Part III: Biological Physics and Complex Fluids (Michaelmas, 2016)
E Lauga (December 3, 2016)

Example Sheet #4
Work due noon Thursday 19 January 2017;
Example class Monday 23 January 2017, MR12, 2pm.

1. *Electro-osmosis.*
Consider two identical, parallel conducting plates at $y = \pm h$. An electric field $E_{\text{ext}}\mathbf{e}_x$ is applied in the fluid along the x direction and drives the electro-osmotic flow of an ionic solution of screening length κ^{-1}.
(a) If both plates are held at potential ϕ_0, calculate the flow between the plates. What does the flow look like in the limit $\kappa h \gg 1$? Find an approximate solution in the limit of overlapping Debye layers, $\kappa h \ll 1$ and interpret physically your result.
(b) How do the results change if the plates are held at opposite potentials so that $\phi(\pm h) = \pm\phi_0$?

2. *Electrophoresis.*
(a) Consider two incompressible solutions of Stokes equations with velocities $\mathbf{u}$ and $\hat{\mathbf{u}}$, stress fields $\boldsymbol{\sigma}$ and $\hat{\boldsymbol{\sigma}}$ and body forces $\mathbf{f}$ and $\hat{\mathbf{f}}$. These solutions are assumed to be taking place in the same fluid volume V bounded by the same surface S with normal $\mathbf{n}$ into the fluid. Prove the reciprocal theorem for Stokes flows i.e.

$$\iint_S \hat{\mathbf{u}} \cdot \boldsymbol{\sigma} \cdot \mathbf{n}\, \mathrm{d}S - \iint_S \mathbf{u} \cdot \hat{\boldsymbol{\sigma}} \cdot \mathbf{n}\, \mathrm{d}S = \iiint_V \hat{\mathbf{f}} \cdot \mathbf{u} \mathrm{d}V - \iiint_V \mathbf{f} \cdot \hat{\mathbf{u}} \mathrm{d}V. \tag{1}$$

(b) Apply the theorem using two specific solutions: For the $\mathbf{u}, \boldsymbol{\sigma}, \mathbf{f}$ problem consider a rigid no-slip particle of potential ϕ_0 held in place by an external force and subject to a flow at infinity $\mathbf{U}_0$. For the $\hat{\mathbf{u}}, \hat{\boldsymbol{\sigma}}, \hat{\mathbf{f}}$ problem consider the same stationary rigid no-slip particle with same potential where, in that case, an external force $\mathbf{F}_0$ is applied to keep it stationary when subject to an external electric field $\mathbf{E}_0$ (denote the field $\mathbf{E}$ and the net charge density ρ_E in the fluid domain). By carefully paying attention to the contribution of the surface integrals at infinity, show that

$$-\mathbf{U}_0 \cdot \mathbf{F}_0 = \iiint_V \rho_E \mathbf{E} \cdot \mathbf{u} \mathrm{d}V. \tag{2}$$

(c) [Difficult] Show that in the thin Debye layer limit (i.e. $\kappa R \gg 1$ where R is any dimension of the particle), when using the locally-planar solution for the charge density along with a Taylor expansion for the velocity near S, Eq. (2) becomes

$$\mathbf{U}_0 \cdot \mathbf{F}_0 = \frac{\epsilon\phi_0}{\mu} \iint_S \mathbf{E} \cdot \boldsymbol{\sigma} \cdot \mathbf{n} \mathrm{d}S. \tag{3}$$

Using vector calculus, show that this leads to

$$\mathbf{U}_0 \cdot \mathbf{F}_0 = -\frac{\epsilon\phi_0}{\mu} \mathbf{E}_0 \cdot \mathbf{F}_0, \tag{4}$$

and deduce the value of the electrophoretic mobility of the particle.

3. *Diffusio-osmosis*.*
The reciprocal theorem, Eq. (1), can be used to characterise diffusio-osmotic flow induced inside micro-channels. Consider a straight two-dimensional channel with walls at $y = \pm h$. The wall at $y = h$ is a no-slip surface. Due to self-generated chemical gradients along the surface at $y = -h$, a slip velocity $\mathbf{u}^s = u^s(x)\mathbf{e}_x$ is induced there and we want to compute the

resulting two-dimensional flow rate, Q. The slip velocity is assumed to be periodic along x with period L.
(a) Apply the result of Eq. (1) taking, for the hat problem, the simplest no-slip solution you know for flow in a two-dimensional channel. Deduce Q as a surface integral on $\mathbf{u}^s$.
(b) The wall at $y = -h$ generates chemical gradients through the prescribed rate of production of a chemical species of concentration c as

$$-D\frac{\partial c}{\partial y}\bigg|_{x,y=-h} = A(x), \tag{5}$$

where A is a prescribed periodic function of period L and D is the diffusivity of the species. The concentration on the upper wall is set to zero, $c(x,h) = 0$. The chemical species is assumed to be a solution to the diffusion equation in the channel, $\nabla^2 c = 0$. The diffusio-osmotic mobility, $M(x)$, on the wall at $y = -h$ is a prescribed periodic function of period L. Using a decomposition in complex Fourier series for A and M calculate the flow rate induce in the channel. Deduce that if M and A are proportional to each other, the flow rate is always zero.

4. *Kinematics of locomotion.*
Suppose that an organism swims with velocity, $\mathbf{U}$, and rotation rate, $\boldsymbol{\Omega}$, both of which are constant when computed in the swimmer's frame of reference. Show that the trajectories of the swimmer in the lab frame are in general helices, for which you will explicitly compute the pitch and radius. Show that depending on the relationship between $\mathbf{U}$ and $\boldsymbol{\Omega}$, other types of trajectories are also possible.

5. *Locomotion in isotropic media.*
Consider a swimmer whose shape is slender and fully described by the motion of its centerline. Assume it is located in a hypothetical fluid in which the hydrodynamic force per unit length resisting the swimmer's motion, $\mathbf{f}$, is proportional to the local value of its centerline velocity relative to the fluid, $\mathbf{u}$,

$$\mathbf{f} = -\alpha\mathbf{u}, \tag{6}$$

where α is a constant with dimensions of viscosity. This is a configuration known as "isotropic drag". The swimmer centerline is assumed to be inextensible and of total length L, and is assumed to be of uniform mass per unit length. Show that in that situation, force-free swimming implies that no instantaneous motion of the swimmer's center of mass is possible.

6. *Locomotion by surface motion.*
The integral theorem in Eq. (1) may be exploited in the context of swimming microorganisms. Consider a spherical organism of radius a, swimming at speed $\mathbf{U}$ by instantaneously imposing a velocity distribution $\mathbf{u}'$ along its spherical boundary (measured in the swimming frame). By applying the reciprocal theorem to the free-swimming problem (flow $\mathbf{u}$) and to the problem of solid-body translation at speed $\mathbf{V}$ (flow $\hat{\mathbf{u}}$, which is the standard Stokes drag problem, and for which $\hat{\boldsymbol{\sigma}} \cdot \mathbf{n}$ is in fact constant everywhere along the sphere), show that

$$\mathbf{U} = -\frac{1}{4\pi a^2}\int_S \mathbf{u}' dS. \tag{7}$$

7. *Artificial chemical swimmer.*
Flows induced by diffusio-osmosis may be exploited to design artificial swimmers. Consider a spherical body of radius a which is chemically active and is able to generate its own chemical gradient. Assume that outside the diffuse layer, a chemical species of concentration c diffuses and has a fixed concentration c_∞ far from the sphere. Near the spherical surface right outside

the diffuse layer, the chemical species is produced at a fixed rate in a spatially-dependent (but axisymmetic) manner written in spherical coordinate as

$$-D\mathbf{n}\cdot\nabla c|_{r=a} = A(\theta), \quad A(\theta) = \sum_{n=0}^{\infty} A_n P_n(\cos\theta), \tag{8}$$

where the coefficients A_n are prescribed, D is the diffusivity and where P_n refers to the Legendre polynomial of order n. Assume that the local diffusio-osmotic mobility, M, is constant along the spherical surface and ignore advective transport by the flow so that that c satisfies the diffusion equation.
(a) Solve for the concentration field.
(b) Deduce the local distribution of tangential diffusio-osmotic velocities induced by the chemical gradients.
(c) Using the result of #7, calculate the value of the swimming velocity of the active particle as a function of the coefficients A_n (you will need to look up some properties of Legendre polynomials).

8. *Optimal locomotion*.*
In lectures we derived the swimming velocity, U, of an infinite planar flagellum deforming as a traveling wave. Compute the rate of working, $\dot{W}$, of the flagellum against the fluid per unit wavelength. Using notation from lectures, show that it can be written as

$$\frac{\dot{W}}{c_\perp \Lambda V^2} = \frac{\rho}{\alpha^2} - \frac{\rho^2}{1+\beta(\rho-1)},$$

where $\alpha = \lambda/\Lambda$, and where β, ρ and V were defined in lectures.

Using this value for $\dot{W}$, we can define a swimming efficiency, $\mathcal{E}$, as the ratio between the useful work of translation to the total work, i.e.

$$\mathcal{E} = \frac{c_\parallel \Lambda U^2}{\dot{W}}.$$

Compute the value of $\mathcal{E}$. The result, which is dimensionless, depends only on the values of ρ, α, and β. Justify why we have $\alpha^2 \leq \beta$. Use this inequality to find an upper-bound for $\mathcal{E}$, denoted $\tilde{\mathcal{E}}$, as a function of β and ρ only. Show that there exists a value of β which leads to a maximum value for $\tilde{\mathcal{E}}$ as $\tilde{\mathcal{E}}_{\max} = (1-\sqrt{\rho})^2$. Compute the typical value for $\tilde{\mathcal{E}}_{\max}$ using a good estimate for ρ. What are the shapes for which this maximum efficiency is obtained? Justify your answer by examining the inequalities you had to use to derive the maximum efficiency.

9. *A paper from the literature.*
Read the paper: G. I. Taylor, 1951, "Analysis of the Swimming of Microscopic Organisms", Proc. R. Soc. A, **209**, pp. 447-461 available on the course website. What is the motivation behind Eq. (1)? Why is the flow a solution to Eq. (3)? Verify that Eq. (5) is solution to the equation. Use a symmetry argument to explain why at first order in $b\sigma$ the swimming velocity is necessarily zero. The final formula of Eq. (33) predicts swimming in which direction compared to the direction of propagation of the wave? What is the physics captured by Fig. 4?