A forced colloidal particle

$$\zeta \dot{x} = F(x - v_T t) \qquad \text{let } \begin{matrix} y = x - v_T t \\ \dot{y} = \dot{x} - v_T \end{matrix} \Big\} \; \dot{x} = \dot{y} + v_T$$

$$\downarrow$$

$$\frac{dy}{dt} = \frac{F}{\zeta} - v_T \quad \Rightarrow \quad \int_{x_R}^{-x_L} \frac{dy}{F/\zeta - v_T} = \int dt = \Delta t \qquad \text{assume } v_T > \max\left\{\frac{F(y)}{\zeta}\right\}$$

x_R ← in order of encounter

$$\Delta x = \Delta y + v_T \Delta t = \int_{x_R}^{-x_L} dy \left\{1 + \frac{v_T}{F/\zeta - v_T}\right\} = \int_{-x_L}^{x_R} dy \, \frac{F(y)}{\zeta v_T - F(y)}$$

$$\Rightarrow \boxed{\Delta x = \int_{-x_L}^{x_R} dy \, \frac{F(y)}{\zeta v_T - F(y)}} \qquad \text{note, } \frac{F(y)}{\zeta v_T - F(y)} > \frac{F(y)}{\zeta v_T}$$

$$\hookrightarrow > \int_{-x_L}^{x_R} dy \, \frac{F(y)}{\zeta v_T} = \frac{1}{\zeta v_T} \int dy \, F(y) = 0 \quad \text{since } F(y) = -\partial U / \partial y$$

U = trapping potential

$\therefore \underline{\Delta x > 0}$ particle experiences backwards force for a shorter time than forwards force. as long as $v_T - F/\zeta > 0$ Δt is finite

$$\Delta t = \int_{-x_L}^{x_R} \frac{dy}{v_T - F(y)/\zeta}$$

— could worry about special case $v_T - F(y)/\zeta \sim y^{\alpha}$ w/constraint on α

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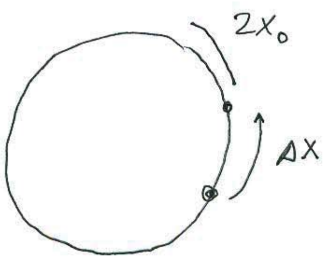
for large v_T

$$\Delta x = \int \frac{dy \, F}{\zeta v_T (1 - F/\zeta v_T)} \simeq \int dy \, \frac{(F + F^2/\zeta v_T + \dots)}{\zeta v_T}$$

$$\simeq \int \frac{dy}{(\zeta v_T)^2} F^2(y) \quad \text{manifestly} > 0 .$$

circular orbit :



$\Delta\theta = \frac{\Delta X}{R}$: after 1^{st} kick, particle is caught again w/ extra time $\frac{2\pi R - 2X_0}{v_T}$

$$= \frac{1}{f_T}\left(1 - \frac{X_0}{\pi R}\right) \qquad f_T = \left(\frac{2\pi R}{v_T}\right)^{-1}$$

$\therefore$ angular frequency is

$$\frac{\Delta X}{\Delta t + \frac{1}{f_T}\left(1 - \frac{X_0}{\pi R}\right)} \cdot \frac{1}{2\pi R}$$

for large trap velocity, $\Delta t \simeq \frac{1}{v_T}\int dy = \frac{2X_0}{v_T} = \frac{1}{f_T} \cdot \frac{X_0}{\pi R}$

then $$f_p = \frac{\Delta X}{2\pi R} \frac{1}{\frac{1}{f_T}\frac{X_0}{\pi R} + \frac{1}{f_T}\left(1 - \frac{X_0}{\pi R}\right)} \simeq \frac{\Delta X}{2\pi R} f_T$$

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triangular potential

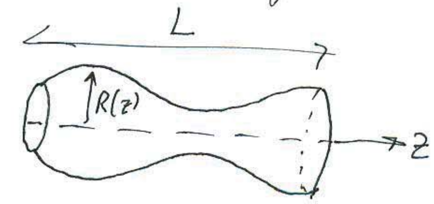
$$\Delta X = \int_{X_0}^{0} dy \frac{F}{\zeta v_T - F} + \int dy \frac{(-F)}{\zeta v_T + F} = \frac{2X_0 v_c^2}{v_T^2 - v_c^2}$$

$$\Delta t = \frac{2X_0 v_T}{v_T^2 - v_c^2}$$

then (some algebra)

$$\boxed{\frac{f_p}{f_c} = \left(\frac{X_0}{\pi R}\right) \frac{f_T/f_c}{(f_T/f_c)^2 - 1} \left\{ 1 + \frac{(X_0/\pi R)}{(f_T/f_c)^2 - 1} \right\}^{-1}}$$

Fluctuations of circular objects



volume $V = \int dz\, \pi R^2(z)$

surface area $S = \int dz\, 2\pi R \sqrt{1 + R_z^2}$

write $R = \rho_0 + u_q \sin qz$ and adjust ρ_0 to conserve volume up to $O(u_q^2)$
(for consistency with application of equipartition theorem).

$$R^2 = \rho_0^2 + 2\rho_0 u_q \sin qz + u_q^2 \sin^2 qz$$

$$V = \underbrace{\pi R_0^2 L}_{\text{initially}} = \pi \rho_0^2 L + \underset{\uparrow \atop \int \sin qz = 0}{0} + \pi u_q^2 \cdot \tfrac{1}{2} L = \pi L\left(\rho_0^2 + \tfrac{1}{2} u_q^2\right)$$

$$\therefore\ \rho_0 = R_0\left[1 - u_q^2/2R_0^2\right]^{1/2} \simeq R_0\left(1 - \frac{1}{4}\frac{u_q^2}{R_0^2} + \dots\right)$$

$$\therefore\ \boxed{\rho_0 \simeq R_0 - \frac{1}{4}\frac{u_q^2}{R_0} + \dots}$$

look at surface area: $R\sqrt{1 + R_z^2} = (\rho_0 + \zeta)\left(1 + \tfrac{1}{2}\zeta_z^2 + \dots\right)$ where $\zeta = u_q \sin qz$

$$= \rho_0 + u_q \sin qz + \tfrac{1}{2}\rho_0 q^2 u_q^2 \cos^2 qz + \dots$$

$$\therefore\ S = 2\pi\rho_0 L + 2\pi \cdot \tfrac{1}{2}\rho_0 q^2 u_q^2 \cdot \tfrac{1}{2} L = 2\pi L \rho_0\left(1 + \tfrac{1}{4} q^2 u_q^2 + \dots\right)$$

$$= 2\pi L R_0\left(1 - \frac{1}{4}\frac{u_q^2}{R_0^2} + \dots\right)\left(1 + \frac{1}{4} q^2 u_q^2 + \dots\right)$$

hence, the energy is

$$\sigma S = 2\pi L R_0 \sigma\left\{1 + \frac{1}{4R_0^2}\left(q^2 R_0^2 - 1\right) u_q^2 + \dots\right\}$$

$$= \text{const} + \underbrace{\frac{\pi}{2}\frac{L}{R_0}\sigma\left[(qR_0)^2 - 1\right] u_q^2}_{= \frac{1}{2}k_B T \text{ when averaged}} + \dots$$

equipartition $\Rightarrow$ $$\boxed{\langle u_q^2\rangle = \frac{k_B T R_0}{\pi L \sigma\left[(qR_0)^2 - 1\right]}}$$

clearly, for $qR_0 < 1$ there is a problem – the Rayleigh instability!

circular domain

write $\underset{\sim}{r}(\theta) = (\rho_0+\zeta)\,\hat{e}_r$ $\qquad$ $\underset{\sim}{r}_\theta = \zeta_\theta\,\hat{e}_r + (\rho_0+\zeta)\,\hat{e}_\theta$

area $A = \int d\theta\, \frac{1}{2}\, \underset{\sim}{r}\times \underset{\sim}{r}_\theta = \int d\theta\, \frac{1}{2}(\rho_0+\zeta)^2$

initial area $= \pi R_0^2 = \pi\rho_0^2 + 0 + \frac{1}{2}\int d\theta\, u_q^2 \sin^2 q\theta$

$$= \pi\rho_0^2 + \frac{1}{2}u_q^2\cdot\frac{1}{\cancel{2}}\cdot\cancel{2}\pi = \pi\rho_0^2 + \frac{\pi}{2}u_q^2$$

$$\Rightarrow \underline{\rho_0 = R_0\left(1-u_q^2/2R_0^2\right)^{1/2}} \quad \text{as above}$$

length of fluctuating domain:

$$L = \int d\theta\sqrt{g} = \int d\theta\left[(\rho_0+\zeta)^2+\zeta_\theta^2\right]^{1/2} = \rho_0\left[\left(1+\frac{\zeta}{\rho_0}\right)^2+\frac{\zeta_\theta^2}{\rho_0^2}\right]^{1/2}$$

$g = \underset{\sim}{r}_\theta\cdot \underset{\sim}{r}_\theta$

$$\left[1+\frac{2\zeta}{\rho_0}+\frac{\zeta^2}{\rho_0^2}+\frac{\zeta_\theta^2}{\rho_0^2}\right]^{1/2} \simeq 1+\frac{\zeta}{\rho_0}+\frac{1}{2}\left(\cancel{\frac{\zeta^2}{\rho_0^2}}+\frac{\zeta_\theta^2}{\rho_0^2}\right) - \frac{1}{8}\cdot\cancel{\frac{4\zeta^2}{\rho_0^2}}+\dots$$

so, $L = \int d\theta\,\rho_0\left\{1+\frac{\zeta}{\rho_0}+\frac{1}{2}\frac{\zeta_\theta^2}{\rho_0^2}+\dots\right\} \simeq 2\pi\rho_0 + 0 + \frac{1}{2\rho_0}\cdot\cancel{2}\pi\cdot\frac{1}{\cancel{2}}\cdot q^2\cdot u_q^2$

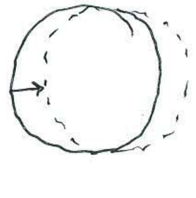
energy:

$$\gamma L = 2\pi\rho_0\gamma + \frac{\gamma}{2\rho_0}\pi q^2 u_q^2 = 2\pi\gamma R_0\left(1-\frac{1}{4}\frac{u_q^2}{R_0^2}+\dots\right) + \frac{\pi}{2}\gamma q^2\frac{u_q^2}{R_0} \quad \text{to leading order}$$

$$= \boxed{2\pi\gamma R_0 + \frac{\pi\gamma}{2R_0}\left((qR_0)^2-1\right)u_q^2}$$

equipartition $\quad \frac{\pi\gamma}{2R_0}\left[(qR_0)^2-1\right]\langle u_q^2\rangle = \frac{1}{2}k_BT \qquad \boxed{\langle u_q^2\rangle = \frac{k_BT\,R_0}{\pi\gamma\left[(qR_0)^2-1\right]}}$

<u>note</u>: $qR_0=1$ mode is a <u>translation</u>; no energy cost



Nonlinear diffusion

$$c_t = D\left(c^p c_x\right)_x \qquad c = \frac{M^{2/(2+p)}}{(Dt)^{1/(2+p)}} F(\xi) \qquad \xi = \frac{x}{(M^p D t)^{1/(2+p)}}$$

$$\int c(x,t)\,dx = M \qquad c_x(0,t) = 0 \qquad c(x\to\infty, t) \to 0$$

$$c_t = -\frac{1}{(2+p)} \frac{M^{2/(2+p)}}{(Dt)^{1/(2+p)} \cdot t} F(\xi) + \frac{M^{2/(2+p)}}{(Dt)^{1/(2+p)}} F'(\xi) \cdot \left(-\frac{1}{(2+p)} \frac{\xi}{t}\right)$$

$$= \frac{M^{2/(2+p)}}{(Dt)^{1/(2+p)}} \cdot \frac{1}{t} \left\{ -\frac{1}{(2+p)} \xi F' - \frac{1}{(2+p)} F \right\} = \boxed{\frac{-M^{2/(2+p)}}{(Dt)^{1/(2+p)} t (2+p)} (\xi F)'}$$

$$c^p c_x = \frac{M^{2p/(2+p)}}{(Dt)^{p/(2+p)}} F^p \frac{M^{2/(2+p)}}{(Dt)^{1/(2+p)}} \cdot \frac{1}{(M^p D t)^{1/(2+p)}} = \boxed{\frac{M}{Dt} F^p F'}$$

$$\rightarrow D(c^p c_x)_x = D \frac{M}{Dt} (F^p F')' \cdot \frac{1}{(M^p D t)^{1/(2+p)}}$$

$$\therefore \frac{M^{2/(2+p)}}{t (Dt)^{1/(2+p)}} (F^p F')' = -\frac{M^{2/(2+p)}}{(Dt)^{1/(2+p)} t (2+p)} (\xi F)' \Rightarrow \boxed{(F^p F')' = -\frac{1}{(2+p)} (\xi F)'}$$

hence $F^p F' = -\frac{1}{(2+p)} \xi F + \text{const}$: by the boundary conditions at $\xi = 0$, const $= 0$

$$\Rightarrow \boxed{F^p F' = -\frac{1}{(2+p)} \xi F}$$

note the normalisation

$$\int c\,dx = \frac{M^{2/(2+p)}}{(Dt)^{1/(2+p)}} \int \underbrace{dx\, F(\xi) \frac{(M^p D t)^{1/(2+p)}}{(M^p D t)^{1/(2+p)}}}_{d\xi} = M \int d\xi\, F(\xi)$$

so we take F normalised to unity

solving the last boxed eqn. $F^p F' = -\frac{1}{(2+p)}\xi F \Rightarrow dF\, F^{p-1} = -\frac{1}{(2+p)}\xi\, d\xi$

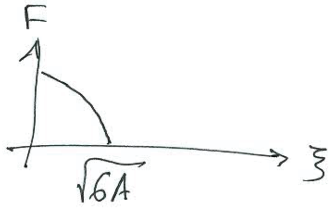
$$\frac{F^p}{p} = -\frac{\xi^2}{2(2+p)} + \text{const} \quad \text{or} \quad \boxed{F(\xi) = \left[A - \frac{p}{2(2+p)}\xi^2\right]^{1/p}}$$

for $0 \leqslant \xi \leqslant \left[\frac{2(2+p)}{p}A\right]^{1/p}$

special case of $p = 1$

$$F(\xi) = A - \frac{1}{6}\xi^2 \qquad 0 \leqslant \xi \leqslant \sqrt{6A}$$

$$= 0 \quad \text{outside}$$



normalisation

$$\int_0^{\sqrt{6A}} \left(A - \frac{1}{6}\xi^2\right) d\xi = 1 = A\sqrt{6A} - \frac{1}{18}(6A)^{3/2} = A^{3/2} \cdot \frac{2}{3}\sqrt{6} = A^{3/2} \cdot \frac{2\sqrt{2}}{\sqrt{3}}$$

$$\therefore A = \left(\frac{\sqrt{3}}{2\sqrt{2}}\right)^{2/3} = \left(\frac{3}{8}\right)^{1/3}.$$

FitzHugh - Nagumo model

i) homogeneous fixed point

$$\left.\begin{aligned} u(1-u)(u-a)+v=0 \\ bu-cv=0 \end{aligned}\right\} \quad v=\frac{b}{c}u \qquad \text{so } u=0 \text{ or } (1-u)(u-a)+\frac{b}{c}=0$$

$$u^2-(1+a)u+a-b/c=0$$

$$u^* = \frac{1+a \pm \sqrt{(1+a)^2-4a+4b/c}}{2}$$

$$= \frac{1+a \pm \sqrt{(1-a)^2+4b/c}}{2}$$

$v^* = \frac{b}{c}u^*$: choose + sol'n so $u^* > 0$.

stability matrix

$$J = \begin{pmatrix} ? & 1 \\ b & -c \end{pmatrix} \qquad J_{11} = (1-u)(u-a)+u(1-u)-u(u-a) \quad @\, u^*$$

$$= -3u^2+2u+2ua-a$$

but $u^2 = \frac{b}{c}-a+(1+a)u$

so $J_{11} = -\frac{3b}{c}+2a-u(1+a)$ after some algebra

now, $u^* > \frac{1+a}{2}$ so $2a-u^*(1+a) < 2a-\frac{(1+a)^2}{2}$

$$< -\tfrac{1}{2}(1-a)^2 < 0$$

thus $J = \begin{pmatrix} - & + \\ + & - \end{pmatrix}$ $\quad \mathrm{Tr}\, J < 0$

and since $J_{11} < -\frac{b}{c}$ $\quad$ Det > 0 $\Big\}$ stable fixed pt.

(ii) $u_t = -\gamma\dot{\phi} \quad u_{xx} = \ddot{\phi} \quad v_t = -\gamma\dot{\psi} \quad \cdot = \frac{d}{d\xi} \quad \xi = x-\gamma t$

$$\boxed{\begin{aligned} -\gamma\dot{\phi} &= \ddot{\phi}+\phi(1-\phi)(\phi-a)+\psi \\ -\gamma\dot{\psi} &= b\phi-c\psi \end{aligned}}$$

iii) if $b = c = 0$ $\quad \dot{\psi} = 0 \Rightarrow \psi = \text{const} = 0$ by statement of problem

$$\Rightarrow -\gamma \dot{\phi} = \ddot{\phi} + \phi(1-\phi)(\phi - a)$$

set $\phi = \dfrac{1}{1+e^{-\alpha\xi}}$ $\quad \dot{\phi} = \dfrac{\alpha e^{\cdots}}{(1+e^{\cdots})^2} = \alpha(\phi - \phi^2)$ $\quad$ (useful trick!)

$$\therefore \ddot{\phi} = \alpha \dot{\phi}(1-2\phi) \Rightarrow \boxed{-\gamma \dot{\phi} = \alpha \dot{\phi}(1-2\phi) + \frac{\dot{\phi}}{\alpha}(\phi - a)}$$

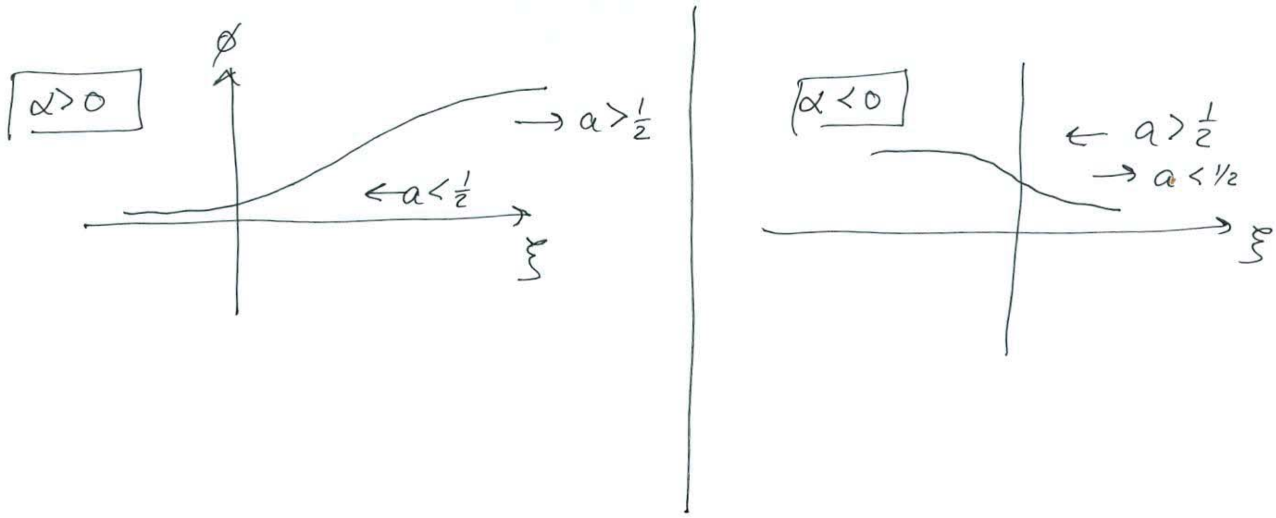
either $\dot{\phi} = 0$ or

$$\phi(1-2\alpha^2) + (\alpha^2 + \gamma\alpha - a) = 0$$

holds $\forall \xi$ if $1 - 2\alpha^2 = 0$ $\quad \alpha = \pm \frac{1}{\sqrt{2}}$

and $\alpha^2 + \gamma\alpha - a = 0$ $\quad \gamma = \pm\sqrt{2}\left(a - \frac{1}{2}\right)$

direction?, depends on γ



Chemotaxis

$$n_t = D n_{xx} + bn\left(1 - \frac{n}{n_0}\right) - \partial_x\left(\chi(a)\, n\, a_x\right)_x$$

$$a_t = D_A a_{xx} + hn - ka \qquad \chi(a) = \frac{\chi_0 K}{(K+a)^2}$$

clearly, set $u = n/n_0$, let $T = bt$ $\frac{\partial}{\partial t} = b\frac{\partial}{\partial T}$ k^{-1} is a time, so set

$$a = h n_0 k^{-1} v$$

diffusive scalings: $x = \sqrt{Db^{-1}}\, X$

$$\partial_x = \frac{1}{\sqrt{\ }}\partial_X \quad \text{etc.}$$

algebra $\longrightarrow$ $\beta = \frac{\chi_0 K k}{D h n_0}$ $\alpha = \frac{Kk}{hn_0}$ $\gamma = k/b$ $\delta = D_A/D$

$$\boxed{\begin{aligned} \mathring{u} &= u'' + u(1-u) - \beta\left[\frac{uv'}{(\alpha+v)^2}\right]' \\ \mathring{v} &= \delta v'' + \gamma(u - v) \end{aligned}}$$

homogeneous steady state $u = v = 1$ by inspection

stability matrix $J = \begin{pmatrix} -1 & 0 \\ \gamma & -\gamma \end{pmatrix}$ $\left.\begin{matrix} Tr < 0 \\ Det > 0 \end{matrix}\right\}$ hence stable

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now allow spatial instability e^{ikx}

$$J = \begin{pmatrix} -1 & 0 \\ \gamma & -\gamma \end{pmatrix} + \begin{pmatrix} -k^2 & 0 \\ 0 & -\delta k^2 \end{pmatrix} + \begin{pmatrix} 0 & \hat\beta k^2 \\ 0 & 0 \end{pmatrix} \qquad \hat\beta = \frac{\beta}{(1+\alpha)^2}$$

$$= \begin{pmatrix} -(1+k^2) & \hat\beta k^2 \\ \gamma & -(\gamma + \delta k^2) \end{pmatrix} \qquad Tr < 0 \text{ always as usual}$$

$$Det = (1+k^2)(\gamma + \delta k^2) - \gamma\hat\beta k^2 = \boxed{\underbrace{\delta}_{\text{"a"}}(k^2)^2 + \underbrace{(\gamma + \delta - \gamma\hat\beta)}_{\text{"b"}} k^2 + \underbrace{\gamma}_{\text{"c"}}} \quad (*)$$

can $*$ ever be negative for $k^2 > 0$?

if so, need $\gamma + \delta - \gamma\hat{\beta} < 0$ "b"

and $(\gamma+\delta-\gamma\hat{\beta})^2 - 4\gamma\delta > 0$ "b^2-4ac"

i.e. $-(\gamma+\delta-\gamma\hat{\beta}) > 2\sqrt{\delta}\sqrt{\gamma}$

or $\gamma\hat{\beta} > \gamma + 2\sqrt{\delta}\sqrt{\gamma} + \delta = (\sqrt{\gamma}+\sqrt{\delta})^2$

$$\boxed{\frac{\gamma\beta}{(1+\alpha)^2} > (\sqrt{\gamma}+\sqrt{\delta})^2}$$ as required $(*)$

$$k_c^2 = \frac{"-b"}{2a} = \frac{-\gamma-\delta+\gamma\hat{\beta}}{2\delta} = \frac{\gamma\beta - (\gamma+\delta)(1+\alpha)^2}{2\delta(1+\alpha)^2}$$

$k_c = \frac{2\pi}{\lambda_c}$ ← wavelength

if $\alpha = \gamma = \delta = 1$

$$k_c^2 = \frac{\beta-8}{8}$$

$$\lambda_c = 2\pi\sqrt{\frac{8}{\beta-8}}$$

and note that $*$ gives $\beta > 16$

A Turing instability

$$u_t = u_{xx} + \frac{u^2}{v} - bu$$

$$v_t = d v_{xx} + u^2 - v$$

homogeneous fixed point.

$$\frac{u^2}{v} - bu = 0 \quad u^2 - v = 0$$

$$\Rightarrow \boxed{u^* = b^{-1} \quad v^* = b^{-2}}$$

$$J\Big|_{u^*, v^*} = \begin{pmatrix} b & -b^2 \\ 2/b & -1 \end{pmatrix} \qquad \begin{aligned} Det &= b \\ Tr &= b-1 \end{aligned} \qquad \text{so} \quad \boxed{\begin{array}{c} 0<b<1 \\ \text{stable} \end{array}}$$

spatial instability $\sim e^{ikx}$

$$J = \begin{pmatrix} b-k^2 & -b^2 \\ 2/b & -1-dk^2 \end{pmatrix}$$

$Tr < 0$ of course

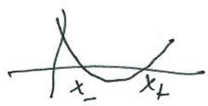
$$\begin{aligned} Det &= (b-k^2)(-1-dk^2) + 2b \\ &= d(k^2)^2 + (1-bd)k^2 + b \end{aligned}$$

can this be negative for any $k^2 > 0$?

need $1-bd<0$ i.e. $bd>1$

also $(1-bd)^2 > 4bd$

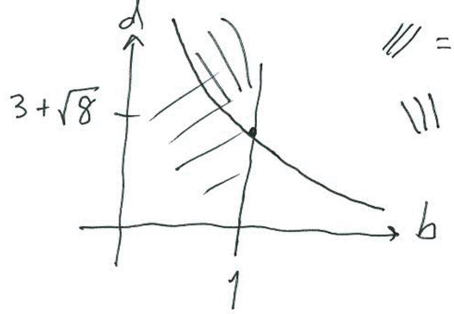
$$1 - 2x + x^2 > 4x$$
$$x^2 - 6x + 1 > 0$$



$x_\pm = 3 \pm \sqrt{8}$

x_+ relevant

so $\boxed{bd > 3 + \sqrt{8}}$



/// = homg. stability

\\\ Turing instability

critical wavenumber $\quad k_c^2 = \frac{-(1-bd)}{2d} = \frac{1+\sqrt{2}}{d}$

$$\lambda_c = 2\pi\sqrt{\frac{d}{1+\sqrt{2}}}$$

Phytoplankton – zooplankton

$$u_t = u_{xx} + u(1+u-\gamma v)$$

$$v_t = d v_{xx} + v(\beta u - v)$$

want $u, v > 0$ only

fixed point: $u_0 = \dfrac{1}{\beta\gamma - 1}$ $\quad v_0 = \beta u_0 = \dfrac{\beta}{\beta\gamma - 1}$ $\quad$ clearly $\beta\gamma > 1$

stability

$$J = u_0 \begin{pmatrix} 1 & -\gamma \\ \beta^2 & -\beta \end{pmatrix}$$

$\mathrm{Det} = u_0^2 \beta(\gamma\beta - 1) > 0$

$\mathrm{Tr} = u_0(1-\beta) < 0 \Rightarrow \boxed{\beta > 1}$

again, with perturbations $\sim e^{ikx}$

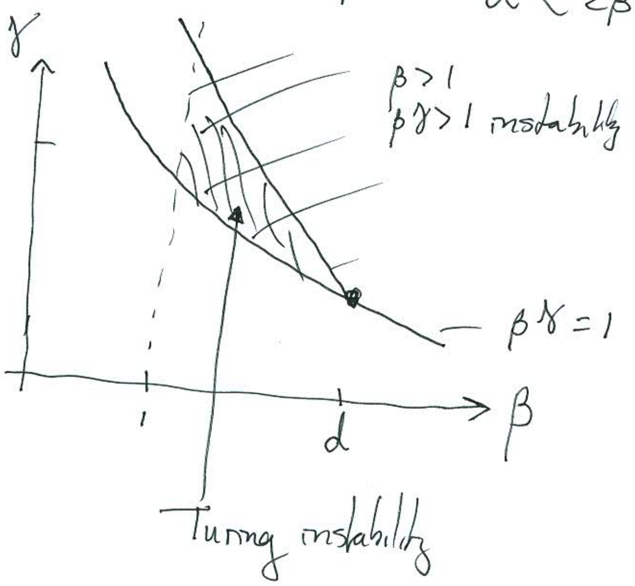
$$\boxed{J = u_0 \begin{pmatrix} 1 & -\gamma \\ \beta^2 & -\beta \end{pmatrix} - k^2 \begin{pmatrix} 1 & 0 \\ 0 & d \end{pmatrix}}$$

so $\mathrm{Tr} < 0$ still and $\mathrm{Det} = d(k^2)^2 + (\beta - d) u_0 k^2 + \beta(\gamma\beta - 1) u_0^2$

for Turing instability, need this < 0 for some $k^2 > 0$

i.e. need $\boxed{d > \beta}$ and $\boxed{(\beta - d)^2 > 4\beta d(\gamma\beta - 1)}$ (*)

i.e. $\gamma < \dfrac{1}{\beta} + \dfrac{1}{d}\left(\dfrac{\beta - d}{2\beta}\right)^2 = \dfrac{1}{d}\left(\dfrac{\beta + d}{2\beta}\right)^2$ and this $> \dfrac{1}{\beta}$



$$k_c^2 = \frac{d - \beta}{2d} = \sqrt{\frac{\beta(\gamma\beta - 1)}{d}}$$

but (*) $\rightarrow \beta < \dfrac{d}{2\sqrt{\gamma d} - 1}$

so $k_c^2 = \dfrac{\sqrt{\gamma d} - 1}{2\sqrt{\gamma d} - 1}$