Part III: Biological Physics and Complex Fluids (Michaelmas, 2016)
E Lauga (October 30, 2016)

Example Sheet #2
Work due 7 November 2016, 8.30am; Example class 7 November 2016, 2pm

1. *Generalised Newtonian fluid*.*
Solve for the velocity profile of a power law fluid (constitutive relationship, $\eta = K\dot{\gamma}^{n-1}$, $\dot{\gamma} \geq 0$) driven by a pressure gradient along a cylindrical circular tube of radius R. Denote the length of the pipe L and the pressure drop Δp. Express the mean flow velocity, $\bar{U}$, as a function of the magnitude of the wall shear-stress, σ_w. Express your solution for the flow in a dimensionless form as $u(r)/\bar{U}$. Sketch the flow profile for different values of n.

2. *Generalised Newtonian fluid.*
Show that any generalised Newtonian fluid under steady three-dimensional extension (extension rate, $\dot{\epsilon} > 0$) has an extensional viscosity equal to three times the shear viscosity for a steady shear flow of the same fluid at shear rate $\dot{\gamma} = \alpha\dot{\epsilon}$, for a value of α which you will determine.

3. *Generalised Newtonian fluid.*
Two immiscible power-law fluid layers are sandwiched between two plates. The bottom plate is stationary while the top plate is moving at steady speed V. The bottom fluid layer has thickness h_1 and viscosity $\eta_1 = K_1\dot{\gamma}^{n_1-1}$ while the top fluid layer, of thickness h_2, has viscosity $\eta_2 = K_2\dot{\gamma}^{n_2-1}$ ($\dot{\gamma} \geq 0$). Solve for the steady flow profile between the plates using carefully-stated boundary conditions (you might introduce $\tilde{V}$ to denote the velocity at the boundary between both layers). Solve the case ($n_1 = 1/2, n_2 = 1$) exactly.

4. *Linear viscoelastic fluid.*
A Maxwell fluid with multiple relaxation times has a relaxation modulus given by

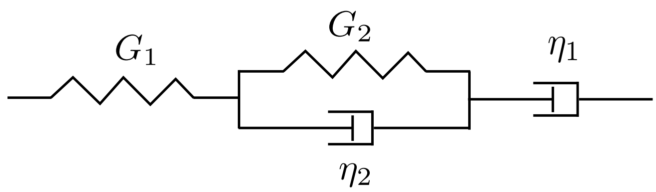
$$G(t) = \sum_i \frac{\eta_i}{\lambda_i} e^{-t/\lambda_i}. \tag{1}$$

Calculate $G'(\omega)$ and $G''(\omega)$ for this fluid.
For a fluid with many relaxation time scales, the sum in Eq. (1) can be written as a continuum spectrum as

$$G(t) = \int_0^\infty H(\lambda) e^{-t/\lambda} \frac{\mathrm{d}\lambda}{\lambda}, \tag{2}$$

where the function H is given. Calculate $G'(\omega)$ and $G''(\omega)$ for this fluid as integrals of H. In the special case where $H(\lambda) = H_0\lambda_0^2/(\lambda_0^2 + \lambda^2)$, where λ_0 is a fixed parameter, evaluate analytically $G''(\omega)$.

5. *Linear viscoelastic fluid*.*



Consider the linear viscoelastic fluid modelled by the spring-dashpot system illustrated in the figure above. Derive the differential relationship between the overall stress, σ, and overall shear-rate, $\dot{\gamma}$, for such a fluid. Write it in the form

$$(1 + \mathcal{D})\,\sigma = \eta\,(1 + \mathcal{L})\,\dot{\gamma}, \tag{3}$$

where $\mathcal{D}$ and $\mathcal{L}$ are two differential operators which you will identify and η is a viscosity to be determined.

6. *Objectivity.*
Consider a general linear viscoelastic fluid where the deviatoric stress, $\boldsymbol{\sigma}$, is linearly related to the shear-rate tensor, $\dot{\boldsymbol{\gamma}}$, as

$$\boldsymbol{\sigma} = \int_{-\infty}^{t} G(t-t')\dot{\boldsymbol{\gamma}}(t')\mathrm{d}t'.$$

This fluid undergoes steady shear at rate $\dot{\gamma}$ on a planar rotating table. The table is described by vectors $\mathbf{e}_x$ and $\mathbf{e}_y$ and rotates at constant rate Ω along the $\mathbf{e}_z$ vector such that at $t=0$ the rotating frame coincides with the laboratory frame $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$. Compute the viscosity of the fluid in the rotating frame and in the laboratory frame and evaluate them at $t=0$. Show that they are not identical in general if $\Omega \neq 0$.

7. *Nonlinear constitutive relationships*.*
Consider an Upper Convected Maxwell fluid under steady, three-dimensional extension (extension rate, $\dot{\epsilon} > 0$). Compute all components of the stress tensor and determine the value of the extensional viscosity, η_{ext}. Show that there is a critical extension rate, $\dot{\epsilon}$, at which the extensional viscosity blows up.

8. *Nonlinear constitutive relationships.*
Multiple constitutive models of polymeric fluids exist which involve objective derivatives. One of them is the *Johnson-Segalman-Oldroyd* fluid. For this fluid, the relationship between the deviatoric stress and the shear rate tensor is given by the first-order differential form

$$\boldsymbol{\sigma} + \lambda_1 \overset{\square}{\boldsymbol{\sigma}} = \eta[\dot{\boldsymbol{\gamma}} + \lambda_2 \overset{\square}{\dot{\boldsymbol{\gamma}}}], \tag{4}$$

where the so-called *Gordon-Schwalter* convected derivative for a tensor $\boldsymbol{\sigma}$ is defined as

$$\overset{\square}{\boldsymbol{\sigma}} \equiv \overset{\triangledown}{\boldsymbol{\sigma}} + \frac{a}{2}(\dot{\boldsymbol{\gamma}} \cdot \boldsymbol{\sigma} + \boldsymbol{\sigma} \cdot \dot{\boldsymbol{\gamma}}), \tag{5}$$

where a is a small, nonzero dimensionless parameter. Calculate the steady shear viscosity for this fluid. What are the conditions on the values of λ_1 and λ_2 for the fluid to be shear-thinning?

9. *A paper from the literature.*
Read pages 66-73 of the book chapter: “Dynamics of complex biofluids” by C. Hohenegger and M. J. Shelley [`http://math.cims.nyu.edu/faculty/shelley/papers/HS2011.pdf`]

The purpose of this chapter is to demonstrate how to “derive” a constitutive relationship, instead of postulating it empirically. The final result is 3.27, which you might recognise as the UCM model studied in class.

What is the physical meaning of Ψ in 3.1 and 3.24? What is assumed about the dumbbell in order to be able to write 3.13? Explain the meaning of each term in the 3.14. Why is there an extra stress term in 3.15? What are the meanings of $\dot{\mathbf{X}}_c$ and $\dot{\mathbf{R}}$ in 3.24?