Examples Sheet #3

1. *Nonlinear Diffusion.* For the nonlinear diffusion equation $C_t = D(C^p C_x)_x$, where p and D are strictly positive constants, show that the self similar solution defined in $x \geq 0$ of the form

$$C(x,t) = \frac{M^{2/(2+p)}}{(Dt)^{1/(2+p)}} F(\xi), \qquad \xi = \frac{x}{(M^p Dt)^{1/(2+p)}}$$

which satisfies $\int_0^\infty C(x,t)\ \mathrm{d}x = M$, $C_x(0,t) = 0$ and $C(x \to \infty, t) \to 0$ for some constant $M > 0$, is given by

$$F(\xi) = \left[A - \frac{p\xi^2}{2(2+p)} \right]^{1/p}, \qquad 0 < \xi < \left[\frac{2(2+p)A}{p} \right]^{1/2}$$

and $F = 0$ otherwise. For the case $p = 1$, prove that $A = (3/8)^{1/3}$.

2. *The Fitzhugh-Nagumo model.* The Fitzhugh-Nogumo model for nerve impulse propagation is given by

$$u_t = u_{xx} + u(1-u)(u-a) + v$$
$$v_t = bu - cv; \quad (b, c > 0, 0 < a < 1).$$

(i) Show that the homogeneous system has a stable fixed point for which neither u nor v is zero. (ii) Seek a travelling wave solution of the equations of the form

$$u = \phi(\xi), v = \psi(\xi), \xi = x - \gamma t.$$

Deduce the ODEs to be satisfied by ϕ, ψ. (iii) Verify that if $b = c = \psi(0) = 0$ then there is a solution of the form

$$\phi = \frac{1}{1 + e^{-\alpha\xi}},$$

for two possible values of α which should be found. What is the wave speed γ in each case? In what direction does the wave propagate?

3. *BZ reaction.* A restricted version of the Belousov-Zhabotinskii reaction between two chemical species with concentrations $u(x,t), v(x,t)$ is

$$u_t = u_{xx} + u(1 - u - rv)$$

$$v_t = dv_{xx} - buv\ ,$$

where r, b, d are positive constants.

Seek travelling wave-front solutions with constant speed such that $(u, v) \to (0, 1)$ as $x \to \infty$, $(u, v) \to (1, 0)$ as $x \to -\infty$. Suppose that, at time $t = 0$, both u and $1 - v$ tend to zero like e^{-ax} as $x \to +\infty$. Show that, if $d = 1$, there is a particular relation between b and r for which both equations reduce to the Fisher equation for a single variable, and determine the possible wave-front speeds in that case, distinguishing between the cases $a < 1$ and $a > 1$.

4. *Chemotaxis.* Consider the chemostatic system

$$n_t = Dn_{xx} + bn(1 - \frac{n}{n_0}) - (\chi(a) n a_x)_x$$

$$a_t = D_A a_{xx} + hn - ka,$$

where $\chi(a) = \chi_0 K/(K+a)^2$. Find a scaling such that this reduces to

$$\dot{u} = u'' + u(1-u) - \beta \left[\frac{uv'}{(\alpha + v)^2} \right]'$$

$$\dot{v} = \delta v'' + \gamma(u - v),$$

where $\cdot$ and $'$ refer to differentiation with the scaled t and x respectively, and $\alpha, \beta, \gamma, \delta$ are positive constants. Show that the uniform, steady solution $u = v = 1$ is unstable if

$$\frac{\beta\gamma}{(1+\alpha)^2} > (\sqrt{\gamma} + \sqrt{\delta})^2,$$

and find the wavenumber at which the system first becomes unstable as χ_0 is increased from zero, in the case $\alpha = \gamma = \delta = 1$.

5. *Turing instability.* Investigate the possibility of Turing instability for the reaction-diffusion system.

$$\frac{\partial u}{\partial t} = \nabla^2 u + \frac{u^2}{v} - bu,$$

$$\frac{\partial v}{\partial t} = d\nabla^2 v + u^2 - v.$$

In particular, find the region of the parameter space (b, d) in which Turing instability can occur, and give a value for the critical wavenumber at the onset of instability.

6. *Phytoplankton-zooplankton.* A space-dependent phytoplankton - zooplankton model can be reduced to the following equations

$$u_t = \nabla^2 u + u + u^2 - \gamma uv$$

$$v_t = d\nabla^2 v + \beta uv - v^2.$$

Find the regions in the $\beta - \gamma$ plane (a) in which there is a stable, homogeneous state (u_0, v_0) in which neither u_0 nor v_0 is zero and (b) in which that state may be unstable to a Turing instability. In case (b), for what values of d will the instability occur, and what is the critical wavenumber for the onset of the instability?