Examples Sheet #3

1. *Euler buckling.* Analyze the buckling of an elastic rod of modulus A under thrusting forces f at each end. Assume (a) clamped and (b) hinged boundary conditions. Give an analytical treatment near the primary bifurcation, and a complete numerical treatment beyond. Just beyond the bifurcation the bent rod behaves like a spring. Find the spring constant.

2. *Boundary conditions on elastic filaments.* By a careful application of functional differentiation, find the "natural" boundary conditions on the curvature of a planar elastica, and on the curvature and torsion of an elastic filament in three dimensions. For a weakly curved filament in the plane, using the Monge representation of the shape, which choices of boundary conditions yield a self-adjoint operator ∂_{4x}?

3. *Waves on an elastic filament.* Consider an elastic filament with bending modulus A in a viscous fluid. For small displacements $y(x,t)$ from a straight configuration the simplest equation of motion is $\zeta y_t = -A y_{xxxx}$, where ζ is a drag coefficient. Imagine that the left end of the filament is constrained to move as $y(x=0,t) = a\cos(\omega t)$. With appropriate boundary conditions at the free right end, find the shape $y(x,t)$ for a semi-infinite filament. Explain the meaning of a characteristic length you find. Try to find the solution for very short filaments. (This is a difficult problem!)

4. *Twisting instability.* As in problem 1, but for a rod which also has a twist elastic constant C and imposed twist density Ω.

5. *Langevin equation with inertia.* In class we considered the overdamped Langevin equation. Now we'll work with the full randomly-forced Newton's law dynamics. Let $\mathbf{u}$ be the velocity of a particle of mass m and friction constant ζ. The Langevin equation is

$$m\frac{d\mathbf{u}}{dt} = -\zeta\mathbf{u} + \mathbf{f} \ ,$$

where $\mathbf{f}(t)$ is the random force.
(a) If $\mathbf{u}_0$ is the velocity at time $t=0$, find the average velocity $\langle \mathbf{u} \rangle$ at later times.
(b) Find $\langle \mathbf{u}^2 \rangle$, using equipartition to fix the amplitude of the noise term, assuming as usual that it is delta function correlated in time.
(c) It is possible (give it a try) to show that the moments of $\mathbf{U} \equiv \mathbf{u} - \mathbf{u}_0$ are those of a Gaussian distribution. You'll need to make some assumptions about the higher-order correlations of $\mathbf{f}$. Show that the distribution of velocity becomes a Maxwell-Boltzmann distribution at long times.
(d) Find the statistical properties (mean and variance) of the displacement $\mathbf{r} - \mathbf{r}_0$. Comment on the time-dependence of the short-time and long-time limits of the variance.