Part III: Biological Physics and Complex Fluids (Michaelmas, 2016)
R E Goldstein (02/10/16)

Example Sheet #3

1. *A forced particle.* A microsphere of radius a and drag coefficient ζ is constrained to move along the x-axis, and is acted on by an optical trap which is moving in the positive x-direction at velocity v_T. When the trap is located at a point x_0 it exerts a force $F(x - x_0)$, so the overdamped dynamics of the particle is

$$\zeta \dot{x} = F(x - v_T t) \ .$$

Suppose that the trap has compact support, so that $F(x) = 0$ for $x < -X_L$ and for $x > X_R$. If the trap starts to the left of the particle, find the particle's net displacement Δx after the trap has passed it by, and the time Δt spent by the particle interacting with the trap. What is the condition that assures that the particle does not remain trapped as $t \to \infty$? Assuming this is the case, show that whatever the form of $F(y)$ the net displacement is always in the direction of the trap motion, and suggest a heuristic explanation for this result. Find the asymptotic behaviour of Δx for large trap velocities.

The trap is now moved around a circle of radius $R \gg a$. Derive the particle's net rotational frequency f_p as a function of the trap angular frequency $f_T = v_T/(2\pi R)$, the displacement Δx in each kick, the interaction time Δt and the potential width $2X_0 = X_R - X_L$. Confirm that in the regime of suitably large trap velocity, which you should define precisely, one obtains the intuitive result $f_p \simeq (\Delta x/2\pi R) f_T$. Specializing to the case of a triangular trapping potential, with $F(y) = F$ for $-X_0 < x < 0$ and $F(y) = -F$ for $0 < x < X_0$, obtain an explicit expression for f_p/f_c as a function of the two quantities $\alpha = X_0/(\pi R)$ and $\beta = f_T/f_c$, where $2\pi R f_c = F/\zeta$.

2. *Fluctuations.* A long cylindrical vesicle of radius R_0, aligned along the z-axis, is subject to a tension $\sigma \gg \kappa/R_0^2$, where κ is the bending modulus. Thus, its energy is well-approximated by $\sigma \mathcal{S}$, where $\mathcal{S}$ is the total surface area of the vesicle. Assuming that fluctuations in the radius preserve axisymmetry, so the fluctuating radius $R(z)$ does not depend on the cylindrical polar angle, find the spectrum of thermal fluctuations as a function of the longitudinal wavevector q, at fixed enclosed volume of fluid. You may take $R(z) = \rho_0 + u_q \sin qz$, where ρ_0 is to be determined by volume conservation. Explain the significance of your result for $qR_0 < 1$.

A circular inclusion of radius R_0 in a lipid membrane consists of a distinct phase from the surrounding lipids, so there is a line tension γ between the two. Find the spectrum of thermal fluctuations in the radius, at fixed enclosed area, as above. Explain the significance of the result for the mode with $qR_0 = 1$.

3. *Nonlinear Diffusion.* For the nonlinear diffusion equation $C_t = D(C^p C_x)_x$, where p and D are strictly positive constants, show that the self similar solution defined in $x \geq 0$ of the form

$$C(x,t) = \frac{M^{2/(2+p)}}{(Dt)^{1/(2+p)}} F(\xi), \qquad \xi = \frac{x}{(M^p D t)^{1/(2+p)}}$$

which satisfies $\int_0^\infty C(x,t)\ \mathrm{d}x = M$, $C_x(0,t) = 0$ and $C(x \to \infty, t) \to 0$ for some constant $M > 0$, is given by

$$F(\xi) = \left[A - \frac{p\xi^2}{2(2+p)} \right]^{1/p}, \qquad 0 < \xi < \left[\frac{2(2+p)A}{p} \right]^{1/2}$$

and $F = 0$ otherwise. For the case $p = 1$, prove that $A = (3/8)^{1/3}$.

4. *The Fitzhugh-Nagumo model.* The Fitzhugh-Nogumo model for nerve impulse propagation is given by

$$u_t = u_{xx} + u(1-u)(u-a) + v$$
$$v_t = bu - cv; \quad (b, c > 0, 0 < a < 1).$$

(i) Show that the homogeneous system has a stable fixed point for which neither u nor v is zero. (ii) Seek a travelling wave solution of the equations of the form

$$u = \phi(\xi), v = \psi(\xi), \xi = x - \gamma t.$$

Deduce the ODEs to be satisfied by ϕ, ψ. (iii) Verify that if $b = c = \psi(0) = 0$ then there is a solution of the form

$$\phi = \frac{1}{1 + e^{-\alpha\xi}},$$

for two possible values of α which should be found. What is the wave speed γ in each case? In what direction does the wave propagate?

5. *Chemotaxis.* Consider the chemostatic system

$$n_t = Dn_{xx} + bn(1 - \frac{n}{n_0}) - (\chi(a)na_x)_x$$
$$a_t = D_A a_{xx} + hn - ka,$$

where $\chi(a) = \chi_0 K/(K+a)^2$. Find a scaling such that this reduces to

$$\dot{u} = u'' + u(1-u) - \beta \left[\frac{uv'}{(\alpha+v)^2}\right]'$$
$$\dot{v} = \delta v'' + \gamma(u - v),$$

where $\cdot$ and $\prime$ refer to differentiation with the scaled t and x respectively, and $\alpha, \beta, \gamma, \delta$ are positive constants. Show that the uniform, steady solution $u = v = 1$ is unstable if

$$\frac{\beta\gamma}{(1+\alpha)^2} > (\sqrt{\gamma} + \sqrt{\delta})^2,$$

and find the wavenumber at which the system first becomes unstable as χ_0 is increased from zero, in the case $\alpha = \gamma = \delta = 1$.

6. *Turing instability.* Investigate the possibility of Turing instability for the reaction-diffusion system.

$$\frac{\partial u}{\partial t} = \nabla^2 u + \frac{u^2}{v} - bu,$$
$$\frac{\partial v}{\partial t} = d\nabla^2 v + u^2 - v.$$

In particular, find the region of the parameter space (b, d) in which Turing instability can occur, and give a value for the critical wavenumber at the onset of instability.

7. *Phytoplankton-zooplankton.* A space-dependent phytoplankton - zooplankton model can be reduced to the following equations

$$u_t = \nabla^2 u + u + u^2 - \gamma uv$$
$$v_t = d\nabla^2 v + \beta uv - v^2.$$

Find the regions in the $\beta - \gamma$ plane (a) in which there is a stable, homogeneous state (u_0, v_0) in which neither u_0 nor v_0 is zero and (b) in which that state may be unstable to a Turing instability. In case (b), for what values of d will the instability occur, and what is the critical wavenumber for the onset of the instability?