1 + i)\int_0^\infty e^{-2xv^2}\,dv = \frac{1}{2}\sqrt{\frac{\pi}{x}}\,e^{i\pi/4}. \tag{6.6.6}$$

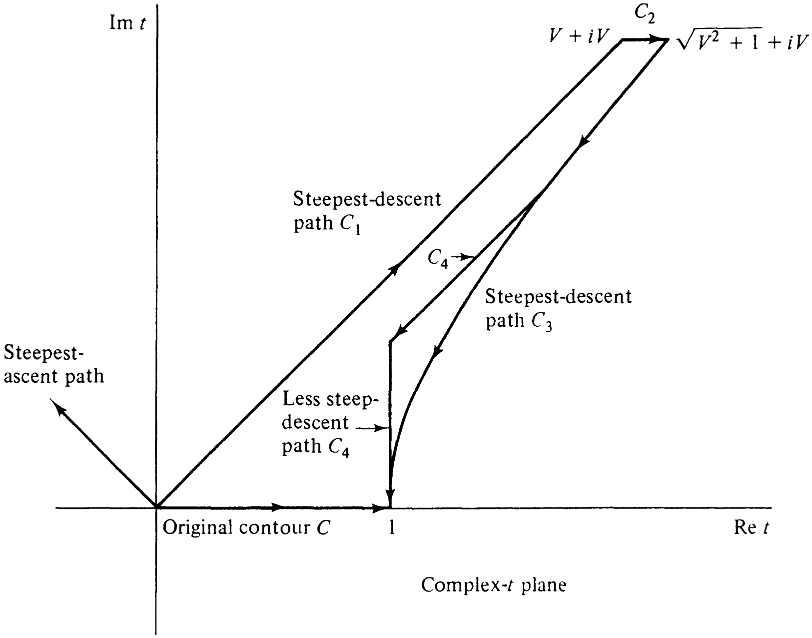

Figure 6.6 The Fourier integral $I(x)$ in Example 2 becomes a pair of Laplace integrals if the original contour C is distorted into $C_1 + C_2 + C_3$ and V is allowed to approach ∞. To simplify the evaluation of the integral along C_3, we can replace the lower part of the contour C_3 by C_4.

This contribution is precisely the leading behavior of $I(x)$ as $x \to +\infty$ that we found using the method of stationary phase.

Now we evaluate the contribution to $I(x)$ from the integral on C_3. Note that if we substitute $t = \sqrt{v^2 + 1} + iv$, $0 \le v < \infty$, then $\rho(t) = it^2 = i - 2v\sqrt{v^2 + 1}$. This verifies that C_3 is a curve of constant phase; it is also a curve of steepest descent. An easy way to obtain the full asymptotic expansion of the integral over C_3 is to use Watson's lemma. To do this, the integral must be expressed in the form $\int_0^\infty f(s)e^{-xs}\,ds$. This motivates the change of variables from t to s where s is defined by

$$\rho(t) = it^2 = i - s; \tag{6.6.7}$$

observe that $s = 2v\sqrt{v^2 + 1}$ is real and satisfies $0 \le s < \infty$ along C_3. Since $t = (1 + is)^{1/2}$, $dt/ds = \frac{1}{2}i(1 + is)^{-1/2}$, so

$$\int_{C_3} e^{ixt^2}\,dt = \tfrac{1}{2}ie^{ix}\int_0^\infty \frac{e^{-xs}}{\sqrt{1 + is}}\,ds.$$

To apply Watson's lemma, we use the Taylor expansion

$$(1 + is)^{-1/2} = \sum_{n=0}^{\infty} (-is)^n\Gamma(n + \tfrac{1}{2})/n!\,\Gamma(\tfrac{1}{2}).$$

We obtain

$$\int_{C_3} e^{ixt^2}\,dt \sim \frac{1}{2} i e^{ix} \sum_{n=0}^{\infty} (-i)^n \frac{\Gamma(n+\frac{1}{2})}{\Gamma(\frac{1}{2})x^{n+1}}, \qquad x \to +\infty. \tag{6.6.8}$$

Combining this result with that in (6.6.6) gives the full asymptotic expansion of $I(x)$ as $x \to +\infty$:

$$I(x) \sim \frac{1}{2}\sqrt{\frac{\pi}{x}}\, e^{i\pi/4} - \frac{1}{2} i e^{ix} \sum_{n=0}^{\infty} (-i)^n \frac{\Gamma(n+\frac{1}{2})}{\Gamma(\frac{1}{2})x^{n+1}}, \qquad x \to +\infty. \tag{6.6.9}$$

Finally, we mention an alternative way to obtain the result in (6.6.8) for the integral on C_3. The substitution in (6.6.7) is an exact parametrization of the curve C_3 in terms of the real parameter s. However, as we know from our discussion of Laplace's method in Sec. 6.4, it is only the immediate neighborhood of the maximum at $t = 1$ that contributes to the full asymptotic expansion of the integral on C_3. Therefore, it is not necessary to follow the curve C_3 exactly. It is correct to shift the integration path C_3 to one which still passes through the maximum at $t = 1$ and which is a descent contour in the sense that $|e^{x\rho(t)}|$ decreases and rejoins C_3 for large $|t|$. Any deformation of C_3 of this kind does not change the value of the integral because the integrand is analytic. For the present example, a convenient alternative to C_3 is a contour C_4 which originates at $t = 1$, goes vertically upward parallel to the imaginary-t axis, and then rejoins the contour C_3 at any point in the upper half plane (see Fig. 6.6). Only the vertical straight-line portion of C_4 in the immediate vicinity of $t = 1$ contributes to the full asymptotic expansion of the integral. We can parametrize the straight-line portion of C_4 near $t = 1$ by $t = 1 + iv$, where v is real and $0 \le v \le \varepsilon$ with ε small. Thus,

$$\begin{aligned}\int_{C_3} e^{ixt^2}\,dt &= \int_{C_4} e^{ixt^2}\,dt \sim i\int_0^{\varepsilon} e^{ix(1+iv)^2}\,dv\\ &= ie^{ix}\int_0^{\varepsilon} e^{-2xv}e^{-ixv^2}\,dv, \qquad x \to +\infty.\end{aligned}$$

Using Laplace's method

$$\begin{aligned}\int_0^{\varepsilon} e^{-2xv}e^{-ixv^2}\,dv &\sim \int_0^{\varepsilon} e^{-2xv} \sum_{n=0}^{\infty} \frac{(-ix)^n v^{2n}}{n!}\,dv\\ &\sim \sum_{n=0}^{\infty} \frac{(-i)^n(2n)!}{2^{2n+1}n!\,x^{n+1}}, \qquad x \to +\infty.\end{aligned}$$

Since $(2n)!/(2^{2n}n!) = \Gamma(n+\frac{1}{2})/\Gamma(\frac{1}{2})$, we have reproduced (6.6.8) exactly.

This alternative calculation, in which we have replaced the curved path C_3 by a path C_4 which begins as a straight line, is an important computational device that is frequently helpful in the method of steepest descents. Note that C_4 is neither a curve of constant phase nor a curve of steepest descent, although it is a curve of descent of $|e^{x\rho(t)}|$. Other descent curves could be used instead of C_4 (see Prob. 6.58).

Example 3 *Sophisticated example of the method of steepest descents.* What is the leading behavior of the generalized Fourier integral

$$I(x) = \int_0^1 \exp\,(ixe^{-1/s})\,ds \tag{6.6.10}$$

as $x \to +\infty$? This is a sophisticated example because $s = 0$ is an infinite-order stationary point; i.e., all derivatives of $e^{-1/s}$ vanish as $s \to 0+$. We know from our discussion of the method of stationary phase that if the first nonvanishing derivative of ψ in (6.5.1) at a stationary point is $\psi^{(p)}$, then $I(x)$ must vanish like $x^{-1/p}$ as $x \to +\infty$. Therefore, we expect that if the integrand has an infinite-order stationary point, $I(x)$ vanishes less rapidly than any power of $1/x$ as $x \to +\infty$. However, the Riemann-Lebesgue lemma guarantees that $I(x)$ does indeed vanish as $x \to +\infty$.

How fast does $I(x)$ in (6.6.10) vanish? It is hard to apply the method of stationary phase to $I(x)$ directly. (Try it!) However, the method of steepest descents provides a relatively easy approach. We begin by making the substitution $t = e^{-1/s}$:

$$I(x) = \int_0^{1/e} \frac{e^{ixt}}{t(\ln t)^2}\, dt.$$

The form of this integral is similar to that of the integral (6.6.3) considered in Example 1. Therefore, as in Example 1, we shift the contour $C: 0 \le t \le 1/e$ to two vertical lines parallel to the imaginary-t axis:

$$I(x) = \int_{C_1} \frac{e^{ixt}}{t(\ln t)^2}\, dt - \int_{C_3} \frac{e^{ixt}}{t(\ln t)^2}\, dt, \tag{6.6.11}$$

where C_1 is the path $t = iv$ $(0 \le v < \infty)$ and C_3 is the path $t = 1/e + iv$ $(0 \le v < \infty)$. Now we find the leading behavior of each of the integrals on the right in (6.6.11).

The integral on the path C_3 requires only a straightforward application of Laplace's method. We substitute $t = 1/e + iv$ $(0 \le v < \infty)$ and obtain

$$\int_{C_3} \frac{e^{ixt}}{t(\ln t)^2}\, dt = ie^{ix/e} \int_0^\infty \frac{e^{-xv}}{(1/e + iv)[\ln (1/e + iv)]^2}\, dv \sim ie^{ix/e} e/x, \qquad x \to +\infty. \tag{6.6.12}$$

The integral on C_1 is more difficult. We simplify the integral by substituting $t = iv$ $(0 \le v < \infty)$ and perform one integration by parts:

$$I_1(x) = \int_{C_1} \frac{e^{ixt}}{t(\ln t)^2}\, dt = \int_0^\infty \frac{e^{-xv}}{v[\ln (iv)]^2}\, dv = -x \int_0^\infty \frac{e^{-xv}}{\ln (iv)}\, dv. \tag{6.6.13}$$

The integral on the right side of (6.6.13) is a Laplace integral; we can restrict the range of integration to the vicinity of $v = 0$ without altering its asymptotic expansion as $x \to +\infty$. Thus,

$$I_1(x) \sim -x \int_0^\varepsilon \frac{e^{-xv}}{\ln (iv)}\, dv, \qquad x \to +\infty.$$

This integral does not yield to a straightforward application of Laplace's method because the integrand vanishes at $v = 0$. Moreover, the conventional treatment of a moving maximum [see the derivation of the Stirling series for $\Gamma(x)$ given in Example 10 of Sec. 6.4] does not work because the moving maximum of the integrand is too broad (see Prob. 6.47). A good way to proceed is to substitute $r = xv$ and thus obtain

$$I_1(x) \sim -\int_0^{\varepsilon x} \frac{e^{-r}}{\ln r - \ln x + i\pi/2}\, dr, \qquad x \to +\infty,$$

where we have used the relation $\ln (iv) = \ln v + i\pi/2$. Next, we argue that the immediate vicinity of the origin, say $0 \le r \le 1/x^{1/2}$, does not contribute to the asymptotic expansion of the integral as $x \to +\infty$. To prove this we bound the contribution to $I_1(x)$ from $0 \le r \le 1/x^{1/2}$:

$$\left| \int_0^{1/x^{1/2}} \frac{e^{-r}}{\ln r - \ln x + i\pi/2}\, dr \right| \le \frac{2}{\pi x^{1/2}},$$

because $|\ln r - \ln x + i\pi/2| \ge \pi/2$ and $|e^{-r}| \le 1$. This contribution to $I_1(x)$ is negligible because, as we shall see, the full asymptotic expansion of $I_1(x)$ is a series in inverse powers of $\ln x$. Thus,

$$I_1(x) \sim -\int_{1/x^{1/2}}^{\varepsilon x} \frac{e^{-r}}{\ln r - \ln x + i\pi/2}\, dr, \qquad x \to +\infty. \tag{6.6.14}$$

To expand the integral in (6.6.14), we Taylor expand the integrand in powers of $1/\ln x$:

$$\frac{1}{\ln r - \ln x + i\pi/2} = -\frac{1}{\ln x} \sum_{n=0}^{\infty} \left(\frac{i\pi/2 + \ln r}{\ln x} \right)^n, \qquad x^{-1/2} \le r \le \varepsilon x,\ x \to +\infty.$$

Thus, $$I_1(x) \sim \frac{1}{\ln x} \sum_{n=0}^{\infty} \int_{1/x^{1/2}}^{\varepsilon x} e^{-r}\left(\frac{i\pi/2 + \ln r}{\ln x}\right)^n dr, \qquad x \to +\infty. \tag{6.6.15}$$

The range of each of the integrals in (6.6.15) can be extended to $0 \le r < \infty$ with an error smaller than any inverse power of ln x. (Why?) Evaluating the first two integrals, we obtain

$$I_1(x) \sim \frac{1}{\ln x} + \frac{i\pi/2 - \gamma}{(\ln x)^2} + \cdots, \tag{6.6.16}$$

where we have used $\int_0^\infty \ln re^{-r}\, dr = -\gamma$. The coefficient of the general term in (6.6.16) may be expressed in terms of derivatives of $\Gamma(t)$ at $t = 1$ (see Prob. 6.59).

Combining the results (6.6.12) and (6.6.16) with (6.6.11), we obtain the final result

$$I(x) \sim \frac{1}{\ln x} + \frac{i\pi/2 - \gamma}{(\ln x)^2} + \cdots, \qquad x \to +\infty. \tag{6.6.17}$$

One could not have guessed this result from a cursory inspection of the original integral in (6.6.10)! Does this asymptotic series diverge? (See Prob. 6.60.) In Table 6.1 we compare numerical values of $I(x)$ with the asymptotic results for $I(x)$ given in (6.6.17).

Formal Discussion of Steepest-Descent Paths in the Complex Plane

In the previous three introductory examples, we have shown that deforming contours of integration in the complex-t plane can facilitate the asymptotic evaluation of integrals. It is now appropriate to give a more general discussion of steepest-descent (constant-phase) contours.

We begin by recalling the role of the gradient in elementary calculus. If $f(u, v)$ is a differentiable function of two variables, then the gradient of f is the vector $\nabla f = (\partial f/\partial u, \partial f/\partial v)$. This vector points in the direction of the most rapid change of f at the point (u, v). In terms of the gradient, the directional derivative df/ds in the direction of the unit vector $\mathbf{n}$ is $df/ds = \mathbf{n} \cdot \nabla f$. This directional derivative is the rate of change of f in the direction $\mathbf{n}$. Thus, the largest directional derivative is in the direction $\mathbf{n} = \nabla f/|\nabla f|$ and has magnitude $|\nabla f|$. On a two-dimensional contour plot of $f(u, v)$, the vector ∇f is perpendicular to the contours of constant f

Table 6.1 Comparison between the exact value of the integral $I(x)$ in (6.6.10) and one-term and two-term asymptotic approximations to $I(x)$ in (6.6.17) obtained using the method of steepest descents

ln x	Exact value of $I(x)$	One-term asymptotic approximation	Two-term asymptotic approximation
0	$0.9814 + 0.1467i$	∞	∞
2	$0.3077 + 0.5419i$	0.5000	$0.3557 + 0.3927i$
4	$0.2499 + 0.0643i$	0.2500	$0.2139 + 0.0982i$
6	$0.1428 + 0.0423i$	0.1667	$0.1506 + 0.0436i$
8	$0.1146 + 0.0227i$	0.1250	$0.1160 + 0.0245i$
10	$0.0935 + 0.0143i$	0.1000	$0.0942 + 0.0157i$
12	$0.0790 + 0.0100i$	0.0833	$0.0793 + 0.0109i$

(level curves). Note that the directional derivative in the direction of the tangents to a level curve is 0.

We will now give a formal proof that constant-phase contours are also steepest contours. Let $\rho(t) = \phi(t) + i\psi(t)$ be an analytic function of the complex variable $t = u + iv$. Also, for the moment, we restrict ourselves to regions of the complex-t plane in which $\rho'(t) \neq 0$.

We define a constant-phase contour of $e^{x\rho(t)}$ where $x > 0$ as a contour on which $\psi(t)$ is constant. A steepest contour is defined as a contour whose tangent is parallel to $\nabla |e^{x\rho(t)}| = \nabla e^{x\phi(t)}$, which is parallel to $\nabla\phi$. That is, a steepest contour is one on which the magnitude of $e^{x\rho(t)}$ is changing most rapidly with t.

Now we will show that if $\rho(t)$ is analytic, then constant-phase contours are steepest contours. If $\rho(t)$ is analytic, then it satisfies the Cauchy-Riemann equations

$$\partial\phi/\partial u = \partial\psi/\partial v, \qquad \partial\phi/\partial v = -\partial\psi/\partial u.$$

Therefore,

$$(\partial\phi/\partial u)(\partial\psi/\partial u) + (\partial\phi/\partial v)(\partial\psi/\partial v) = 0.$$

However, this equation can be written in vector form as $\nabla\phi \cdot \nabla\psi = 0$, so $\nabla\phi$ is perpendicular to $\nabla\psi$ and the directional derivative in the direction of $\nabla\phi$ satisfies $d\psi/ds = 0$. Thus, ψ is constant on contours whose tangents are parallel to $\nabla\phi$, showing that constant-phase contours are also steepest contours.

There is a slightly more sophisticated way to establish that constant-phase contours are steepest contours. It is well known that an analytic function $\rho(t)$ is a conformal (angle-preserving) mapping from the complex-t plane (u, v) to the complex-ρ plane (ϕ, ψ) if $\rho'(t) \neq 0$. Therefore, since lines of constant u are perpendicular to lines of constant v, lines of constant ϕ are perpendicular to lines of constant ψ. But lines of constant ϕ are also perpendicular to steepest curves of ϕ. This reestablishes the identity of steepest and constant-phase contours.

In the above two arguments, it was necessary to assume that $\rho'(t) \neq 0$. In the second argument, this condition was necessary because a map is not conformal at a point where $\rho'(t) = 0$. Where was this condition used in the first argument?

Saddle Points

When the contour of integration in (6.6.1) is deformed into constant-phase contours, the asymptotic behavior of the integral is determined by the behavior of the integrand near the local maxima of $\phi(t)$ along the contour. These local maxima of $\phi(t)$ may occur at endpoints of constant-phase contours (see Examples 1 to 3) or at an interior point of a constant-phase contour. If $\phi(t)$ has an interior maximum then the directional derivative along the constant-phase contour $d\phi/ds = |\nabla\phi|$ vanishes. The Cauchy-Riemann equations imply that $\nabla\phi = \nabla\psi = 0$ so $\rho'(t) = 0$ at an interior maximum of ϕ on a constant-phase contour.

A point at which $\rho'(t) = 0$ is called a *saddle point*. Saddle points are special because it is only at such a point that two distinct steepest curves can intersect. When $\rho'(t_0) \neq 0$, there is only one steepest curve passing through t and its tangent is parallel to $\nabla\phi$. In the direction of $\nabla\phi$, $|e^{x\rho}|$ is increasing so this portion of the curve is a steepest-*ascent* curve; in the direction of $-\nabla\phi$, $|e^{x\rho}|$ is decreasing so this portion of the curve is a steepest-*descent* curve. On the other hand, when $\rho'(t_0) = 0$ there are two or more steepest-ascent curves and two or more steepest-descent curves emerging from the point t_0.

To study the nature of the steepest curves emerging from a saddle point, let us study the region of the complex-t plane near t_0.

Example 4 *Steepest curves of* e^{xt^2} *near the saddle point* $t = 0$. Here $\rho(t) = t^2$. Observe that $\rho'(t) = 2t$ vanishes at $t = 0$, which verifies that 0 is a saddle point. We substitute $t = u + iv$ and identify the real and imaginary parts of $\rho(t)$:

$$\rho(t) = u^2 - v^2 + 2iuv, \qquad \phi(t) = u^2 - v^2, \qquad \psi(t) = 2uv.$$

Since $\rho(0) = 0$, the constant-phase contours that pass through $t = 0$ must satisfy $\psi(t) = 0$ everywhere. The constant-phase contours $u = 0$ (the imaginary axis) and $v = 0$ (the real axis) cross at the saddle point $t = 0$.

All four curves that emerge from $t = 0$, (a) $u = 0$ with v positive, (b) $u = 0$ with v negative, (c) $v = 0$ with u positive, and (d) $v = 0$ with u negative, are steepest curves because $\rho'(t) \neq 0$ except at $t = 0$. Which of these four curves are steepest-ascent curves and which are steepest-descent curves? On curves (a) and (b), $\phi(t) = -v^2$, so ϕ is decreasing away from $t = 0$; these curves are steepest-descent curves. On curves (c) and (d), $\phi(t) = u^2$, so ϕ is increasing away from $t = 0$; these curves are steepest-ascent curves. A plot showing these steepest-ascent and -descent curves as well as the level curves of ϕ away from $t = 0$ is given in Fig. 6.7.

Example 5 *Steepest curves of* $e^{ix \cosh t}$ *near the saddle point* $t = 0$. Here $\rho(t) = i \cosh t$, so $\rho'(t) = i \sinh t$ vanishes at $t = 0$. If we substitute $t = u + iv$ and use the identity

$$\cosh (u + iv) = \cosh u \cos v + i \sinh u \sin v,$$

we obtain the real and imaginary parts of $\rho(t)$:

$$\phi(t) = -\sinh u \sin v, \qquad \psi(t) = \cosh u \cos v.$$

Since $\rho(0) = i$, the constant-phase contours passing through $t = 0$ must satisfy $\psi(t) = \text{Im } \rho(t) = 1$. Thus, the constant-phase contours through $t = 0$ are given by

$$\cosh u \cos v = 1.$$

Other constant-phase contours (steepest-descent and -ascent curves) are given by $\cosh u \cos v = c$, where c is a constant. On Fig. 6.8 we plot the constant-phase contours for various values of c. Observe that steepest curves never cross except at saddle points.

Example 6 *Steepest curves of* $e^{x(\sinh t - t)}$ *near the saddle point at* $t = 0$. Here $\rho(t) = \sinh t - t$, so $\rho'(t) = \cosh t - 1$ vanishes at $t = 0$. Note that $\rho''(t) = \sinh t$ also vanishes at 0 and that the lowest nonvanishing derivative of ρ at $t = 0$ is $\rho'''(t)$. We call such a saddle point a third-order saddle point. At $t = 0$ six constant-phase contours meet. To find these contours we substitute $t = u + iv$ and identify the real and imaginary parts of ρ:

$$\rho = \phi + i\psi = (\sinh u \cos v - u) + i(\cosh u \sin v - v).$$

But $\rho(0) = 0$. Thus, constant-phase contours passing through $t = 0$ satisfy $\cosh u \sin v - v = 0$. Solutions to this equation are $v = 0$ (the u axis) and $u = \text{arc cosh } (v/\sin v)$.

In Prob. 6.61 you are asked to verify that (a) a total of six steepest paths emerge from $t = 0$; (b) paths emerge at 60° angles from adjacent paths; (c) as t moves away from 0, the paths alternate between steepest-ascent and steepest-descent paths; (d) the paths approach $\pm\infty$, $\pm\infty + i\pi$, $\pm\infty - i\pi$. All these results are shown on Fig. 6.9.

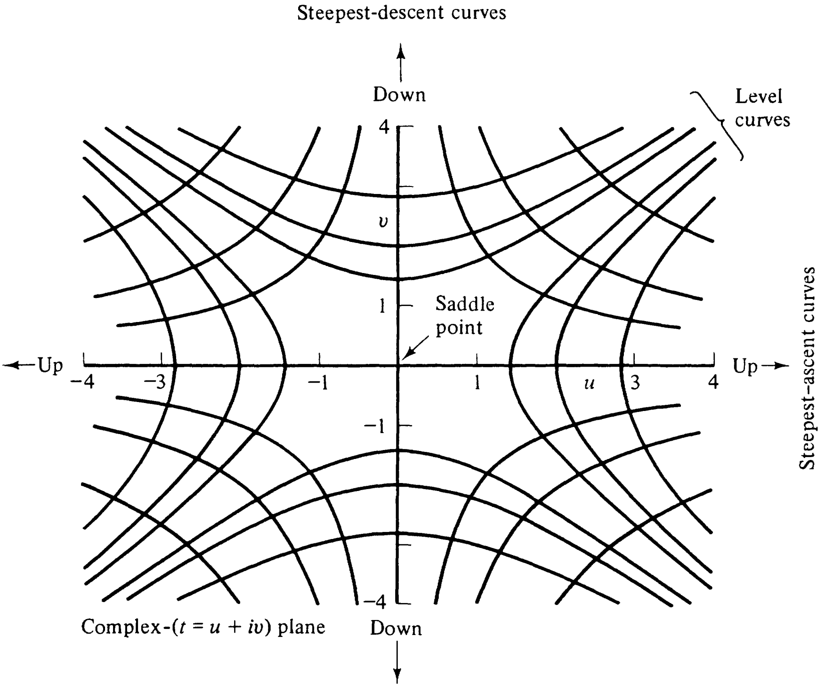

Figure 6.7 Steepest curves of e^{xt^2} near the saddle point at $t = 0$ in the complex-t plane. The steepest curves satisfy uv = constant. The level curves of ϕ satisfy $u^2 - v^2$ = constant and are orthogonal to the steepest curves.

Example 7 *Steepest curves of* $e^{x(\cosh t - t^2/2)}$ *near the saddle point at* $t = 0$. Here $\rho(t) = \cosh t - t^2/2$. Note that $\rho'(t)$, $\rho''(t)$, and $\rho'''(t)$ all vanish at $t = 0$. The first nonvanishing derivative of $\rho(t)$ at $t = 0$ is $d^4\rho/dt^4$, so we call $t = 0$ a fourth-order saddle point. Eight constant-phase curves meet at $t = 0$. Note that

$$\rho(t) = \cosh u \cos v + (v^2 - u^2)/2 + i(\sinh u \sin v - uv).$$

Thus, constant-phase contours emerging from $t = 0$ satisfy $\psi = \sinh u \sin v - uv = 0$. Solutions to this equation are $u = 0$ (the imaginary axis), $v = 0$ (the real axis), and $(\sinh u)/u = v/\sin v$.

In Prob. 6.62 you are asked to verify the results on Fig. 6.10. Namely, that (*a*) eight steepest paths emerge from $t = 0$, all equally spaced at 45° from each other; (*b*) as t moves away from 0, the paths alternate between steepest-ascent and steepest-descent paths; (*c*) the four steepest-ascent paths lie on the u and v axes; (*d*) the four steepest-descent paths approach $\pm\infty + i\pi$, $\pm\infty - i\pi$.

Steepest-Descent Approximation to Integrals with Saddle Points

We have seen that by shifting the integration contour so that it follows a path of constant phase we can treat an integral of the form in (6.6.1) by Laplace's method.

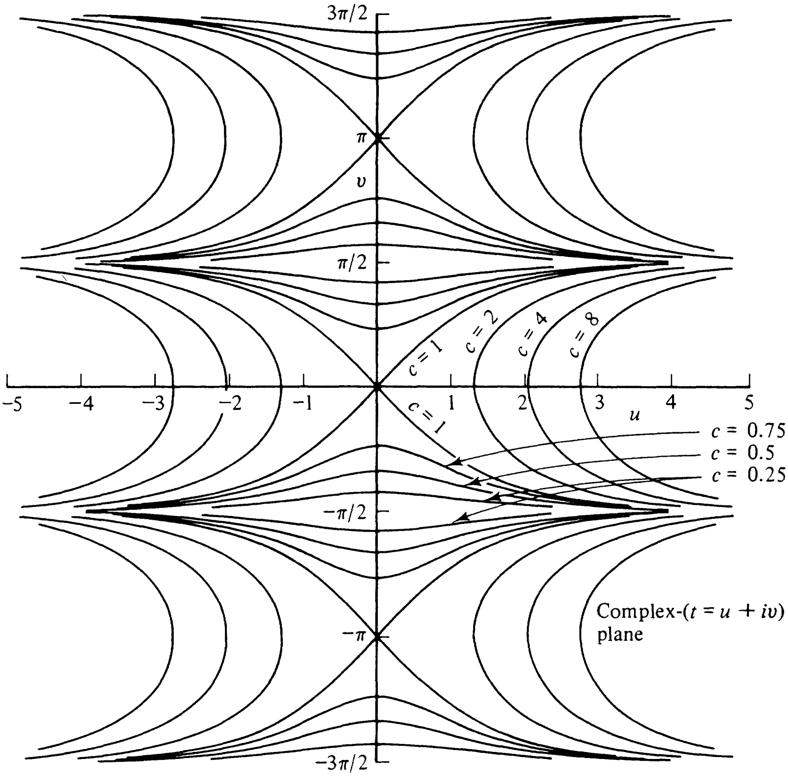

Figure 6.8 Constant-phase (steepest) contours of $\exp(ix \cosh t)$ in the complex-$(t = u + iv)$ plane. Constant-phase contours satisfy $(\cosh u)(\cos v) = c$, where c is a constant. Saddle points lying at $t = 0$ and $t = \pm i\pi$ are shown.

What happens when the constant-phase contour passes through a saddle point? In the following examples we encounter this situation.

Example 8 *Asymptotic expansion of* $J_0(x)$ *as* $x \to +\infty$. A standard integral representation for $J_0(x)$ [see (6.5.13)] is $J_0(x) = \int_{-\pi/2}^{\pi/2} \cos(x \cos\theta)\, d\theta/\pi$, which can be transformed into

$$J_0(x) = \text{Re}\, \frac{1}{i\pi} \int_{-i\pi/2}^{i\pi/2} dt\, e^{ix \cosh t} \tag{6.6.18a}$$

by substituting $t = i\theta$.

We can certainly use the method of stationary phase to find the leading behavior of this integral as $x \to +\infty$ (see Prob. 6.54). However, it is better to use the method of steepest descents to find the higher-order corrections to the leading behavior. (Why?)

To apply the method of steepest descents we extend the contour to infinity. Note that the integrals

$$\frac{1}{i\pi} \int_{-\infty - i\pi/2}^{-i\pi/2} dt\, e^{ix \cosh t}, \tag{6.6.18b}$$

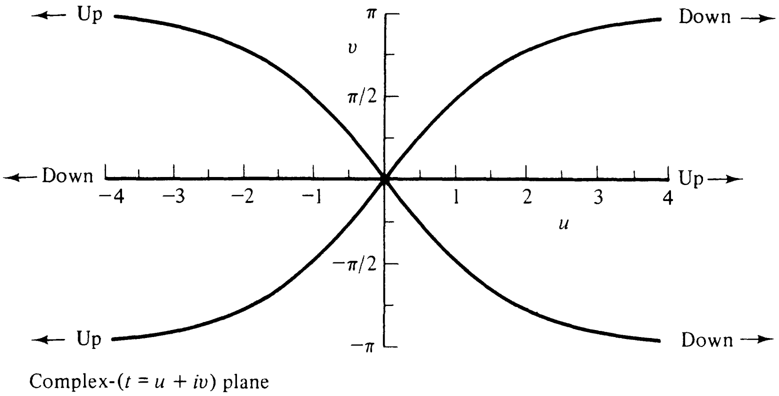

Figure 6.9 Steepest curves of $\exp[x(-t + \sinh t)]$ near the third-order saddle point at $t = 0$. The plot indicates that three steepest-descent curves and three steepest-ascent curves meet at $t = 0$.

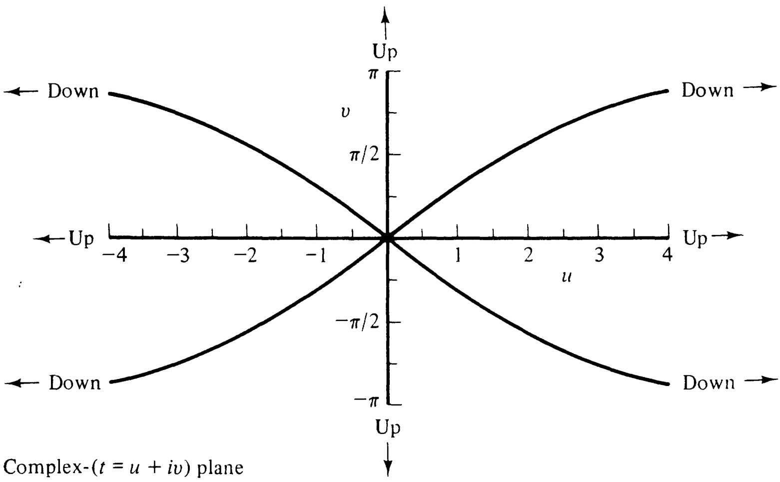

Figure 6.10 Steepest curves of $\exp[x(-\frac{1}{2}t^2 + \cosh t)]$ near the fourth-order saddle point at $t = 0$. The graph shows that four steepest-descent curves and four steepest-ascent curves meet at $t = 0$. In Example 12 the structure of the saddle point is the same as the one in this graph shifted by $i\pi$; the steepest-descent curve used in Example 12 consists of the curves in the third and fourth quadrants of this figure.

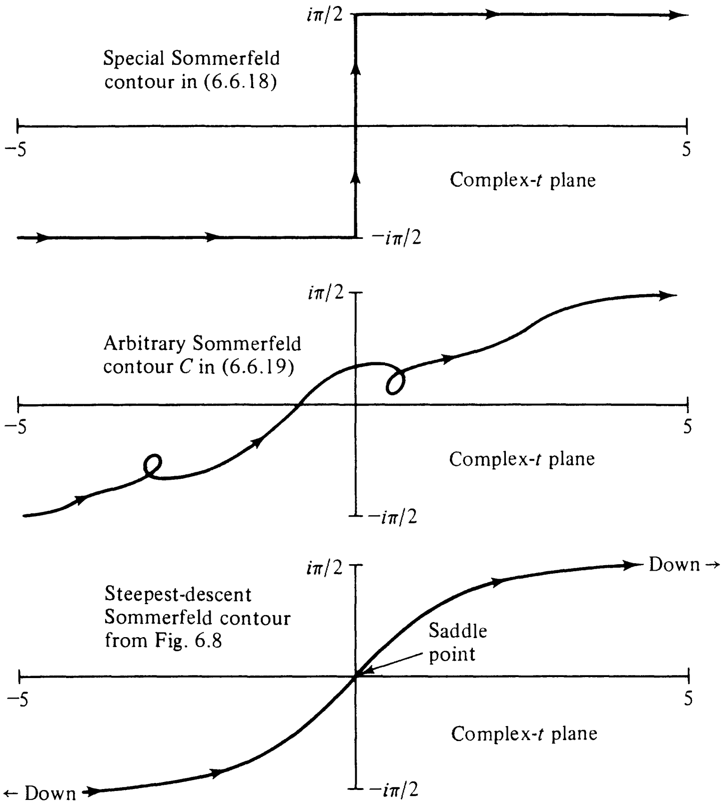

Figure 6.11 To find the asymptotic behavior of $J_0(x)$ as $x \to +\infty$ we first represent $J_0(x)$ as an integral along the special contour in (6.6.18*a*, *b*, *c*). Second, we observe that any Sommerfeld contour C from $-\infty - i\pi/2$ to $+\infty + i\pi/2$ is equally good. Third, to approximate the integral in (6.6.19) we choose that Sommerfeld contour which is also a path of steepest descent through the saddle point at $t = 0$.

where the contour extends along a line parallel to and below the real axis, and

$$\frac{1}{i\pi}\int_{i\pi/2}^{\infty + i\pi/2} dt\, e^{ix\cosh t}, \tag{6.6.18c}$$

where the contour extends along a line parallel to and above the real axis, are convergent and pure imaginary (see Prob. 6.63). Thus, we have constructed the rather fancy representation

$$J_0(x) = \text{Re}\,\frac{1}{i\pi}\int_C dt\, e^{ix\cosh t}, \tag{6.6.19}$$

where C is *any* contour which ranges from $-\infty - i\pi/2$ to $+\infty + i\pi/2$ (see Fig. 6.11). Such a contour is called a Sommerfeld contour.

From here on the steepest-descent analysis is easy because there is a curve of constant phase which ranges from $-\infty - i\pi/2$ to $+\infty + i\pi/2$ (see Fig. 6.8)! We have seen in Example 5 that this curve passes through a saddle point at $t = 0$. The equation for this curve is $\cosh u \cos v = 1$. Note that $|e^{ix \cosh t}|$ *attains its maximum value on the contour at the saddle point at* $t = 0$. Thus, we know from our study of Laplace's method that as $x \to +\infty$ the entire asymptotic expansion is determined by a small neighborhood about $t = 0$.

To find the leading behavior of $J_0(x)$ as $x \to \infty$, we approximate the steepest-descent path in a small neighborhood of $t = 0$ by the straight line $t = (1 + i)s$ (s real) and approximate $\cosh t$ near $s = 0$ by $\cosh t \sim 1 + is^2$ $(s \to 0)$. Thus,

$$J_0(x) \sim \operatorname{Re}\,[(1 + i)/i\pi] \int_{s=-\varepsilon}^{\varepsilon} e^{ix - xs^2}\, ds, \qquad x \to +\infty.$$

Extending the limits of integration to ∞ and evaluating the integral gives

$$J_0(x) \sim \operatorname{Re}\,[(1 + i)/i\pi]e^{ix}\sqrt{\pi/x} = \sqrt{2/\pi x}\,\cos(x - \pi/4), \qquad x \to +\infty.$$

To find the full asymptotic expansion of $J_0(x)$ as $x \to +\infty$, we use Watson's lemma. It is simplest to parametrize the integration path in terms of $\phi = \operatorname{Re}\rho(t)$. We know that along the steepest-descent contour $\rho(t) = i + \phi(t)$, where $\phi(t)$ is real and ranges from $\phi = 0$ at $t = 0$ to $\phi = -\infty$ at $t = \pm(\infty \pm i\pi/2)$. Also, we have $\phi(t) = i \cosh t - i$, so $d\phi = i \sinh t\, dt$. Thus, $dt = d\phi/i\sqrt{-\phi^2 - 2i\phi}$. Substituting this result into (6.6.19) and replacing ϕ by $-\phi$ gives

$$J_0(x) = \operatorname{Re} \frac{e^{ix - i\pi/4}}{\pi}\sqrt{2}\int_0^{\infty} \frac{d\phi}{\sqrt{\phi}}\, e^{-\phi x}\left(1 - \frac{i\phi}{2}\right)^{-1/2}.$$

To apply Watson's lemma, we expand the square root:

$$\left(1 - \frac{i\phi}{2}\right)^{-1/2} = \sum_{n=0}^{\infty} \frac{(i\phi/2)^n \Gamma(n + \frac{1}{2})}{n!\, \Gamma(\frac{1}{2})}$$

and integrate term by term:

$$J_0(x) \sim \operatorname{Re} \frac{e^{ix - i\pi/4}}{\pi^{3/2}}\sqrt{2}\sum_{n=0}^{\infty} \frac{[\Gamma(n + \frac{1}{2})]^2}{n!\,\sqrt{x}}\left(\frac{i}{2x}\right)^n, \qquad x \to +\infty.$$

Thus, the full asymptotic expansion of $J_0(x)$ is given by

$$J_0(x) = \sqrt{\frac{2}{x\pi}}\,[\alpha(x)\cos(x - \pi/4) + \beta(x)\sin(x - \pi/4)], \tag{6.6.20}$$

where

$$\alpha(x) \sim \sum_{k=0}^{\infty} \frac{[\Gamma(2k + \frac{1}{2})]^2(-1)^k}{\pi(2k)!\,(2x)^{2k}}, \qquad x \to +\infty,$$

and

$$\beta(x) \sim \sum_{k=0}^{\infty} \frac{[\Gamma(2k + \frac{3}{2})]^2(-1)^{k+1}}{\pi(2k + 1)!\,(2x)^{2k+1}}, \qquad x \to +\infty.$$

The trick of adding the contour integrals (6.6.18*b*,*c*) to (6.6.18*a*) to derive (6.6.19) could have been avoided by deforming the contour from $-i\pi/2$ to $i\pi/2$ into three constant-phase contours: C_1: $t = -i\pi/2 + u$ $(-\infty < u \le 0)$; C_2: $\cosh u \cos v = 1$; and C_3: $t = i\pi/2 + u$ $(0 \le u < \infty)$. The contributions from C_1 and C_3 cancel exactly in this problem.

Example 9 *Asymptotic expansion of* $\Gamma(x)$ *as* $x \to +\infty$. In Example 10 of Sec. 6.4 we used Laplace's method to show that

$$\Gamma(x) \sim x^x e^{-x}\sqrt{2\pi/x} \tag{6.6.21}$$

[see (6.4.39)]. In this example we use the method of steepest descents to rederive this result from a complex-contour integral representation of $\Gamma(x)$ [see Prob. 2.6(f)]:

$$\frac{1}{\Gamma(x)} = \frac{1}{2\pi i}\int_C e^t t^{-x}\,dt, \tag{6.6.22}$$

where C is a contour that begins at $t = -\infty - ia$ ($a > 0$), encircles the branch cut that lies along the negative real axis, and ends up at $-\infty + ib$ ($b > 0$) (see Fig. 6.12). The branch cut is present when x is nonintegral because t^{-x} is a multivalued function. The advantage of (6.6.22) over the integral representation used in Example 10 is that it converges for *all* complex values of x and not just those x for which $\operatorname{Re} x > 0$. Nevertheless, in this example we will only investigate the behavior of $\Gamma(x)$ in the limit $x \to +\infty$.

We begin our analysis by making the same substitution that was made in Example 10 of Sec. 6.4; namely, $t = xs$. This substitution converts the integrand from one that has a movable saddle point to one that has a fixed saddle point. (Why?) The resulting integral representation is

$$\frac{1}{\Gamma(x)} = \frac{1}{2\pi i x^{x-1}}\int_C ds\, e^{x(s-\ln s)}. \tag{6.6.23}$$

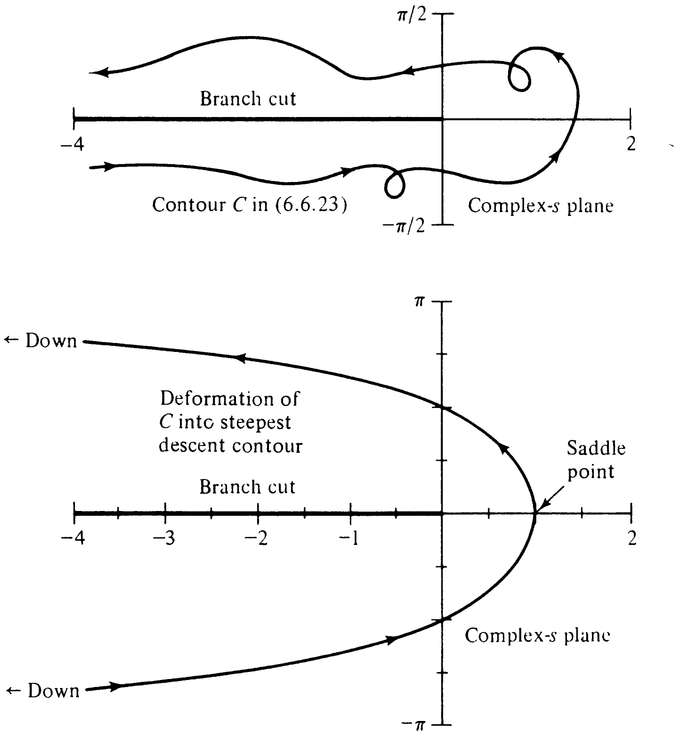

Figure 6.12 To find the asymptotic behavior of $\Gamma(x)$ as $x \to +\infty$, we represent $\Gamma(x)$ as the integral in (6.6.23) along a contour C in the complex-s plane which goes around the branch cut on the negative real axis. Then we distort C into a steepest-descent contour which passes through the saddle point at $s = 1$.

For this integral $\rho(s) = s - \ln s$. Thus, $\rho'(s) = 1 - 1/s$ and $\rho' = 0$ when $s = 1$. So there is a simple (second-order) saddle point at $s = 1$.

To ascertain the structure of the saddle point we let $s = u + iv$ and identify the real and imaginary parts of ρ: $\rho(s) = u - \ln \sqrt{u^2 + v^2} + i(v - \arctan v/u)$. At $s = 1$, $\rho = 1$. Therefore, paths of constant phase (steepest curves) emerging from $s = 1$ must satisfy

$$v - \arctan v/u = 0.$$

There are two solutions to this equation: $v = 0$ and $u = v \cot v$. These two curves are shown on Fig. 6.12. In Prob. 6.64 you are asked to verify that (*a*) the steepest-descent curves are correctly shown on Fig. 6.12; (*b*) as s moves away from $s = 1$, steepest-descent curves emerge from $s = 1$ initially parallel to the $\operatorname{Im} s = v$ axis; (*c*) the steepest-descent curves cross the v axis at $\pm i\pi/2$ and approach $s = -\infty \pm i\pi$.

To use the method of steepest descents, we simply shift the contour C so that it is just the steepest-descent contour on Fig. 6.12 which passes through the saddle point at $s = 1$. Let us review why we choose such a contour. In general, we always choose a steepest-descent contour because on such a contour we can apply the techniques of Laplace's method directly to complex integrals. If the steepest-descent contour is finite and does not pass through a saddle point, then the maximum value of $|e^{x\rho}|$ must occur at an endpoint of the contour and we need only perform a local analysis of the integral at this endpoint. However, in the present example the contour has no endpoint and is infinitely long. It is crucial that it pass through a saddle point because $|e^{x\rho}|$ reaches its maximum at the saddle point and decays exponentially as $s \to \infty$ along *both* of the steepest-descent curves. If there were no saddle point, then, although $|e^{x\rho}|$ would decrease in one direction along the contour, it would increase in the other direction and the integral would not even converge!

Now we proceed with the asymptotic expansion of the integral in (6.6.23). We can approximate the steepest-descent contour in the neighborhood of $s = 1$ by the straight line $s = 1 + iv$. This gives the Laplace integral

$$\frac{1}{\Gamma(x)} \sim \frac{1}{2\pi x^{x-1}} \int_{-\varepsilon}^{\varepsilon} dv\, e^{x(1 - v^2/2)}, \qquad x \to +\infty,$$

which we evaluate by letting $\varepsilon \to \infty$:

$$\frac{1}{\Gamma(x)} \sim \frac{1}{2\pi x^{x-1}} \frac{e^x}{\sqrt{x}} \sqrt{2\pi}, \qquad x \to +\infty.$$

We thereby recover the result in (6.6.21).

Example 10 *Steepest-descents approximation of a real integral where Laplace's method fails.* In this example we consider the real integral

$$I(x) = \int_0^1 dt\, e^{-4xt^2} \cos(5xt - xt^3) \tag{6.6.24}$$

in the limit $x \to +\infty$. This integral is *not* a Laplace integral because the argument of the cosine contains x. Nonetheless, one might think that one could use the ideas of Laplace's method to approximate the integral. To wit, one would argue that as $x \to +\infty$, the contribution to the integral is localized about $x = 0$. Thus, a very naive approach is simply to replace the argument of the cosine by 0. If this reasoning were correct, then we would conclude that

$$I(x) \sim \int_0^1 dt\, e^{-4xt^2} \sim \sqrt{\frac{\pi}{16x}}, \qquad x \to +\infty. \qquad \text{(WRONG)}$$

This result is clearly incorrect because e^{-xt^2} does not become exponentially small until t is larger than $1/\sqrt{x}$. Thus, when $t \sim 1/\sqrt{x}$ $(x \to +\infty)$, the argument of the cosine is *not* small. In particular, the term $5xt$ is large and the cosine oscillates rapidly. This suggests that there is destructive interference and that $I(x)$ decays much more rapidly than $\sqrt{\pi/16x}$ as $x \to +\infty$.

Can we correct this approach by including the $5xt$ term but neglecting the xt^3 term? After all, when t lies in the range from 0 to $1/\sqrt{x}$, the term $xt^3 \to 0$ as $x \to +\infty$. Thus, xt^3 does not even shift the phase of the cosine more than a fraction of a cycle. If we were to include just the $5xt$ term, we would obtain

$$\begin{aligned} I(x) &\sim \int_0^1 dt\, e^{-4xt^2} \cos(5xt), \qquad x \to +\infty, \\ &\sim \int_0^\infty dt\, e^{-4xt^2} \cos(5xt), \qquad x \to +\infty, \\ &= \tfrac{1}{2} \int_{-\infty}^{\infty} dt\, e^{-4xt^2 + 5ixt} \\ &= \tfrac{1}{2} \int_{-\infty}^{\infty} dt\, e^{-x(2t - 5i/4)^2 - 25x/16} \\ &= \tfrac{1}{4}\sqrt{\pi/x}\, e^{-25x/16}, \qquad x \to +\infty. \qquad \text{(WRONG)} \end{aligned}$$

Although this result is exponentially smaller than the previous wrong result, it is also wrong! It is incorrect to neglect the xt^3 term (see Prob. 6.65).

But if we cannot neglect even the xt^3 term, then how can we make any approximation at all? It should not be necessary to do the integral exactly to find its asymptotic behavior!

The correct approach is to use the method of steepest descents to approximate the integral at a saddle point in the complex plane. To prepare for this analysis we rewrite the integral in the following convenient form:

$$\begin{aligned} I(x) &= \tfrac{1}{2} \int_{-1}^{1} dt\, e^{-4xt^2 + 5ixt - ixt^3} \\ &= \tfrac{1}{2} e^{-2x} \int_{-1}^{1} dt\, e^{x\rho(t)}, \end{aligned} \tag{6.6.25}$$

where

$$\rho(t) = -(t - i)^2 - i(t - i)^3. \tag{6.6.26}$$

Our objective now is to find steepest-descent (constant-phase) contours that emerge from $t = 1$ and $t = -1$, to distort the original contour of integration t: $-1 \to 1$ into these contours, and then to use Laplace's method. To find these contours we substitute $t = u + iv$ and identify the real and imaginary parts of ρ:

$$\begin{aligned} \rho(t) &= \phi + i\psi \\ &= -v^3 + 4v^2 - 5v + 3u^2 v - 4u^2 + 2 + i(3uv^2 - 8uv + 5u - u^3). \end{aligned} \tag{6.6.27}$$

Note that the phase of $\psi = \mathrm{Im}\,\rho$ at $t = 1$ and at $t = -1$ is different: $\mathrm{Im}\,\rho(-1) = -4$, $\mathrm{Im}\,\rho(1) = 4$. Thus, there is no single constant-phase contour which connects $t = -1$ to $t = 1$.

Our method is similar to that used in Examples 1 and 2. We follow steepest-descent contours C_1 and C_2 from $t = -1$ and from $t = 1$ out to ∞. Next, we join these two contours at ∞ by a third contour C_3 which is also a path of constant phase. C_3 must pass through a saddle point because its endpoints lie at ∞; otherwise, the integral along C_3 will not converge (see the discussion in Example 9).

There are two saddle points in the complex plane because $\rho'(t) = -2(t - i) - 3i(t - i)^2 = 0$ has two roots, $t = i$ and $t = 5i/3$. The contour C_3 happens to pass through the saddle point at i. On Fig. 6.13 we plot the three constant-phase contours C_1, C_2, and C_3. It is clear that the original contour C can be deformed into $C_1 + C_2 + C_3$. (In Prob. 6.66 you are to verify the results on Fig. 6.13.)

The asymptotic behavior of $I(x)$ as $x \to +\infty$ is determined by just three points on the contour $C_1 + C_2 + C_3$: the endpoints of C_1 and C_2 at $t = -1$ and at $t = +1$ and the saddle point at i. However, the contributions to $I(x)$ at $t = \pm 1$ are exponentially small compared with

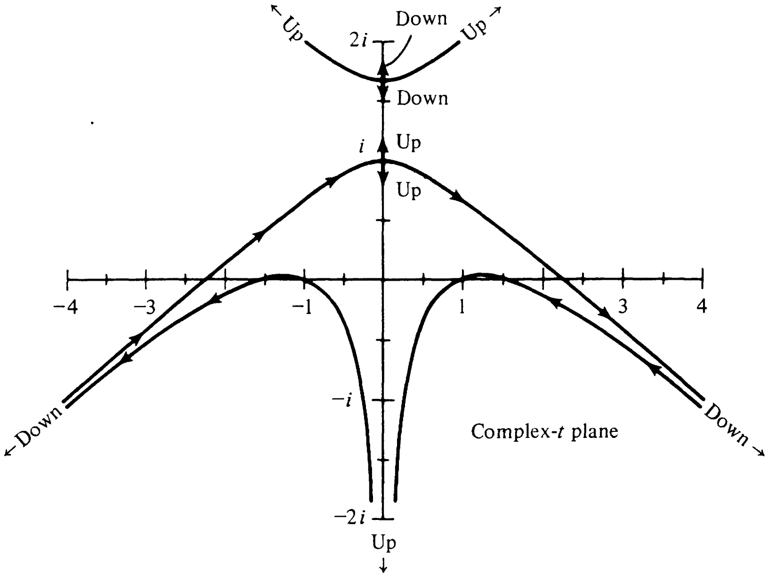

Figure 6.13 To approximate the integral in (6.6.25) by the method of steepest descents we deform the original contour connecting the points $t = -1$ to $t = 1$ along the real axis into the three distinct steepest-descent contours above, one of which passes through a saddle point at $t = i$. Steepest-ascent and -descent curves near a second saddle point at $t = 5i/3$ and steepest-ascent curves going from 1 and -1 to $-i\infty$ are also shown, but these curves play no role in the calculation.

that at $t = i$ (see Prob. 6.67). Near $t = i$ we can approximate the contour C_3 by the straight line $t = i + u$ and $\rho(t)$ by $\rho(t) \sim -u^2$ $(u \to 0)$. Thus,

$$\begin{aligned} I(x) &\sim \tfrac{1}{2}e^{-2x}\int_{-\varepsilon}^{\varepsilon} e^{-xu^2}\,du, && x \to +\infty,\\ &\sim \tfrac{1}{2}e^{-2x}\sqrt{\pi/x}, && x \to +\infty. \end{aligned} \tag{6.6.28}$$

This, finally, is the correct asymptotic behavior of $I(x)$! This splendid example certainly shows the subtlety of asymptotic analysis and the power of the method of steepest descents.

Example 11 *Steepest-descents analysis with a third-order saddle point.* In Example 5 of Sec. 6.5 and Prob. 6.55 we showed that

$$J_x(x) \sim \frac{1}{\pi}2^{-2/3}3^{-1/6}\Gamma\left(\frac{1}{3}\right)x^{-1/3}, \qquad x \to +\infty. \tag{6.6.29}$$

Here we rederive (6.6.29) using the method of steepest descents.

We begin with the complex-contour integral representation for $J_\nu(x)$:

$$J_\nu(x) = \frac{1}{2\pi i}\int_C dt\, e^{x\sinh t - \nu t}, \tag{6.6.30}$$

where C is a Sommerfeld contour that begins at $+\infty - i\pi$ and ends at $+\infty + i\pi$. Setting $\nu = x$ gives

$$J_x(x) = \frac{1}{2\pi i}\int_C dt\, e^{x(\sinh t - t)}. \tag{6.6.31}$$

For this integral $\rho(t) = \sinh t - t$ has a third-order saddle point at $t = 0$.

We have already analyzed the steepest curves of this $\rho(t)$ in Example 6 (see Fig. 6.9). Note that we can deform the contour C so that it follows steepest-descent paths to and from the saddle point at $t = 0$.

The contribution to $J_x(x)$ as $x \to +\infty$ comes entirely from the neighborhood of the saddle point. In the vicinity of the saddle point we can approximate the contours approaching and leaving $t = 0$ by the straight lines $t = re^{-i\pi/3}$ and $t = re^{i\pi/3}$. Substituting into (6.6.30) gives

$$J_x(x) \sim \frac{1}{2\pi i}\int_{r=\varepsilon}^{0} dr\, e^{-i\pi/3}e^{-xr^3/6} + \frac{1}{2\pi i}\int_{r=0}^{\varepsilon} dr\, e^{i\pi/3}e^{-xr^3/6}, \qquad x \to +\infty.$$

To evaluate these integrals, we replace ε by ∞:

$$J_x(x) \sim \frac{e^{i\pi/3} - e^{-i\pi/3}}{2\pi i}\int_{r=0}^{\infty} dr\, e^{-xr^3/6}, \qquad x \to +\infty,$$

$$= \frac{\sin(\pi/3)}{\pi}(6/x)^{1/3}\int_0^\infty e^{-r^3}\, dr, \qquad x \to +\infty.$$

But $\int_0^\infty e^{-r^3}\, dr = \frac{1}{3}\int_0^\infty e^{-s}s^{-2/3}\, ds = \frac{1}{3}\Gamma(\frac{1}{3})$. Thus,

$$J_x(x) \sim \frac{\sin(\pi/3)}{3\pi}\left(\frac{6}{x}\right)^{1/3}\Gamma\left(\frac{1}{3}\right),$$

which reproduces the result in (6.6.29).

Example 12 *Steepest-descents analysis with a fourth-order saddle point.* What is the leading asymptotic behavior of the real integral

$$I(x) = \int_0^\infty dt\, \cos(x\pi t)\, e^{-x(\cosh t + t^2/2)} \tag{6.6.32}$$

as $x \to +\infty$? To analyze $I(x)$ we first rewrite the integral as

$$I(x) = \frac{1}{2}\int_{-\infty}^{\infty} e^{x(i\pi t - \cosh t - t^2/2)}$$

$$= \frac{1}{2}e^{-x\pi^2/2}\int_{-\infty}^{\infty} dt\, e^{x[\cosh(t - i\pi) - (t - i\pi)^2/2]}. \tag{6.6.33}$$

For this integral $\rho(t) = \cosh(t - i\pi) - (t - i\pi)^2/2$ has a fourth-order saddle point at $t = i\pi$ (see Example 7). The steepest-descent contours from this saddle point are drawn in Fig. 6.10 (the saddle point in Fig. 6.10 is shifted downward by π).

To approximate $I(x)$ we shift the original integration path t: $-\infty \to +\infty$ from the real axis into the complex plane so that it follows a steepest-descent curve passing through the saddle point. The asymptotic behavior of $I(x)$ is completely determined by the contribution from the saddle point. In the neighborhood of the saddle point at $i\pi$, we can approximate the steepest-descent contour by the straight lines $t = i\pi + re^{i\pi/4}$ to the left of the saddle point and $t = i\pi + re^{-i\pi/4}$ to the right of the saddle point. In terms of r, (6.6.33) becomes

$$I(x) \sim \frac{1}{2}e^{-x\pi^2/2}\left[\int_{-\varepsilon}^{0} e^{i\pi/4}\, dr\, e^{x(1 - r^4/24)} + \int_0^{\varepsilon} e^{-i\pi/4}\, dr\, e^{x(1 - r^4/24)}\right], \qquad x \to +\infty,$$

$$= e^{-x\pi^2/2 + x}\cos(\pi/4)\int_0^\infty dr\, e^{-xr^4/24}, \qquad x \to +\infty.$$

But $\int_0^\infty dr\, e^{-r^4} = \frac{1}{4}\int_0^\infty dr\, r^{-3/4}e^{-r} = \frac{1}{4}\Gamma(\frac{1}{4})$. Thus, we obtain the final result that

$$I(x) \sim \tfrac{1}{4}e^{x(1 - \pi^2/2)}(6/x)^{1/4}\Gamma(\tfrac{1}{4}). \tag{6.6.34}$$

This result could not have been obtained by performing a Laplace-like analysis of the real integral in (6.6.32). Suppose, for example, we argue that as $x \to +\infty$ the contribution to (6.6.32) comes entirely from the neighborhood of the origin $t = 0$. Then it would seem valid to replace $\cosh t$ by $1 + t^2/2$, the first two terms in its Taylor series. If we do this, we obtain an integral which we can evaluate exactly:

$$\int_0^\infty dt \cos(x\pi t)\, e^{-x(1+t^2)} = \tfrac{1}{2}\int_{-\infty}^{\infty} dt\, e^{ix\pi t - x(1+t^2)}$$

$$= \tfrac{1}{2}\int_{-\infty}^{\infty} dt\, e^{-x(t-i\pi/2)^2} e^{-x(1+\pi^2/4)}$$

$$= \tfrac{1}{2}\sqrt{\pi/x}\, e^{-x(1+\pi^2/4)}.$$

But this does not agree with (6.6.34) and is therefore *not* the asymptotic behavior of $I(x)$ as $x \to +\infty$! What is wrong with this argument? (See Prob. 6.68.)

Steepest Descents for Complex x and the Stokes Phenomenon

Until now, x in (6.6.1) has been treated as a large *real* parameter. However, the method of steepest descents can be used to treat problems where x is complex. As we have already seen in Secs. 3.7 and 3.8, an asymptotic relation is valid as $x \to \infty$ in a wedge-shaped region of the complex-x plane. At the edge of the wedge, the asymptotic relation ceases to be valid and must be replaced by another asymptotic relation. This change from one asymptotic relation to another is called the Stokes phenomenon.

The Stokes phenomenon usually surfaces in the method of steepest descents in a relatively simple way. For example, as x rotates in the complex plane, the structure of steepest-descent paths can change abruptly. When this happens, the asymptotic behavior of the integral changes accordingly. The integral representation of Ai (x) behaves in this manner (see Prob. 6.75). The Stokes phenomenon can also appear when the contribution from an endpoint of the contour suddenly becomes subdominant relative to the contribution from a saddle point (or vice versa). We consider this case in the next example.

Example 13 *Reexamination of Example 10 for complex x.* In this example we explain how the Stokes phenomenon arises in the integral (6.6.25). It is essential that the reader master Example 10 before reading further.

The integral $I(x)$ in (6.6.25) exhibits the Stokes phenomenon at $\arg x = \pm \arctan \frac{1}{2} \doteq 26.57°$ and at $\pm\pi$. When $|\arg x| < \arctan \frac{1}{2}$, the contribution to $I(x)$ from the saddle point at $t = i$ dominates the endpoint contributions. As in (6.6.28), this gives

$$I(x) \sim \tfrac{1}{2} e^{-2x}\sqrt{\pi/x}, \qquad x \to \infty,\ |\arg x| < \arctan \tfrac{1}{2}. \tag{6.6.35}$$

When $\arctan \frac{1}{2} < \arg x < \pi$, the endpoint contribution from $t = -1$ dominates. We obtain (see Prob. 6.69)

$$I(x) \sim \frac{i-4}{68x} e^{-4(i+1)x}, \qquad x \to \infty,\ \arctan \tfrac{1}{2} < \arg x < \pi. \tag{6.6.36}$$

When $-\pi < \arg x < -\arctan \frac{1}{2}$, the endpoint contribution from $t = 1$ dominates, giving

$$I(x) \sim -\frac{i+4}{68x} e^{4(i-1)x}, \qquad x \to \infty,\ -\pi < \arg x < -\arctan \tfrac{1}{2}. \tag{6.6.37}$$

It is interesting to see what happens to the steepest-descent contours as x is rotated into the complex-x plane. We have plotted the steepest-descent contours for $I(x)$ for $\arg x = 0°, 30°, 75°$, and $135°$ in Figs. 6.13 to 6.16. Observe that as $\arg x$ increases from $0°$ to $75°$, the contours through the endpoints at $t = \pm 1$ and the saddle point at $t = i$ tilt and distort slightly. Note that the asymptotes of these contours at ∞ rotate by $-(\arg x)/3$ as $\arg x$ increases. This is so because

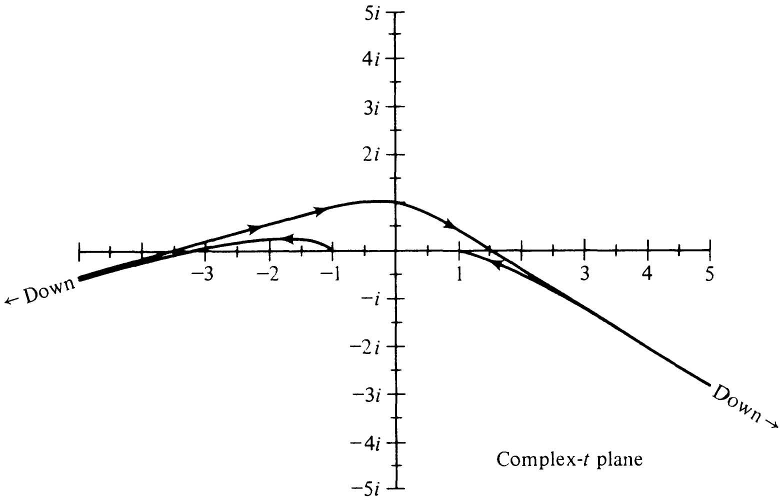

Figure 6.14 Steepest-descent path for $I(x)$ in (6.6.25) when $\arg x = 30°$. (See Fig. 6.13.)

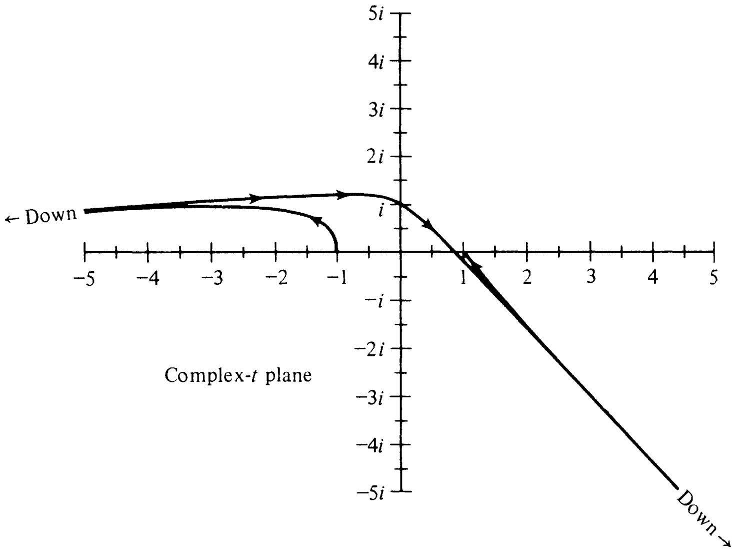

Figure 6.15 Steepest-descent path for $I(x)$ in (6.6.25) when $\arg x = 75°$.

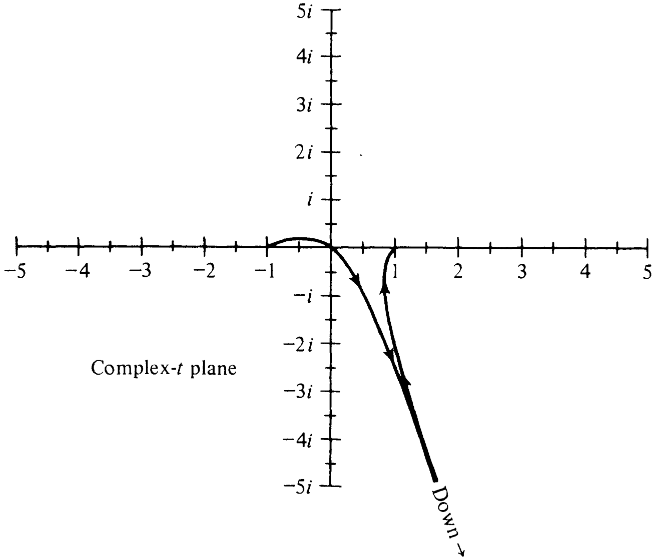

Figure 6.16 Steepest-descent path for $I(x)$ in (6.6.25) when $\arg x = 135°$. Note that the steepest-descent path no longer passes through the saddle point at $t = i$ as it does in Figs. 6.13 to 6.15.

$\text{Im}\,[x\rho(t)]$ must be constant at $t = \infty$. The constancy of $\text{Im}\,[x\rho(t)]$ on the steepest-descent contours implies that the endpoint contours passing through $t = \pm 1$ rotate by $-\arg x$ near $t = \pm 1$ and that the contour through $t = i$ rotates by $-(\arg x)/2$ near $t = i$. There is *no* abrupt or discontinuous change in the configuration of the steepest-descent contours as $\arg x$ increases past arc tan $\frac{1}{2}$. In this example the Stokes phenomenon is not associated with any discontinuity in the structure of the steepest-descent path. It occurs because the contribution from the saddle point becomes subdominant with respect to the contribution from the endpoint as $\arg x$ increases past arc tan $\frac{1}{2}$.

When $\arg x$ reaches $\pi - \arc\tan 2 \doteq 116.57°$, there is a discontinuous change in the steepest-descent path for $I(x)$ (see Prob. 6.69). As illustrated in Fig. 6.16, when $\arg x = 135°$, the steepest-descent contour no longer passes through the saddle point at $t = i$. When $\arg x > 116.57°$, the steepest-descent contours from $t = \pm 1$ meet at ∞, so it is no longer necessary to join them by a constant-phase contour passing through the saddle point at i. The abrupt disappearance of the saddle-point contour from the steepest-descent path when $\arg x$ increases beyond $116.57°$ does not affect the asymptotic behavior of $I(x)$ because the saddle-point contribution from $t = i$ is subdominant when $\text{arc tan}\,\frac{1}{2} < |\arg x| < \pi$.

6.7 ASYMPTOTIC EVALUATION OF SUMS

In this section we discuss methods for finding the asymptotic behavior of sums which depend on a large parameter x. We consider four methods in all: truncating the sum after a finite number of terms, approximating the sum by a Riemann integral, Laplace's method for sums, and the Euler-Maclaurin sum formula. The