Nonlinear diffusion

$$C_t = D\left(C^p C_x\right)_x \qquad C = \frac{M^{2/(2+p)}}{(Dt)^{1/(2+p)}} F(\xi) \qquad \xi = \frac{x}{(M^p D t)^{1/(2+p)}}$$

$$\int C(x,t)\,dx = M \qquad C_x(0,t) = 0 \qquad C(x\to\infty, t) \to 0$$

$$C_t = -\frac{1}{(2+p)} \frac{M^{2/(2+p)}}{(Dt)^{1/(2+p)}\cdot t} F(\xi) + \frac{M^{2/(2+p)}}{(Dt)^{1/(2+p)}} F'(\xi)\cdot\left(-\frac{1}{(2+p)}\frac{\xi}{t}\right)$$

$$= \frac{M^{2/(2+p)}}{(Dt)^{1/(2+p)}} \cdot \frac{1}{t}\left\{-\frac{1}{(2+p)}\xi F' - \frac{1}{(2+p)} F\right\} = \boxed{\frac{-M^{2/(2+p)}}{(Dt)^{1/(2+p)}\,t(2+p)}(\xi F)'}$$

$$C^p C_x = \frac{M^{2p/(2+p)}}{(Dt)^{p/(2+p)}} F^p \,\frac{M^{2/(2+p)}}{(Dt)^{1/(2+p)}} \cdot \frac{1}{(M^p D t)^{1/(2+p)}} = \boxed{\frac{M}{Dt} F^p F'}$$

$$\to D(C^p C_x)_x = D\frac{M}{Dt}(F^pF')' \cdot \frac{1}{(M^pDt)^{1/(2+p)}}$$

$$\therefore \frac{M^{2/(2+p)}}{t(Dt)^{1/(2+p)}}(F^pF')' = -\frac{M^{2/(2+p)}}{(Dt)^{1/(2+p)}\,t(2+p)}(\xi F)' \Rightarrow \boxed{(F^pF')' = -\frac{1}{(2+p)}(\xi F)'}$$

hence $F^pF' = -\frac{1}{(2+p)}\xi F + \text{const}$: by the boundary conditions at $\xi = 0$, const $= 0$

$$\Rightarrow \boxed{F^pF' = -\frac{1}{(2+p)}\xi F}$$

note the normalisation

$$\int C\,dx = \frac{M^{2/(2+p)}}{(Dt)^{1/(2+p)}} \int \underbrace{dx\, F(\xi)\,\frac{(M^pDt)^{1/(2+p)}}{(M^pDt)^{1/(2+p)}}}_{d\xi} = M\int d\xi\, F(\xi)$$

so we take F normalised to unity

solving the last boxed eqn. $F^p F' = -\frac{1}{(2+p)}\xi F \Rightarrow dF\,F^{p-1} = -\frac{1}{(2+p)}\xi\, d\xi$

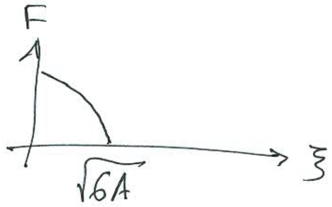
$$\frac{F^p}{p} = -\frac{\xi^2}{2(2+p)} + \text{const} \quad \text{or} \quad \boxed{F(\xi) = \left[A - \frac{p}{2(2+p)}\xi^2\right]^{1/p}}$$

for $0 \leq \xi \leq \left[\frac{2(2+p)}{p}A\right]^{1/p}$

special case of $p=1$

$$F(\xi) = A - \frac{1}{6}\xi^2 \qquad 0 \leq \xi \leq \sqrt{6A}$$

$$= 0 \quad \text{outside}$$



normalisation

$$\int_0^{\sqrt{6A}} \left(A - \frac{1}{6}\xi^2\right) d\xi = 1 = A\sqrt{6A} - \frac{1}{18}(6A)^{3/2} = A^{3/2}\cdot\frac{2}{3}\sqrt{6} = A^{3/2}\cdot\frac{2\sqrt{2}}{\sqrt{3}}$$

$$\therefore A = \left(\frac{\sqrt{3}}{2\sqrt{2}}\right)^{2/3} = \left(\frac{3}{8}\right)^{1/3}$$

FitzHugh - Nagumo model

i) homogeneous fixed point

$$\left.\begin{aligned} u(1-u)(u-a)+v&=0 \\ bu-cv&=0 \end{aligned}\right\} \quad v=\frac{b}{c}u \qquad \text{so } u=0 \text{ or } (1-u)(u-a)+\frac{b}{c}=0$$

$$u^2-(1+a)u+a-b/c=0$$

$$u^* = \frac{1+a \pm \sqrt{(1+a)^2-4a+4b/c}}{2}$$

$$= \frac{1+a \pm \sqrt{(1-a)^2+4b/c}}{2}$$

$$v^* = \frac{b}{c}u^* \quad : \text{choose } + \text{ sol'n so } u^*>0.$$

stability matrix

$$J = \begin{pmatrix} ? & 1 \\ b & -c \end{pmatrix} \qquad J_{11} = (1-u)(u-a)+u(1-u)-u(u-a) \quad @\, u^*$$

$$= -3u^2+2u+2ua-a$$

$$\text{but } u^2 = \frac{b}{c}-a+(1+a)u$$

$$\text{so } J_{11} = -\frac{3b}{c}+2a-u(1+a) \quad \text{after some algebra}$$

$$\text{now, } u^* > \frac{1+a}{2} \quad \text{so } 2a-u^*(1+a) \leq 2a-\frac{(1+a)^2}{2}$$

$$\leq -\frac{1}{2}(1-a)^2 < 0$$

thus $J = \begin{pmatrix} - & + \\ + & - \end{pmatrix}$ $\quad \mathrm{Tr}\, J < 0$

and since $J_{11} < -\frac{b}{c}$ $\quad$ $\left.\mathrm{Det} > 0\right\}$ stable fixed pt.

(ii) $u_t = -\gamma\dot{\phi}$ $\quad u_{xx} = \ddot{\phi}$ $\quad v_t = -\gamma\dot{\psi}$ $\quad \cdot = \frac{d}{d\xi}$ $\quad \xi = x-\gamma t$

$$\boxed{\begin{aligned} -\gamma\dot{\phi} &= \ddot{\phi}+\phi(1-\phi)(\phi-a)+\psi \\ -\gamma\dot{\psi} &= b\phi-c\psi \end{aligned}}$$

iii) if $b = c = 0$ $\quad \dot{\psi} = 0 \Rightarrow \psi = \text{const} = 0$ by statement of problem

$$\Rightarrow -\gamma \dot{\phi} = \ddot{\phi} + \phi(1-\phi)(\phi - a)$$

set $\phi = \dfrac{1}{1+e^{-\alpha\xi}}$ $\quad \dot{\phi} = \dfrac{\alpha e^{\cdots}}{(1+e^{\cdots})^2} = \alpha(\phi - \phi^2)$ $\quad$ (useful trick!)

$$\therefore \ddot{\phi} = \alpha \dot{\phi}(1-2\phi) \Rightarrow \boxed{-\gamma \dot{\phi} = \alpha \dot{\phi}(1-2\phi) + \frac{\dot{\phi}}{\alpha}(\phi - a)}$$

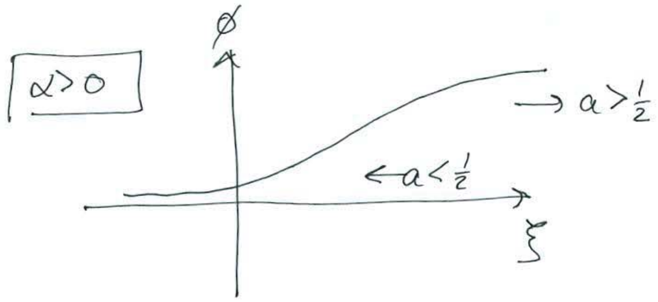
either $\dot{\phi} = 0$ or

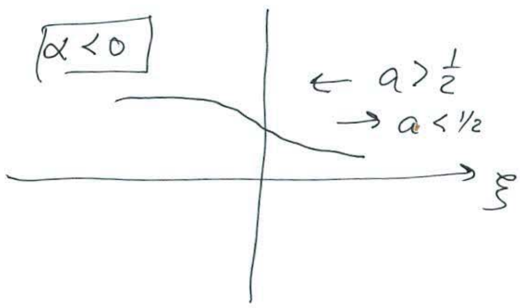
$$\phi(1-2\alpha^2) + (\alpha^2 + \gamma\alpha - a) = 0$$

holds $\forall \xi$ if $1 - 2\alpha^2 = 0 \quad \alpha = \pm\frac{1}{\sqrt{2}}$

and $\alpha^2 + \gamma\alpha - a = 0 \quad \gamma = \pm\sqrt{2}\left(a - \frac{1}{2}\right)$

direction?, depends on γ





Chemotaxis

$$n_t = D n_{xx} + bn\left(1-\frac{n}{n_0}\right) - \partial_x\left(\chi(a)\, n\, a_x\right)_x$$

$$a_t = D_A a_{xx} + hn - ka \qquad \chi(a) = \frac{\chi_0 K}{(K+a)^2}$$

clearly, set $u = n/n_0$, let $T = bt$ $\frac{\partial}{\partial t} = b\frac{\partial}{\partial T}$ k^{-1} is a time, so set $a = hn_0 k^{-1} v$

diffusive scalings: $x = \sqrt{Db^{-1}}\, X$

$\partial_x = \frac{1}{\sqrt{\ }}\partial_X$ etc.

algebra $\longrightarrow$ $\beta = \frac{\chi_0 Kk}{Dhn_0}$ $\alpha = \frac{Kk}{hn_0}$ $\gamma = k/b$ $\delta = D_A/D$

$$\boxed{\begin{aligned} \mathring{u} &= u'' + u(1-u) - \beta\left[\frac{uv'}{(\alpha+v)^2}\right]' \\ \mathring{v} &= \delta v'' + \gamma(u-v) \end{aligned}}$$

homogeneous steady state $u = v = 1$ by inspection

stability matrix $J = \begin{pmatrix} -1 & 0 \\ \gamma & -\gamma \end{pmatrix}$ $\left.\begin{matrix} Tr < 0 \\ Det > 0 \end{matrix}\right\}$ hence stable

---

now allow spatial instability e^{ikx}

$$J = \begin{pmatrix} -1 & 0 \\ \gamma & -\gamma \end{pmatrix} + \begin{pmatrix} -k^2 & 0 \\ 0 & -\delta k^2 \end{pmatrix} + \begin{pmatrix} 0 & \hat{\beta}k^2 \\ 0 & 0 \end{pmatrix} \qquad \hat{\beta} = \frac{\beta}{(1+\alpha)^2}$$

$$= \begin{pmatrix} -(1+k^2) & \hat{\beta}k^2 \\ \gamma & -(\gamma+\delta k^2) \end{pmatrix} \qquad Tr < 0 \text{ always as usual}$$

$$Det = (1+k^2)(\gamma+\delta k^2) - \gamma\hat{\beta}k^2 = \boxed{\underbrace{\delta}_{\text{"a"}}(k^2)^2 + \underbrace{(\gamma+\delta-\gamma\hat{\beta})}_{\text{"b"}}k^2 + \underbrace{\gamma}_{\text{"c"}}} \qquad (*)$$

can * ever be negative for $k^2>0$?

if so, need $\gamma+\delta-\gamma\hat{\beta}<0$ "b"

and $(\gamma+\delta-\gamma\hat{\beta})^2-4\gamma\delta>0$ "b^2-4ac"

i.e. $-(\gamma+\delta-\gamma\hat{\beta})>2\sqrt{\delta}\sqrt{\gamma}$

or $\gamma\hat{\beta}>\gamma+2\sqrt{\delta}\sqrt{\gamma}+\delta=(\sqrt{\gamma}+\sqrt{\delta})^2$

$$\boxed{\frac{\gamma\beta}{(1+\alpha)^2}>(\sqrt{\gamma}+\sqrt{\delta})^2}$$ as required (*)

$$k_c^2=\frac{"-b"}{2a}=\frac{-\gamma-\delta+\gamma\hat{\beta}}{2\delta}=\frac{\gamma\beta-(\gamma+\delta)(1+\alpha)^2}{2\delta(1+\alpha)^2}$$

$k_c=\frac{2\pi}{\lambda_c}$ ← wavelength

if $\alpha=\gamma=\delta=1$

$$k_c^2=\frac{\beta-8}{8}$$

$$\lambda_c=2\pi\sqrt{\frac{8}{\beta-8}}$$

and note that * gives $\beta>16$

A Turing instability

$$u_t = u_{xx} + \frac{u^2}{v} - bu$$

$$v_t = d v_{xx} + u^2 - v$$

homogeneous fixed point.

$$\frac{u^2}{v} - bu = 0 \qquad u^2 - v = 0$$

$$\Rightarrow \boxed{u^* = b^{-1} \quad v^* = b^{-2}}$$

$$J\big|_{u^*,v^*} = \begin{pmatrix} b & -b^2 \\ 2/b & -1 \end{pmatrix} \qquad \begin{aligned} \mathrm{Det} &= b \\ \mathrm{Tr} &= b-1 \end{aligned} \qquad \text{so} \quad \boxed{\begin{matrix} 0<b<1 \\ \text{stable} \end{matrix}}$$

spatial instability $\sim e^{ikx}$

$$J = \begin{pmatrix} b-k^2 & -b^2 \\ 2/b & -1-dk^2 \end{pmatrix}$$

$\mathrm{Tr} < 0$ of course

$$\begin{aligned} \mathrm{Det} &= (b-k^2)(-1-dk^2) + 2b \\ &= d(k^2)^2 + (1-bd)\,k^2 + b \end{aligned}$$

can this be negative for any $k^2 > 0$?

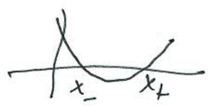
need $1-bd < 0$ i.e. $bd > 1$

also $(1-bd)^2 > 4bd$

$1 - 2x + x^2 > 4x$

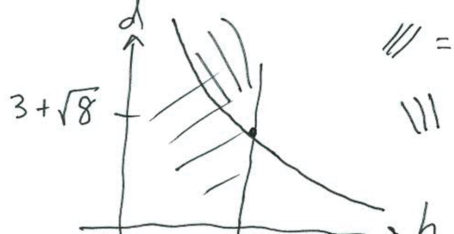
$x^2 - 6x + 1 > 0$

so $\boxed{bd > 3 + \sqrt{8}}$



$x_\pm = 3 \pm \sqrt{8}$

x_+ relevant



/// = homg. stability

\\\ Turing instability

critical wavenumber $\quad k_c^2 = \frac{-(1-bd)}{2d} = \frac{1+\sqrt{2}}{d}$

$$\lambda_c = 2\pi\sqrt{\frac{d}{1+\sqrt{2}}}$$

Phytoplankton – zooplankton

$$u_t = u_{xx} + u(1+u-\gamma v)$$
$$v_t = d\,v_{xx} + v(\beta u - v)$$

want $u, v > 0$ only

fixed point: $u_0 = \frac{1}{\beta\gamma - 1}$ $\quad v_0 = \beta u_0 = \frac{\beta}{\beta\gamma - 1}$ $\quad$ clearly $\beta\gamma > 1$

stability

$$J = u_0 \begin{pmatrix} 1 & -\gamma \\ \beta^2 & -\beta \end{pmatrix}$$

$\mathrm{Det} = u_0^2 \beta(\gamma\beta - 1) > 0$

$\mathrm{Tr} = u_0(1-\beta) < 0 \Rightarrow \boxed{\beta > 1}$

again, with perturbations $\sim e^{ikx}$

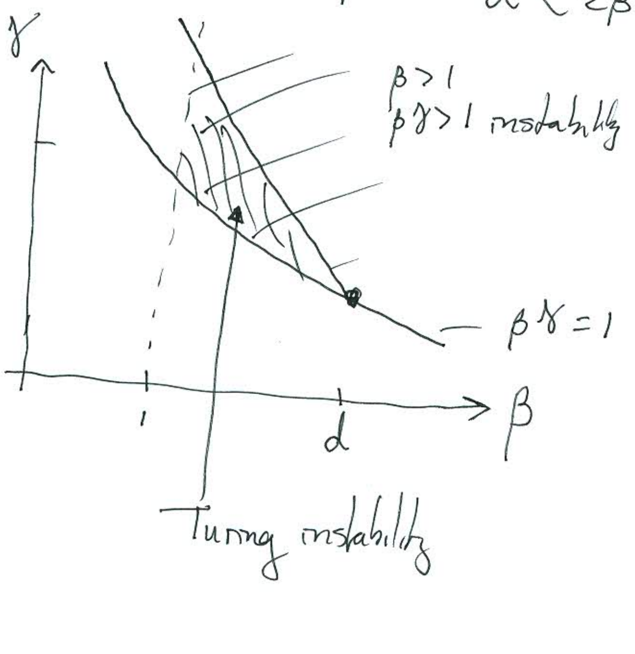
$$\boxed{J = u_0 \begin{pmatrix} 1 & -\gamma \\ \beta^2 & -\beta \end{pmatrix} - k^2 \begin{pmatrix} 1 & 0 \\ 0 & d \end{pmatrix}}$$

so $\mathrm{Tr} < 0$ still and $\mathrm{Det} = d(k^2)^2 + (\beta - d) u_0 k^2 + \beta(\gamma\beta - 1) u_0^2$

for Turing instability, need this < 0 for some $k^2 > 0$

i.e. need $\boxed{d > \beta}$ and $\boxed{(\beta - d)^2 > 4\beta d(\gamma\beta - 1)}$ (*)

i.e. $\gamma < \frac{1}{\beta} + \frac{1}{d}\left(\frac{\beta - d}{2\beta}\right)^2 = \frac{1}{d}\left(\frac{\beta + d}{2\beta}\right)^2$ and this $> \frac{1}{\beta}$



$$k_c^2 = \frac{d - \beta}{2d} = \sqrt{\frac{\beta(\gamma\beta - 1)}{d}}$$

but (*) $\rightarrow \beta < \frac{d}{2\sqrt{\gamma d} - 1}$

so $k_c^2 = \frac{\sqrt{\gamma d} - 1}{2\sqrt{\gamma d} - 1}$