Part III: Soft Matter (Michaelmas, 2007) R E Goldstein (11/11/07)

Examples Sheet #2

1. *Radius of Gyration and Structure Factor for a Gaussian polymer chain.* The radius of gyration R_g of a polymer is defined as

$$R_g^2 = \frac{1}{2N^2}\sum_{n=1}^{N}\langle(\mathbf{R}_n - \mathbf{R}_{\rm com})^2\rangle \ ,$$

where N is the number of links and $\mathbf{R}_{\rm com}$ is the center of mass of the chain. For a Gaussian chain with step length b, find R_g and the scattering function

$$g(\mathbf{k}) = \frac{1}{N}\sum_{m,n}^{N}\langle\exp\left[i\mathbf{k}\cdot(\mathbf{R}_n - \mathbf{R}_m)\right]\rangle \ .$$

Express your result in a scaling form.

2. *Differential geometry of space curves.* Consider a space curve $\mathbf{r}(s)$ in three dimensions, with velocity

$$\mathbf{r}_t = U\hat{\mathbf{n}} + V\hat{\mathbf{b}} + W\hat{\mathbf{t}} \ .$$

(a) Find the time rate of change of the metric g, and length L in terms of U, V, and W.
(b) What is the condition for local arclength conservation?
(c) Derive the time evolution of the curvature $\kappa(s)$ and of the torsion $\tau(s)$.
(d) Find the equation of motion for the Hasimoto/Darboux function

$$\psi = \kappa \exp\left(i\int_s ds'\tau(s')\right)$$

3. *Integrable curve motions – I.* Suppose a plane curve has normal velocity $U = \kappa_s$.
(a) What is the equation of motion for the curvature under the conditions of local arclength conservation?
(b) Express the pde in (a) in a local conservation form. In what other context does this pde arise?
(c) Find as many conserved quantities as you can.
(d) Find a solitonic solution of the pde in (a).

4. *Integrable curve motions – II.* Repeat problem 3 for a space curve with velocity $V = \kappa$, using the Hasimoto/Darboux function.

5. *Electrostatic contributions to elastic energy of surfaces.*
(a) Using Debye-Hückel theory, calculate the electrostatic potential outside, and electrostatic energy of flat, cylindrical, and spherical surfaces with either fixed charge density σ_0 or fixed potential ϕ_0.
(b) By comparing these energies from (a) in the regime that the screening length $\lambda_{DH} \ll R$, where R is the cylinder or sphere radius, deduce the electrostatic contribution to the elastic

modulus k, Gaussian curvature modulus k_G, and spontaneous curvature H_0 in the elastic energy

$$\mathcal{E} = \frac{1}{2}k \int dS \left(H - H_0\right)^2 + \frac{1}{2}k_G \int dSK \ ,$$

where $H = 1/R_1 + 1/R_2$ is the mean curvature (with R_1 and R_2 the principal radii of curvature), and $K = 1/R_1R_2$ is the Gaussian curvature.
(c) Estimate k, k_G, and the spontaneous radius $R_0 = 1/H_0$ for typical values of σ_0 and ϕ_0.

6. *Debye-Hückel theory near a rippled surface.* Membranes and surfaces in physical and biological systems are often not flat and smooth, but may exhibit undulations of various periodicities. Suppose a two-dimensional surface deviates from flatness by a one-dimensional modulation $h(x)$.
(a) If the surface has a constant surface charge density σ_0, find within Debye-Hückel theory the electrostatic potential away from the surface to second order in the displacement h and its derivatives.
(b) Using the results in (a), find the electrostatic potential to second order in h. Compare with the results in Problem 2.

7. *The Poisson-Boltzmann equation in one dimension.*
(a) Find the electrostatic potential that satisfies the Poisson-Boltzmann equation for a 1 : 1 electrolyte with mean concentration c,

$$\nabla^2\phi - \frac{8\pi ce}{\epsilon}\sinh\left(\beta e\phi\right) = 0 \ ,$$

away from a surface with fixed potential ϕ_0.
(b) Find the relationship between the surface charge density and the surface potential, and thus calculate the electrostatic free energy of that surface.
(c) Generalize (a) and (b) to the case of two parallel surfaces, and thus find the energy of interaction as a function of separation. Compare with the weak-field limit.

8. *Correlations in a Smectic Liquid Crystal.* Complete the derivation in class of the correlation function

$$G_n(\mathbf{r}) \propto \langle \exp\left[iq_0\left(u(\mathbf{r}) - u(\mathbf{0})\right]\right\rangle$$

for a smectic liquid crystal, where the energy functional has the form

$$\mathcal{E}[u] = \frac{1}{2}\int d^3r \left\{B\left(\nabla_\parallel u\right)^2 + K\left(\nabla_\perp^2 u\right)^2\right\} \ .$$