Heat and Mass Transfer from Single Spheres in Stokes Flow

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(Received November 27, 1961)

The classical problem of heat and mass transfer from single spheres at low values of the Reynolds number, where the velocity field is given by Stokes' formula, is considered. It is shown, by the use of a singular perturbation technique, how an expansion may be derived for the Nusselt number Nu in terms of the Péclet number Pe which yields an accurate expression for the rate of transfer of energy or matter in the range $0 \leqq \mathrm{Pe} \leqq 1$. It is also established, by studying both the $\mathrm{Pe} \to 0$ and $\mathrm{Pe} \to \infty$ asymptotes, that the functional relation between Nu and Pe as obtained with the Stokes velocity profile is less sensitive to an increase in the Reynolds number than the familiar Stokes law for the drag coefficient.

I. INTRODUCTION

NUMEROUS theoretical investigations have been reported in recent years centering around the classical problem of heat and mass transfer from a solid sphere into a low-Reynolds-number velocity field. And yet, a careful study of the rather extensive literature on the subject will quickly reveal that there still exist a good many disagreements among the various results which have been published so far, and that the precise dependence of the rate of transfer of energy or mass on the principal parameters of the physical system is in this particular case, at least, not completely understood. The purpose of this paper will be therefore to settle some of the existing controversies on this topic which, in view of its practical significance in the important field of small-particle dynamics and low-Reynolds-number flows, is currently attracting an increasing amount of attention.

The main part of our discussion will deal with the solution of the well-known convection equation under conditions where the velocity field around the sphere may be described by Stokes' classical formula. It may then easily be shown, and will shortly become apparent from an inspection of the basic equations themselves, that if such a state of motion is postulated, the average Nusselt number Nu, defined in the usual manner for either mass or heat transfer,[1] will become a function of the Péclet number alone, where the Péclet number Pe is defined as being equal to the product Re Pr or Re Sc, depending on whether heat or mass exchange is being considered. Thus, even though the better part of our analysis will be restricted to systems for which the Reynolds number $\mathrm{Re} < O(1)$, it will also include cases with $\mathrm{Pe} \gg 1$ since for many liquids either Pr or Sc can indeed be very large.

The theoretical determination of the exact functional relation between Nu and Pe is naturally the main point of interest in problems of this general type. According to Kronig and Bruijsten[2] it is possible, for low values of the Péclet number, to expand Nu in the form

$$\mathrm{Nu} = 2 + \tfrac{1}{2}\,\mathrm{Pe} + (581/1920)\,\mathrm{Pe}^2 + \cdots, \qquad (1)$$

where the first term represents the contribution from pure conduction or diffusion in the absence of any convective effects. Breiman[3] on the other hand concluded that, again for small Pe,

$$\mathrm{Nu} = 2 + \tfrac{1}{2}\,\mathrm{Pe} - \tfrac{1}{4}\,\mathrm{Pe}^2 \ln \mathrm{Pe} - 0.0334\,\mathrm{Pe}^2 + \cdots, \qquad (2)$$

which is in disagreement with Eq. (1). This is indeed surprising, since the two investigations were concerned with exactly the same mathematical model and differed only in the method of solution. In other words, whereas Eq. (1) was obtained from a more or less conventional perturbation expansion of the convection equation, Breiman's procedure consisted of using a Green's function technique to transform the problem into an integral equation,[4] which was then solved by iteration. A classical perturbation solution was also attempted by Frisch[5] with, however, an incorrect velocity distribution, so that the expression arrived at by him,

$$\mathrm{Nu} = 2 + O(\mathrm{Pe}^2), \qquad (3)$$

must therefore be discarded.

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[1] E. R. G. Eckert and R. M. Drake, *Heat and Mass Transfer* (McGraw-Hill Book Company, Inc., New York, 1959).

[2] R. Kronig and J. Bruijsten, Appl. Sci. Research **A2**, 439 (1951).

[3] L. Breiman, Norman Bridge Laboratory, California Institute of Technology, Rept. No. 2F-2 (1952).

[4] B. Friedman, *Principles and Techniques of Applied Mathematics* (John Wiley & Sons, Inc., New York, 1956).

[5] H. L. Frisch, J. Chem. Phys. **22**, 123 (1954).

In all the studies mentioned above, only those systems were considered for which the convective effects were relatively minor in comparison with pure conduction or diffusion. At the other extreme, however, are the cases with $\mathrm{Pe} \gg 1$ for which the effects of molecular conduction or diffusion may be neglected everywhere, except for a thin boundary-layer-type region near the fluid-solid interface, where the main part of the temperature or concentration drop will take place. Thus, it is possible to resort to an asymptotic solution almost identical to the well-known Lighthill[6] formula for heat transfer in laminar boundary-layer flows, which reduces[7,8] to

$$\mathrm{Nu} \to 0.991\ \mathrm{Pe}^{\frac{1}{3}} \quad \text{as} \quad \mathrm{Pe} \to \infty. \tag{4}$$

It is clear then that, although the exact functional dependence of Nu on Pe has already been established for the two limiting cases $\mathrm{Pe} = 0$ and $\mathrm{Pe} \to \infty$, the behavior of the function for intermediate Pe values is at present largely unknown, since the two mathematically approximate solutions[9,10] which have recently been proposed for the complete range of Péclet numbers cannot be accepted *a priori* with total confidence.

The present article has therefore a twofold purpose. First, to re-examine the expansion of Nu in terms of the Péclet number for small Pe, and by adding more terms to this series than has up to now been found possible, to shed some light on the form of the complete solution for the classical problem of heat and mass transfer. And secondly, to extend the analysis into the region of somewhat larger Reynolds numbers in order to ascertain to what extent the phenomenon under consideration may be affected by deviations in the velocity profile from the Stokesian creeping-flow formula.

II. BASIC EQUATIONS AND THE METHOD OF SOLUTION

The basic equation for the transport of energy or mass is of course well known.[11,12] In dimensionless form, and for an axisymmetric spherical system,

$$\nabla_r^2 h = \epsilon\left(U_r \frac{\partial h}{\partial r} + \frac{U_\theta}{r}\frac{\partial h}{\partial \theta}\right), \tag{5}$$

with boundary conditions

$$h = 1 \quad \text{at} \quad r = 1; \qquad h = 0 \quad \text{for} \quad r \to \infty,$$

where h is a normalized temperature or concentration, U_r and U_θ are, respectively, the velocity components in the r and θ directions divided by the free-stream velocity U_∞, and r is the distance from the center of the sphere divided by the radius a. Also, ∇^2 is the familiar operator

$$\nabla_r^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial}{\partial \mu}\left[(1-\mu^2)\frac{\partial}{\partial \mu}\right],$$

$$\mu \equiv \cos\theta, \quad \text{and} \quad \epsilon \equiv \tfrac{1}{2}\mathrm{Pe},$$

where

$$\mathrm{Pe} = 2aU_\infty \rho c_p/k \qquad \text{for heat transfer,}$$

whereas

$$\mathrm{Pe} = 2aU_\infty/D \qquad \text{for mass exchange.}$$

(c_p is the specific heat per unit mass, ρ the mass density, k the thermal conductivity, and D the molecular diffusion coefficient.) And finally, in the so-called Stokesian flow region

$$U_r = \left(1 - \frac{3}{2r} + \frac{1}{2r^3}\right)\mu,$$
$$U_\theta = -\left(1 - \frac{3}{4r} - \frac{1}{4r^3}\right)(1-\mu^2)^{\frac{1}{2}}, \tag{6}$$

which are known to provide an adequate approximation to the exact velocity field if $\mathrm{Re} < O(1)$.[13]

One might feel inclined now to attempt a classical perturbation solution of Eq. (5) by expanding h in the form

$$h = \sum_{n=0}^{\infty} \epsilon^n h_n, \tag{7}$$

and determining the functions h_n from the recursion formula

$$\nabla_r^2 h_n = U_r \frac{\partial h_{n-1}}{\partial r} + \frac{U_\theta}{r}\frac{\partial h_{n-1}}{\partial \theta}. \tag{8}$$

This was the approach originally taken by Kronig and Bruijsten[2] who, however, quickly realized that although $h_0 = 1/r$, the remaining functions h_n could not be made to vanish at infinity, and that for $n \geq 2$, they even diverged. The reason for the

[6] M. J. Lighthill, Proc. Roy. Soc. (London) **A202**, 369 (1950).
[7] V. G. Levich, *Physicochemical Hydrodynamics* (Moscow, 1959).
[8] S. K. Friedlander, J. Am. Inst. Chem. Engrs. **7**, 347 (1961).
[9] S. K. Friedlander, J. Am. Inst. Chem. Engrs. **3**, 43 (1957).
[10] T. Yuge, Repts. Inst. High Speed Mech. of Tôhoku Univ. **6**, 143 (1956).
[11] S. Goldstein, *Modern Developments in Fluid Mechanics* (Oxford University Press, Oxford, 1938).
[12] H. Schlichting, *Boundary Layer Theory* (McGraw-Hill Book Company, Inc., New York, 1960), 4th ed.

[13] We shall neglect here the effect of a mass-transfer induced finite interfacial velocity which may, under certain well-defined conditions, rather significantly affect the rate of mass exchange[1] [A. Acrivos, J. Am. Inst. Chem. Engrs. **6**, 410 (1960)].

failure of the classical expansion can easily be explained by the fact that, no matter how small ϵ is, there is always a region far away from the surface of the sphere where both the conduction and the convection terms become of the same order of magnitude, which then prevents Eq. (7) from becoming a uniformly valid approximation to the function h. As a matter of fact, this behavior is strikingly similar to what is observed in the more familiar problem of the velocity distribution around a sphere for low values of the Reynolds number, where it has been noticed, originally by Whitehead,[14] that a straightforward perturbation correction to Stokes' solution will diverge at infinity. It appears reasonable therefore to attempt a solution by the method of singular perturbation expansions which, as was shown by Lagerstrom and Cole,[15] by Kaplun,[16] and by Proudman and Pearson[17] is ideally suited for attacking fluid-mechanical problems similar to the one presently being considered.

Following then a well-established procedure,[17] we construct an "inner" and an "outer" expansion, h and H respectively, in such a way that: (a) The "inner" expansion h satisfies the boundary condition at the solid surface. (b) The "outer" expansion H vanishes at infinity. (c) The two expansions match identically at some arbitrary distance from the surface, and both remain bounded as $\epsilon \to 0$.

Now, upon examination of Eq. (5), it becomes apparent that for small values of ϵ the convection terms can be neglected as a first approximation and that an expansion similar to Eq. (7) can be constructed to represent the "inner" solution near the solid surface. On the other hand, if in Eq. (5) we let $\rho \equiv \epsilon r$ and $h = H$, then

$$\nabla_\rho^2 H = \left(1 - \frac{3\epsilon}{2\rho} + \frac{\epsilon^3}{2\rho^3}\right)\mu \frac{\partial H}{\partial \rho} + \frac{(1-\mu^2)}{\rho}\left(1 - \frac{3\epsilon}{4\rho} - \frac{\epsilon^3}{4\rho^3}\right)\frac{\partial H}{\partial \mu}, \tag{9}$$

where we observe that, as $\epsilon \to 0$, the convection and the conduction terms become of the same order of magnitude and that the resulting equation does not contain the parameter ϵ. We can conclude therefore that Eq. (9) is the proper starting point for generating the "outer" expansion, in the region far from the sphere.

[14] A. N. Whitehead, Quart. J. Math. **23**, 143 (1889).
[15] P. A. Lagerstrom, and J. D. Cole, J. Rat. Mech. Anal. **4**, 817 (1955).
[16] S. Kaplun, J. Math. Mech. **6**, 585 (1957).
[17] I. Proudman, and J. R. A. Pearson, J. Fluid Mech. **2**, 237 (1957).

III. CONSTRUCTION OF THE SOLUTION

We assume next that the "inner" and the "outer" expansion may be represented, respectively, by

$$h(r, \theta) = \sum_{n=0}^{\infty} f_n(\epsilon) h_n(r, \mu), \quad \text{with} \quad f_0(\epsilon) = 1, \tag{10a}$$

and

$$H(\rho, \theta) = \sum_{n=0}^{\infty} F_n(\epsilon) H_n(\rho, \mu), \tag{10b}$$

where the functions $f_n(\epsilon)$ and $F_n(\epsilon)$, not necessarily simple powers of ϵ, are for the moment restricted only by the requirements

$$\lim_{\epsilon \to 0} \frac{f_{n+1}}{f_n} = 0 \quad \text{and} \quad \lim_{\epsilon \to 0} \frac{F_{n+1}}{F_n} = 0.$$

The boundary conditions are

$$h_0(1, \mu) = 1 \quad \text{and} \quad h_n(1, \mu) = 0 \quad \text{for} \quad n \geq 1;$$
$$H_n(\infty, \mu) = 0.$$

A. First Expansion Term

The functions h_0 and H_0 must, respectively, satisfy Eqs. (5) and (9) with $\epsilon = 0$. Thus

$$\nabla_r^2 h_0 = 0 \tag{11}$$

and

$$\nabla_\rho^2 H_0 = \mu \frac{\partial H_0}{\partial \rho} + \frac{1-\mu^2}{\rho}\frac{\partial H_0}{\partial \mu},$$

which, by the substitution

$$H_0 = G_0 \exp(\tfrac{1}{2}\rho\mu),$$

can be transformed into

$$\nabla_\rho^2 G_0 = \tfrac{1}{4} G_0. \tag{12}$$

The general solution of Eqs. (11) and (12) which remains bounded for all θ and which satisfies the imposed boundary conditions is[18]

$$h_0 = (1 - B_0) + \frac{B_0}{r} + \sum_{k=1}^{\infty} B_k (r^{-k-1} - r^k) P_k(\mu), \tag{13}$$

and

$$G_0 = \left(\frac{\pi}{\rho}\right)^{\frac{1}{2}} \sum_{k=0}^{\infty} C_k K_{k+\frac{1}{2}}\left(\frac{\rho}{2}\right) P_k(\mu), \tag{14}$$

where $K_{n+\frac{1}{2}}$ is a modified Bessel function defined as

$$K_{n+\frac{1}{2}}\left(\frac{\rho}{2}\right) = \left(\frac{\pi}{\rho}\right)^{\frac{1}{2}} e^{-\frac{1}{2}\rho} \sum_{m=0}^{n} \frac{(n+m)!}{(n-m)!\, m!\, \rho^m},$$

[18] A. G. Webster, *Partial Differential Equations of Mathematical Physics* (Hafner Publishing Company, New York, 1950).

$P_k(\mu)$ is the appropriate Legendre polynomial, and the constants B_k and C_k are to be determined from the matching requirement that

$$h(r \to \infty, \mu) = H(\rho \to 0, \mu). \qquad (15)$$

If now, $\rho \equiv \epsilon r$, then

$$h_0(r \to \infty, \mu) = (1 - B_0) + \frac{B_0 \epsilon}{\rho} - \sum_{k=1}^{\infty} \frac{B_k \rho^k}{\epsilon^k} P_k(\mu) + O(\epsilon^2),$$

and

$$H(\rho \to 0, \mu) = \frac{\pi F_0(\epsilon)}{\rho}\left[1 + \frac{\rho}{2}(\mu - 1) + \cdots\right] \cdot \sum_{k=0}^{\infty} C_k \frac{(2k)!\, P_k(\mu)}{k!\, \rho^k}, \qquad (16)$$

from which it readily follows that, if $\epsilon \to 0$ so that the additional terms of Eqs. (10a) and (10b) can be neglected:

$$F_0(\epsilon) = \epsilon,$$
$$B_k = 0 \quad \text{for} \quad k \geq 1,$$
$$B_0 = 1, \qquad C_0 = 1/\pi,$$
$$C_k = 0 \quad \text{for} \quad k \geq 1.$$

We can conclude therefore that the first terms of the inner and the outer expansions are given, respectively, by

$$h_0 = \frac{1}{r} \quad \text{and} \quad H_0 = \frac{1}{\rho} \exp\left[\frac{\rho}{2}(\mu - 1)\right], \qquad (17)$$

with

$$F_0(\epsilon) = \epsilon.$$

B. Second Expansion Term

In order to obtain the equation for h_1 we first approximate the convection term by means of h_0 and also assume that $f_1 = \epsilon$, although by this we shall not preclude the possibility that the constants in the solution may themselves depend on ϵ. Thus we deduce from Eqs. (5) and (10a) that

$$\nabla^2 h_1 = -\left(1 - \frac{3}{2r} + \frac{1}{2r^3}\right)\frac{\mu}{r^2},$$

which has, as a particular solution,

$$h_p = \left(\frac{1}{2} - \frac{3}{4r} - \frac{1}{8r^3}\right)\mu.$$

Then, in view of Eq. (13), the general solution which also satisfies the boundary condition $h_1(1, \mu) = 0$, is

$$h_1 = B_0\left(-1 + \frac{1}{r}\right) + \left[\left(-B_1 + \frac{3}{8}\right)r + \frac{B_1}{r^2} + \left(\frac{1}{2} - \frac{3}{4r} - \frac{1}{8r^3}\right)\right]\mu + \sum_{k=2}^{\infty} B_k(r^{-k-1} - r^k)P_k(\mu). \qquad (18)$$

It remains now to evaluate the coefficients B_k from the matching requirement

$$h_0(r \to \infty, \mu) + \epsilon h_1(r \to \infty, \mu) = \epsilon H_0(\rho \to 0, \mu),$$

where H_0 is given by Eq. (17). It can easily be shown that this matching is exact up to and including all $O(\epsilon)$ terms if

$$B_0 = \tfrac{1}{2}, \qquad B_1 = \tfrac{3}{8},$$
$$B_k = 0 \quad \text{for} \quad k \geq 2,$$

which, incidentally, also justifies the original assertion that $f_1(\epsilon) = \epsilon$. Therefore,

$$h_1 = \frac{1}{2}\left(-1 + \frac{1}{r}\right) + \left(\frac{1}{2} - \frac{3}{4r} + \frac{3}{8r^2} - \frac{1}{8r^3}\right)\mu. \qquad (19)$$

The evaluation of H_1 is, on the other hand, somewhat more complicated. We first postulate that $F_1(\epsilon) = \epsilon^2$, then substitute Eq. (10b) into Eq. (9) and, by equating equal powers of ϵ^2, deduce that

$$\nabla_\rho^2 H_1 = \mu \frac{\partial H_1}{\partial \rho} + \frac{1 - \mu^2}{\rho}\frac{\partial H_1}{\partial \mu} - \frac{3}{2}\frac{\mu}{\rho}\frac{\partial H_0}{\partial \rho} - \frac{3}{4}\frac{1 - \mu^2}{\rho^2}\frac{\partial H_0}{\partial \mu}, \qquad (20)$$

which may be rearranged into

$$\nabla_\rho^2 G_1 = \frac{G_1}{4} + \frac{e^{-\frac{1}{2}\rho}}{4\rho^2} \cdot \left[-2 + \left(3 + \frac{6}{\rho}\right)P_1(\mu) - P_2(\mu)\right] \qquad (21)$$

by the substitution

$$H_1 = G_1 \exp\left(\tfrac{1}{2}\rho\mu\right).$$

Naturally, the homogeneous solution to Eq. (21) is again given by Eq. (14), whereas the particular solution may be obtained in a straightforward although somewhat tedious manner.[19] The net result, which also satisfies the boundary condition at infinity, is that

$$H_1 = e^{\frac{1}{2}\rho\mu}\left[\left(\frac{\pi}{\rho}\right)^{\frac{1}{2}} \sum_{k=0}^{\infty} C_k^* K_{k+\frac{1}{2}}\left(\frac{\rho}{2}\right)P_k(\mu) + \sum_{i=0}^{2} R_i(\rho)P_i(\mu)\right], \qquad (22)$$

where

[19] T. D. Taylor, Ph.D. dissertation, University of California, Berkeley (1961).

$$R_0(\rho) = \frac{e^{\frac{1}{2}\rho}}{2\rho}\int_\rho^\infty \frac{e^{-x}}{x}\,dx + \frac{e^{-\frac{1}{2}\rho}}{2\rho}\ln\rho,$$

$$R_1(\rho) = \frac{3}{4\rho}\left(1 - \frac{2}{\rho}\right)e^{\frac{1}{2}\rho}\int_\rho^\infty \frac{e^{-x}}{x}\,dx - \frac{3}{4\rho}\left(1 + \frac{2}{\rho}\right)e^{-\frac{1}{2}\rho}\ln\rho - \frac{3}{2\rho^2}e^{-\frac{1}{2}\rho}, \tag{23}$$

$$R_2(\rho) = \frac{1}{4\rho}\left(1 - \frac{6}{\rho} + \frac{12}{\rho^2}\right)e^{\frac{1}{2}\rho}\int_\rho^\infty \frac{e^{-x}}{x}\,dx + \frac{1}{4\rho}\left(1 + \frac{6}{\rho} + \frac{12}{\rho^2}\right)e^{-\frac{1}{2}\rho}\ln\rho + \frac{3}{2\rho^2}\left(1 + \frac{6}{\rho}\right)e^{-\frac{1}{2}\rho},$$

which, as $\rho \to 0$, reduce to

$$R_0(\rho) \to -\frac{\ln\gamma}{2\rho} - \frac{\ln\rho}{2} + \left(\frac{1}{2} - \frac{\ln\gamma}{4}\right),$$

$$R_1(\rho) \to \frac{3}{2\rho^2}(\ln\gamma - 1) - \frac{3}{4\rho} - \frac{3}{16}(\ln\gamma - 1), \tag{24}$$

$$R_2(\rho) \to \frac{3}{\rho^3}(3 - \ln\gamma) - \frac{1}{8\rho}(3 - \ln\gamma) + \frac{1}{24},$$

where $\ln\gamma \equiv 0.577215$ is the familiar Euler constant.

We now perceive that, in the (ρ, μ) variables, the "inner" expansion given by Eqs. (17) and (19) has the form

$$h_0 + \epsilon h_1 = \epsilon[1/\rho + \tfrac{1}{2}(\mu - 1)] + (\epsilon^2/2\rho)(1 - \tfrac{3}{2}\mu) + O(\epsilon^3), \tag{25}$$

where all the terms ρ^{-m} $(m \geq 1)$ of the complete "inner" solution h are given exactly as shown above up to an accuracy of at least $O(\epsilon^2)$. This is so, because in view of the condition $f_2(\epsilon) < \epsilon$ for $\epsilon \to 0$, the contribution of $f_2(\epsilon)h_2$ to any given term ρ^{-m} $(m \geq 0)$ of h must necessarily be of higher order in ϵ than that of ϵh_1. We may conclude therefore that if the function $\epsilon H_0 + \epsilon^2 H_1$ is to be matched exactly as $\rho \to 0$ with Eq. (25), and since the $O(\epsilon)$ terms have already been matched in the construction of h_1, we must require that all ρ^{-m} $(m \geq 2)$ terms of H_1 be zero and that the term ρ^{-1} of H_1 be equal to $\frac{1}{2}(1 - \frac{3}{2}\mu)$. It is possible then to establish, after making careful use of Eqs. (22)–(24), that

$$\begin{aligned} C_k^* &= 0, \quad \text{for} \quad k \geq 3, \\ C_2^* &= -(3 - \ln\gamma)/4\pi, \\ C_1^* &= 3(1 - \ln\gamma)/4\pi, \\ C_0^* &= (1 + \ln\gamma)/2\pi, \end{aligned} \tag{26}$$

and that, as $\rho \to 0$,

$$\epsilon H_0 + \epsilon^2 H_1 \to \exp\left[\frac{\rho}{2}(\mu - 1)\right] + \frac{\epsilon^2}{2\rho}\left(1 - \frac{3\mu}{2}\right) - \frac{\epsilon^2\ln\rho}{2} + \epsilon^2\left[\left(\frac{1}{8} - \frac{\ln\gamma}{2}\right) + \frac{P_1}{4} - \frac{5P_2}{24}\right] + \cdots, \tag{27}$$

with

$$\frac{\epsilon}{\rho}\exp\left[\frac{\rho}{2}(\mu - 1)\right] = \frac{\epsilon}{\rho}\left[\left(1 - \frac{\rho}{2}\right) + \frac{\rho}{2}P_1 + \rho^2\left(\frac{P_2}{12} - \frac{P_1}{4} + \frac{1}{6}\right) + O(\rho^3)\right].$$

C. Higher-Order Terms

The next term of the "inner" expansion may again be obtained from Eq. (8) if it is assumed that $f_2 = \epsilon^2$. Therefore,

$$\nabla^2 h_2 = U_r\frac{\partial h_1}{\partial r} + \frac{U_\theta}{r}\frac{\partial h_1}{\partial\theta},$$

which, in view of Eqs. (6) and (19), may be rearranged into

$$\nabla^2 h_2 = \sum_{k=0}^{2} Z_k(r)P_k(\mu), \tag{27}$$

where

$$Z_0 = \frac{1}{3r} - \frac{1}{2r^2} + \frac{7}{48r^4} + \frac{1}{8r^5} - \frac{3}{16r^6} + \frac{1}{12r^7},$$

$$Z_1 = -\frac{1}{2r^2} + \frac{3}{4r^3} - \frac{1}{4r^5}, \tag{28}$$

$$Z_2 = -\frac{1}{3r} + \frac{5}{4r^2} - \frac{15}{8r^3} + \frac{65}{48r^4} - \frac{5}{16r^5} - \frac{3}{16r^6} + \frac{5}{48r^7}.$$

As before, the homogeneous solution to Eq. (27) $(h_2)_H$ is the familiar expression

$$(h_2)_H = \sum_{k=0}^{\infty}[\hat{C}_k r^k + \hat{B}_k r^{-(k+1)}]P_k(\mu), \tag{29}$$

whereas the particular solution $(h_2)_P$ becomes

$$(h_2)_P = \sum_{k=0}^{2} L_k(r)P_k(\mu), \tag{30}$$

where

$$L_0(r) = \frac{r}{6} - \frac{\ln r}{2} + \frac{7}{96r^2} + \frac{1}{48r^3} - \frac{1}{64r^4} + \frac{1}{240r^5},$$

$$L_1(r) = \frac{1}{4} - \frac{3}{8r} - \frac{1}{16r^3}, \tag{31}$$

$$L_2(r) = \frac{r}{12} - \frac{5}{24} + \frac{5}{16r} - \frac{65}{192r^2} + \frac{\ln r}{16r^3} - \frac{1}{32r^4} + \frac{5}{672r^5}.$$

Now, since h_2 must be matched with the "outer" expansion as given by Eq. (27), it cannot contain Legendre polynomials of order greater than 2, and therefore

$$\hat{C}_k = \hat{B}_k = 0 \quad \text{for} \quad k \geq 3.$$

Similarly, since $h_2 = 0$ at $r = 1$, we must require that the remaining $\hat{C}_k$'s be so chosen that

$$h_2 = \sum_{k=0}^{2} [\hat{B}_k r^{-(k+1)} - r^k[\hat{B}_k + L_k(1)] + L_k(r)]P_k(\mu). \tag{32}$$

It remains then to evaluate the $\hat{B}_k$'s from the matching requirement between the inner and the outer expansions. It is apparent, however, that since $L_0(r)$ contains a term $\ln r$, the substitution $\rho \equiv r\epsilon$ into $\epsilon^2 h_2$ will generate a net contribution of $O(\epsilon^2 \ln \epsilon)$ which cannot be matched to the "outer" solution given by Eq. (27). It is clear then that the postulated "inner" expansion of h should contain a term $(\epsilon^2 \ln \epsilon)h_2^*$ so that

$$h = h_0 + \epsilon h_1 + (\epsilon^2 \ln \epsilon)h_2^* + \epsilon^2 h_2 + \cdots . \tag{33}$$

If we next substitute Eq. (33) into Eq. (5) and equate terms $O(\epsilon^2 \ln \epsilon)$, we can readily see that

$$\nabla^2 h_2^* = 0,$$

and since, because of the matching requirement with Eq. (27), Eq. (33) cannot contain terms of $O(\epsilon^2 \ln \epsilon)$ as $r \rightarrow \infty$, it immediately follows that

$$h_2^* = \tfrac{1}{2}(1/r - 1). \tag{34}$$

Therefore, with $\rho = r\epsilon$,

$$\epsilon^2 \ln \epsilon h_2^* + \epsilon^2 h_2 = -\sum_{k=0}^{2} \epsilon^{(2-k)}\rho^k[\hat{B}_k + L_k(1)]P_k(\mu) + \frac{\epsilon\rho}{6} + \frac{\epsilon^2 P_1}{4} + P_2\left\{\frac{\epsilon\rho}{12} - \frac{5\epsilon^2}{24}\right\} - \frac{\epsilon^2 \ln \rho}{2} + \frac{\epsilon^3 \ln \epsilon}{2\rho} + O(\epsilon^3), \tag{35}$$

which, when added to Eq. (25), may be matched with Eqs. (27) by letting

$$\hat{B}_2 + L_2(1) = 0, \qquad \hat{B}_1 + L_1(1) = \tfrac{1}{4}, \qquad \hat{B}_0 + L_0(1) = \tfrac{1}{2} \ln \gamma - \tfrac{1}{8}. \tag{36}$$

It becomes apparent, however, that this matching cannot include the $O(\epsilon^3 \ln \epsilon)$ term in Eq. (35), and this can only mean that the next term of the "outer" expansion, Eq. (10b), must be of the form $(\epsilon^3 \ln \epsilon)H_2(\rho, \mu)$. It is also rather obvious that since H_2 and H_0 must satisfy the same differential equation, and since the matching requirement is such that

$$H_2(\rho, \mu) \rightarrow 1/2\rho \quad \text{as} \quad \rho \rightarrow 0,$$

it must follow that

$$H_2 = \tfrac{1}{2}H_0 = (1/2\rho) \exp [\tfrac{1}{2}\rho(\mu - 1)].$$

This would in turn imply that the next term in the "inner" expansion would have to be $(\epsilon^3 \ln \epsilon)h_3$ and that, as a matter of fact, h_3 should be equal to $\frac{1}{2}h_1$, since h_3 and h_1 can easily be shown to satisfy exactly the same differential equation and an identical matching requirement except for a factor of $\frac{1}{2}$.

We can conclude therefore that the "inner" expansion takes the form

$$h = 1/r + (\epsilon + \tfrac{1}{2}\epsilon^3 \ln \epsilon)h_1 + \tfrac{1}{2}(\epsilon^2 \ln \epsilon)(1/r - 1) + \epsilon^2 h_2 + \cdots , \tag{37}$$

where h_1 and h_2 are given, respectively, by Eqs. (19) and (32) and where, it can easily be demonstrated that the higher-order terms of Eq. (37) must be successively $O(\epsilon^3)$, $O[\epsilon^4 (\ln \epsilon)^2]$, $O(\epsilon^4 \ln \epsilon)$, etc. We shall, however, refrain from continuing the perturbation any further because of the excessive algebraic effort which would be required to compute the $O(\epsilon^3)$ contribution.

IV. EXPRESSION FOR THE AVERAGE NUSSELT NUMBER

We next turn our attention to the determination of the average number, Nu. If the diameter of the sphere is chosen as the characteristic length,

$$\text{Nu} = -\int_{-1}^{1} \left(\frac{\partial h}{\partial r}\right)_{r=1} d\mu, \tag{38}$$

where h refers to the "inner" expansion as given by Eq. (37). This in turn may be put into the form

$$h = \sum_{j=0}^{\infty} \phi_j(r, \epsilon)P_j(\mu),$$

so that, in view of the orthogonality relation

$$\int_{-1}^{1} P_j(\mu)\, d\mu = 0, \quad \text{for} \quad j \neq 0,$$

only the term $\phi_0(r, \epsilon)$ will contribute to the integral in Eq. (38). Thus, we can easily deduce from Eqs. (19), (31), (32), and (36)–(38) that

$$\text{Nu} = 2\{1 + \tfrac{1}{2}(\epsilon + \epsilon^2 \ln \epsilon + \tfrac{1}{2}\epsilon^3 \ln \epsilon) + \epsilon^2[\tfrac{1}{2} \ln \gamma - \tfrac{1}{8} - L_0(1) - L_0'(1)] + \cdots\} ,$$

which may be rearranged into

TABLE I. A numerical comparison of Eqs. (1), (2), and (39).

Pe	Nu from Eq. (2)	Nu from Eq. (1)	Nu from Eq. (39)
0.1	2.055	2.053	2.044
0.2	2.115	2.112	2.084
0.3	2.174	2.177	2.124
0.4	2.231	2.248	2.165
0.5	2.284	2.326	2.209
0.6	2.334	2.409	2.259
0.7	2.377	2.498	2.315
0.8	2.414	2.594	2.379
0.9	2.444	2.695	2.451
1.0	2.467	2.803	2.534

$$\mathrm{Nu} = 2 + \tfrac{1}{2}\,\mathrm{Pe} + \tfrac{1}{4}\,\mathrm{Pe}^2 \ln \mathrm{Pe} + 0.03404\,\mathrm{Pe}^2 + \tfrac{1}{16}\,\mathrm{Pe}^3 \ln \mathrm{Pe} + \cdots \qquad (39)$$

by the substitutions

$$\epsilon \equiv \tfrac{1}{2}\,\mathrm{Pe}, \qquad L_0(1) = 239/960, \quad \text{and} \quad L_0'(1) = -\tfrac{1}{2}.$$

Equation (39), which incidentally bears a considerable resemblance to the small Reynolds number expansion for the drag coefficient of a sphere as derived by Proudman and Pearson,[17] summarizes then the principal result of our analysis and allows us to compute the Nusselt number for values of Pe up to about 1. It is indeed apparent that the first two terms of our series remain in complete agreement with the corresponding terms of Eqs. (1) and (2), but that, beyond this point, the earlier results cease to be valid even though Breiman's formula [Eq. (2)] correctly predicts the absolute magnitude but not the sign of the third and fourth terms. It is also worthwhile to point out, however, that the advantage of the singular perturbation technique over other methods of solution consists in providing us with a fairly rigid check on the correctness of all the many steps in the rather involved analysis, since otherwise the matching requirement between the "inner" and the "outer" expansions could not in general be satisfied. The numerical values of Nu, as computed from Eqs. (1), (2), and (39) are shown in Table I for the range $0 \leq \mathrm{Pe} \leq 1$.

V. INFLUENCE OF AN INCREASING REYNOLDS NUMBER

Our discussion has been restricted, up to this point, to systems with strictly speaking vanishingly small Reynolds numbers, since it is known that the velocity components given by Eq. (6) may be used to describe the flow field exactly only in the limit $\mathrm{Re} \to 0$. It is of interest therefore to examine how the introduction of a velocity profile more realistic than Eq. (6) will affect the results arrived at earlier and in particular the functional dependence of Nu on Pe.

It is immediately clear of course, that to attempt a generalization of Eq. (39) would at this stage be prohibitively cumbersome. Instead, what will be studied is the effect of the Reynolds number on the first two terms of Eq. (39) and on the asymptotic solution for $\mathrm{Pe} \to \infty$, in order to provide us with a reasonably clear indication concerning the range of validity of the $\mathrm{Re} \to 0$ result which we have just obtained.

A. Effect of Re on the Solution for Small Pe

Let us now refer once again to Eq. (5), where U_r and U_θ are left unspecified, except for the requirement that

$$U_r \to \mu \quad \text{as} \quad r \to \infty,$$

and

$$U_\theta \to -(1-\mu^2)^{\frac{1}{2}} \quad \text{as} \quad r \to \infty.$$

It is at once apparent that the first terms of the "inner" and the "outer" expansions, h_0 and H_0, respectively, will remain exactly the same as before, and will be given by Eq. (17). The function h_1 will have to be modified, however, since

$$\nabla^2 h_1 = -U_r(r,\mu)/r^2. \qquad (40)$$

The general solution of Eq. (40) is once more of the form

$$h_1 = \sum_{k=0}^{\infty} \phi_k(r) P_k(\mu),$$

but since we shall be mainly interested in obtaining the average Nusselt number, Nu, as defined by Eq. (38), it is clear that only $\phi_0(r)$ needs to be computed. Because of the continuity equation, though,

$$\overline{U}_r(r) \equiv \tfrac{1}{2}\int_{-1}^{1} U_r(r,\mu)\, d\mu = 0 \quad \text{for all } r \text{ and all Re},$$

and therefore

$$d/dr[r^2(d\phi_0/dr)] = 0$$

with boundary conditions $\phi_0(1) = 0$ and $\phi_0 \to -\frac{1}{2}$ as $r \to \infty$ because of the matching requirement with H_0.

We can conclude then that

$$\mathrm{Nu} = 2 + \tfrac{1}{2}\,\mathrm{Pe} + O(\mathrm{Pe}^2 \ln \mathrm{Pe}) \qquad (41)$$

and that the first two terms will remain completely independent of the Reynolds number Re.

B. Effect of Re on the Pe $\rightarrow \infty$ Asymptote

The asymptotic solution for Pe $\rightarrow \infty$ may also be derived rather simply since, by repeating the rigorous arguments of Morgan and Warner[20] concerning the solution to the laminar-boundary-layer energy equation for large Prandtl numbers, it is possible to show that Eq. (6) may be rearranged as Pe $\rightarrow \infty$ into the form

$$[\alpha(\mu)(1-\mu^2)^{\frac{1}{2}}]y\,\frac{\partial h}{\partial \mu} + \beta(\mu)y^2\,\frac{\partial h}{\partial y} = \frac{\partial^2 h}{\partial y^2}, \qquad (42)$$

where

$$y \equiv (r-1)(\tfrac{1}{2}\,\mathrm{Pe})^{\frac{1}{3}},$$

while

$$\alpha(\mu) \equiv \lim_{r\to 1}\left(\frac{U_\theta}{1-r}\right) \quad \text{and} \quad \beta(\mu) \equiv \lim_{r\to 1}\frac{U_r}{(1-r)^2}.$$

It is then easy to establish from the results of Proudman and Pearson[17,21] that if the Reynolds number is defined with the diameter as the characteristic length,

$$\alpha(\mu) = \tfrac{3}{2}(1-\mu^2)^{\frac{1}{2}}[1 + \tfrac{3}{16}\,\mathrm{Re}\,(1-\tfrac{4}{3}\mu) + (9/160)\,\mathrm{Re}^2\ln \mathrm{Re} + O(\mathrm{Re}^2)] \qquad (43)$$

and

$$\beta(\mu) = -\tfrac{1}{2}(d/d\mu)[(1-\mu^2)^{\frac{1}{2}}\alpha(\mu)],$$

so that the solution to Eq. (42) may be derived in a straightforward manner by a similarity transformation method.[22] Thus

$$\mathrm{Nu} = \frac{(\frac{1}{2}\,\mathrm{Pe})^{\frac{1}{3}}}{9^{\frac{1}{3}}\Gamma(\frac{4}{3})}\,\frac{3}{2}\left\{\int_{-1}^{1}[(1-\mu^2)^{\frac{1}{2}}\alpha(\mu)]^{\frac{1}{2}}\,d\mu\right\}^{\frac{2}{3}},$$

which in turn can be simplified into

$$\mathrm{Nu} = 0.991\,\mathrm{Pe}^{\frac{1}{3}} \cdot [1 + \tfrac{1}{16}\,\mathrm{Re} + (3/160)\,\mathrm{Re}^2\ln \mathrm{Re} + O(\mathrm{Re}^2)]$$

And finally, by comparing the above with the Proudman and Pearson[17,21] formula for the drag of a solid sphere, we can conclude that

$$\mathrm{Nu} = 0.991\,\mathrm{Pe}^{\frac{1}{3}}\,[C_D/C_D(s)]^{\frac{1}{3}} \qquad (44)$$

up to but not necessarily including $O(\mathrm{Re}^2)$, where $C_D/C_D(s)$ is the correction to Stokes' law for the drag coefficient.

This intriguing result then not only allows us to predict, in a simple and convenient manner, the effect of increasing Re on the Nusselt number for large Pe, but, together with Eq. (41), permits us to conclude that the functional relation between Nu and Pe, as obtained with the Stokes velocity profile, is considerably less sensitive to an increase in the Reynolds number than the familiar Stokes' law for the drag coefficient.

ACKNOWLEDGMENT

This research was supported in part by a grant from the National Science Foundation and was initiated at Cambridge, England, while the senior author was the recipient of a Guggenheim fellowship.

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