1. $C_t = D(C^p C_x)_x$ $\qquad C = \dfrac{M^{2/(2+p)}}{(Dt)^{1/(2+p)}} F(\xi) \qquad \xi = \dfrac{x}{(M^p D t)^{1/(2+p)}}$

$$\int C(x,t)\,dx = M \qquad C_x(0,t) = 0 \qquad C(x \to \infty, t) \to 0$$

$$C_t = -\frac{1}{(2+p)} \frac{M^{2/(2+p)}}{(Dt)^{1/(2+p)} \cdot t} F(\xi) + \frac{M^{2/(2+p)}}{(Dt)^{1/(2+p)}} F'(\xi) \cdot \left(-\frac{1}{(2+p)} \frac{\xi}{t}\right)$$

$$= \frac{M^{2/(2+p)}}{(Dt)^{1/(2+p)}} \cdot \frac{1}{t} \left\{ -\frac{1}{(2+p)} \xi F' - \frac{1}{(2+p)} F \right\} = \boxed{-\frac{M^{2/(2+p)}}{(Dt)^{1/(2+p)} t (2+p)} (\xi F)'}$$

$$C^p C_x = \frac{M^{2p/(2+p)}}{(Dt)^{p/(2+p)}} F^p \frac{M^{2/(2+p)}}{(Dt)^{1/(2+p)}} \cdot \frac{1}{(M^p D t)^{1/(2+p)}} F' = \boxed{\frac{M}{Dt} F^p F'}$$

$$\rightarrow D(C^p C_x)_x = \cancel{D} \frac{M}{\cancel{D} t} (F^p F')' \cdot \frac{1}{(M^p D t)^{1/(2+p)}}$$

$$\therefore \cancel{\frac{M^{2/(2+p)}}{t (Dt)^{1/(2+p)}}} (F^p F')' = -\frac{\cancel{M^{2/(2+p)}}}{\cancel{(Dt)^{1/(2+p)} t}(2+p)} (\xi F)' \Rightarrow \boxed{(F^p F')' = -\frac{1}{(2+p)} (\xi F)'}$$

hence $F^p F' = -\frac{1}{(2+p)} \xi F + \text{const}$: by the boundary conditions at $\xi = 0$ const $= 0$

$$\Rightarrow \boxed{F^p F' = -\frac{1}{(2+p)} \xi F}$$

note the normalization $\displaystyle \int C\,dx = \frac{M^{2/(2+p)}}{(Dt)^{1/(2+p)}} \int \underbrace{dx\, F(\xi) \frac{(M^p D t)^{1/(2+p)}}{(M^p D t)^{1/(2+p)}}}_{d\xi} = M \int d\xi\, F(\xi)$

so we take F normalized to unity.

solving the last boxed eqn. $F^p F' = -\frac{1}{(2+p)}\xi F \Rightarrow dF\cdot F^{p-1} = -\frac{1}{(2+p)}\xi\, d\xi$

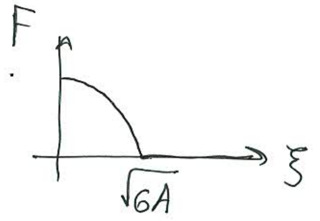
$$\frac{F^p}{p} = -\frac{\xi^2}{2(2+p)} + \text{const} \quad \text{or} \quad \boxed{F(\xi) = \left[A - \frac{p}{2(2+p)}\xi^2\right]^{1/p}}$$

for $0 \le \xi \le \left[\frac{2(2+p)}{p}A\right]^{1/2}$

special case of $p=1$

$$F(\xi) = A - \frac{1}{6}\xi^2 \qquad 0 \le \xi \le \sqrt{6A}$$

$= 0$ outside



normalization

$$\int_0^{\sqrt{6A}} \left(A - \frac{1}{6}\xi^2\right) d\xi = 1 = A\sqrt{6A} - \frac{1}{18}(6A)^{3/2} = A^{3/2}\cdot\frac{2}{3}\sqrt{6} = A^{3/2}\cdot\frac{2\sqrt{2}}{\sqrt{3}}$$

$$\therefore\ A = \left(\frac{\sqrt{3}}{2\sqrt{2}}\right)^{2/3} = \left(\frac{3}{8}\right)^{1/3}$$

2. i) Homogeneous fixed point

$$\left.\begin{array}{l} u(1-u)(u-a)+v=0 \\ bu-cv=0 \end{array}\right\} \quad v=\frac{b}{c}u \qquad \text{so} \quad u=0 \text{ or } (1-u)(u-a)+\frac{b}{c}=0$$

$$u^2-(1+a)u+a-b/c=0$$

$$u^* = \frac{1+a \pm \sqrt{(1+a)^2-4a+4b/c}}{2} = \frac{1+a\pm\sqrt{(1-a)^2+4b/c}}{2}$$

$$v^* = \frac{b}{c}u^*$$

choose + solution so $u^* > 0$

stability matrix

$$J = \begin{pmatrix} ? & 1 \\ b & -c \end{pmatrix}$$

$$J_{11} = (1-u)(u-a)+u(1-u)-u(u-a) \quad \text{at } u^*$$

$$= -3u^2+2u+2ua-a$$

but $u^2 = \frac{b}{c} - a + (1+a)u$

so $J_{11} = -\frac{3b}{c} + 2a - u(1+a)$ after some algebra

now, $u^* > \frac{1+a}{2}$ so $2a-u^*(1+a) \leq 2a - \frac{(1+a)^2}{2}$

$$\leq -\frac{1}{2}(1-a)^2 < 0$$

thus

$$J = \begin{pmatrix} - & + \\ + & - \end{pmatrix} \qquad \left.\begin{array}{l} \operatorname{tr} J < 0 \\ \text{and since } J_{11} < -\frac{b}{c} \quad \text{so } \det > 0 \end{array}\right\} \text{stable fixed point.}$$

ii) $u_t = -\gamma\dot{\phi} \quad u_{xx} = \ddot{\phi} \quad v_t = -\gamma\dot{\psi} \qquad \dot{} = \frac{d}{d\xi} \quad \xi = x-\gamma t$

$$\boxed{\begin{array}{l} -\gamma\dot{\phi} = \ddot{\phi} + \phi(1-\phi)(\phi-a)+\psi \\ -\gamma\dot{\psi} = b\phi - c\psi \end{array}}$$

iii) if $b = c = 0$ $\quad \mathring{\psi} = 0 \Rightarrow \psi = \text{const} = 0$ by statement of problem

$$\Rightarrow -\gamma\mathring{\phi} = \ddot{\phi} + \phi(1-\phi)(\phi-a)$$

set $\phi = \dfrac{1}{1+e^{-\alpha\xi}}$ $\quad \mathring{\phi} = \dfrac{\alpha e^{\cdots}}{(1+e^{\cdots})^2} = \alpha(\phi - \phi^2)$ (useful trick!)

$$\ddot{\phi} = \alpha\mathring{\phi}(1-2\phi)$$

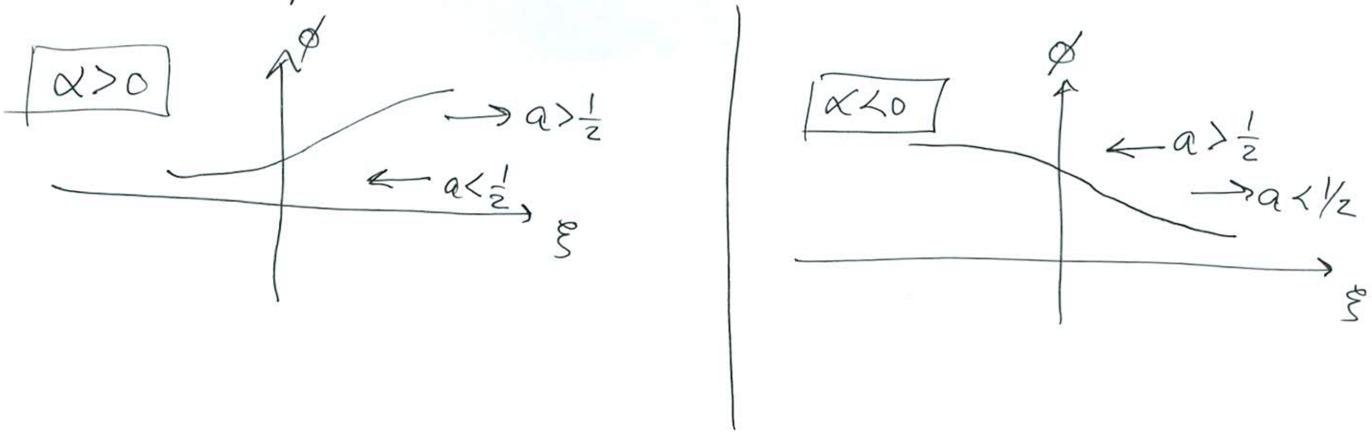
$$\Rightarrow \boxed{-\gamma\mathring{\phi} = \alpha\mathring{\phi}(1-2\phi) + \frac{\mathring{\phi}}{\alpha}(\phi-a)}$$

either $\mathring{\phi} = 0$ or $\phi(1-2\alpha^2) + (\alpha^2 + \gamma\alpha - a) = 0$

holds $\forall\xi$ if $1 - 2\alpha^2 = 0$ $\quad \alpha = \pm\frac{1}{\sqrt{2}}$

and $\alpha^2 + \gamma\alpha - a = 0$ $\quad \gamma = \pm\sqrt{2}(a - \frac{1}{2})$

direction? depends on γ



3. set $u = \phi(\xi) \quad v = \psi(\xi) \quad \xi = x - ct$

$$-c\dot{\phi} = \ddot{\phi} + \phi(1-\phi-r\psi)$$
$$-c\dot{\psi} = d\ddot{\psi} - b\phi\psi$$

$(u,v) \to (0,1) \quad x \to \infty$
$\to (1,0) \quad x \to -\infty$

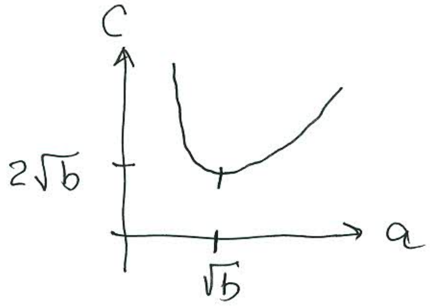
given asymptotic limits let's look at $\boxed{v = 1-u}$

$$u_t = u_{xx} + u(1-u-r(1-u))$$
$$= u_{xx} + (1-r)u(1-u)$$
$$-u_t = -d\,u_{xx} - b\,u(1-u)$$
$$u_t = d\,u_{xx} + bu(1-u)$$
$$d = 1 \Rightarrow \boxed{b = 1-r}$$

to determine wave speeds, we look at the leading edge of front and linearize:

$$u_t \simeq u_{xx} + bu + \dots \qquad ca = b + a^2$$

$$c(a) = \frac{b}{a} + a$$



note: front will move at smallest $c(a)$ with $a < a_{initial}$

can see easily: if we bound by e^{-ax} then stays bounded
can always bound an exponential by a shallower one (smaller a)
front can only be pulled down to shallower

if initial $a > \sqrt{b}$ then heads to $\sqrt{b}$, if $< \sqrt{b}$ stays put

$$c = \begin{cases} a + \frac{b}{a} & a \leq \sqrt{b} \\ 2\sqrt{b} & a \geq \sqrt{b} \end{cases}$$

4. $$n_t = Dn_{xx} + bn\left(1 - \frac{n}{n_0}\right) - \partial_x(\chi(a)\, n a_x)_x$$ (1)

$$a_t = D_A a_{xx} + hn - ka$$

clearly set $u = n/n_0$ let $T = bt$ $\frac{\partial}{\partial t} = b\frac{\partial}{\partial T}$ k^{-1} is a time so set

$a = hn_0 k^{-1} v$

diffusive scalings $x = \sqrt{Db^{-1}}X$

$\partial_x = \frac{1}{\sqrt{\ }}\partial_X$ etc.

algebra
$\Rightarrow$ $\beta = \frac{\chi_0 K k}{D h n_0}$ $\alpha = \frac{Kk}{hn_0}$ $\gamma = k/b$ $\delta = D_A/D$

$$\boxed{\begin{aligned} \dot{u} &= u'' + u(1-u) - \beta\left[\frac{uv'}{(\alpha+v)^2}\right]' \\ \dot{v} &= \delta v'' + \gamma(u-v) \end{aligned}}$$

homogeneous steady state $u = v = 1$ by inspection

stability matrix $J = \begin{pmatrix} -1 & 0 \\ \gamma & -\gamma \end{pmatrix}$ $\left.\begin{matrix} \mathrm{tr} < 0 \\ \det > 0 \end{matrix}\right\}$ hence stable

now allow spatial instability e^{ikx}

$$J = \begin{pmatrix} -1 & 0 \\ \gamma & -\gamma \end{pmatrix} + \begin{pmatrix} -k^2 & 0 \\ 0 & -\delta k^2 \end{pmatrix} + \begin{pmatrix} 0 & \hat{\beta}k^2 \\ 0 & 0 \end{pmatrix} \qquad \hat{\beta} = \frac{\beta}{(1+\alpha)^2}$$

$$= \begin{pmatrix} -(1+k^2) & \hat{\beta}k^2 \\ \gamma & -(\gamma+\delta k^2) \end{pmatrix}$$ tr < 0 always as usual

$$\det = (1+k^2)(\gamma+\delta k^2) - \gamma\hat{\beta}k^2 = \boxed{\underset{\substack{\uparrow \\ \text{"a"} \\ d}}{\delta}(k^2)^2 + \underbrace{(\gamma+\delta-\gamma\hat{\beta})}_{\text{"b"}}k^2 + \underbrace{\gamma}_{\text{"c"}}} \quad *$$

can $(*)$ ever be negative for $k^2 > 0$?

if so, need $\gamma + \delta - \gamma\hat{\beta} < 0$ "b"

and $(\gamma + \delta - \gamma\hat{\beta})^2 - 4\gamma\delta > 0$ "$b^2 - 4ac$"

i.e. $-(\gamma + \delta - \gamma\hat{\beta}) > 2\sqrt{\delta}\sqrt{\gamma}$

or $\gamma\hat{\beta} > \gamma + 2\sqrt{\delta}\sqrt{\gamma} + \delta = (\sqrt{\gamma} + \sqrt{\delta})^2$

$$\boxed{\frac{\gamma\beta}{(1+\alpha)^2} > (\sqrt{\gamma} + \sqrt{\delta})^2}$$ as required $(*)$

$$k_c^2 = "\frac{-b}{2a}" = \frac{-\gamma - \delta + \gamma\hat{\beta}}{2\delta} = \frac{\gamma\beta - (\gamma+\delta)(1+\alpha)^2}{2\delta(1+\alpha)^2}$$

$k_c = \frac{2\pi}{\lambda_c}$ ← wavelength

if $\alpha = \gamma = \delta = 1$

$$k_c^2 = \frac{\beta - 8}{8}$$

$$\lambda_c = 2\pi\sqrt{\frac{8}{\beta - 8}}$$

and note that ~~$*$~~ gives $\underline{\beta > 16}$

5. $u_t = u_{xx} + \frac{u^2}{v} - bu$

$v_t = d\, v_{xx} + u^2 - v$

homogeneous fixed point

$\frac{u^2}{v} - bu = 0 \qquad u^2 - v = 0$

$\Rightarrow \boxed{u^* = b^{-1} \quad v^* = b^{-2}}$

$J\Big|_{u^*,v^*} = \begin{pmatrix} b & -b^2 \\ 2b^{-1} & -1 \end{pmatrix}$

$\det = b$

$\mathrm{tr} = b-1$

so $\boxed{\begin{matrix} 0<b<1 \\ \text{stable} \end{matrix}}$

Spatial instability $\sim e^{ikx}$

$J = \begin{pmatrix} b-k^2 & -b^2 \\ 2b^{-1} & -1-dk^2 \end{pmatrix}$

$\mathrm{tr} < 0$ of course

$\det = (b-k^2)(-1-dk^2) + 2b$

$= d\,(k^2)^2 + (1-bd)k^2 + b$

can this be negative for any $k^2 > 0$?

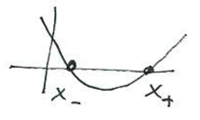
need $1-bd < 0$ i.e. $bd > 1$

also $(1-bd)^2 > 4bd$

so $\boxed{bd > 3 + \sqrt{8}}$

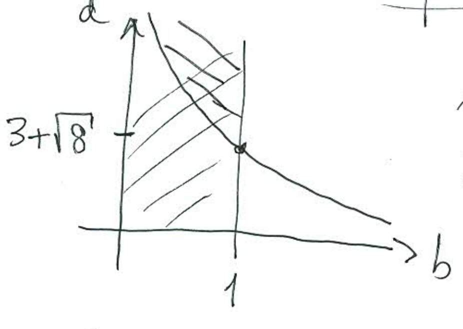
$1 - 2x + x^2 > 4x$

$x^2 - 6x + 1 > 0$



$x_\pm = 3 \pm \sqrt{8}$

x_+ relevant



/// = homog. stability

≡ Turing instability.

critical wavenumber

$k_c^2 = \frac{-(1-bd)}{2d} = \frac{1+\sqrt{2}}{d}$

$\lambda_c = 2\pi \sqrt{\frac{d}{1+\sqrt{2}}}$

6. $u_t = u_{xx} + u(1+u-\gamma v)$

$v_t = d\,v_{xx} + v(\beta u - v)$

want $u, v > 0$ only

fixed point: $u_0 = \dfrac{1}{\beta\gamma - 1}$ $\quad v_0 = \beta u_0 = \dfrac{\beta}{\beta\gamma - 1}$ $\qquad$ clearly $\beta\gamma > 1$

stability $J = u_0 \begin{pmatrix} 1 & -\gamma \\ \beta^2 & -\beta \end{pmatrix}$ $\qquad \det = u_0^2 \beta(\gamma\beta - 1) > 0$ ✓

$\mathrm{tr} = u_0(1-\beta) < 0 \iff \boxed{\beta > 1}$

again, with perturbations $\sim e^{ikx}$

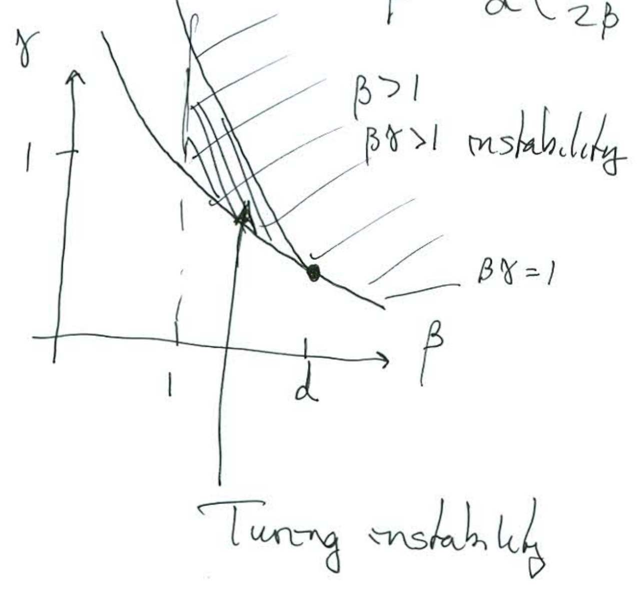
$$\boxed{J = u_0\begin{pmatrix} 1 & -\gamma \\ \beta^2 & -\beta \end{pmatrix} - k^2 \begin{pmatrix} 1 & 0 \\ 0 & d \end{pmatrix}}$$

so $\mathrm{tr} < 0$ still and $\det = d(k^2)^2 + (\beta - d)u_0 k^2 + \beta(\gamma\beta - 1)u_0^2$

for Turing instability, need this < 0 for some $k^2 > 0$

i.e. need $\boxed{d > \beta}$ and $\boxed{(\beta - d)^2 > 4\beta d(\gamma\beta - 1)}$ $(*)$

i.e. $\gamma < \dfrac{1}{\beta} + \dfrac{1}{d}\left(\dfrac{\beta - d}{2\beta}\right)^2 = \dfrac{1}{d}\left(\dfrac{\beta + d}{2\beta}\right)^2$ and this $> \dfrac{1}{\beta}$



$k_c^2 = \dfrac{d - \beta}{2d} = \sqrt{\dfrac{\beta(\gamma\beta - 1)}{d}}$

but $(*) \Rightarrow \beta < \dfrac{d}{2\sqrt{\gamma d} - 1}$

so $k_c^2 = \dfrac{\sqrt{\gamma d} - 1}{2\sqrt{\gamma d} - 1}$