Summary of previously obtained results and results for Fourier type integrals.

I - Asymptotics of real integrals

(In this section the letter R in the itemisation stands for real case.)

1-R Let $f(t)$ be a smooth real function bounded in the interval $[a, b]$ and such that the integral

$$I(x) = \int_a^b f(t)e^{-xt}\, dt$$

exists. Then the asymptotic behavior of this integral as $x \to \infty$ can be obtained using integraton by parts and is given by

$$I(x) \sim \sum_{n=0}^{\infty} \frac{(-1)^n f^{(n)}(b)e^{-xb}}{x^{n+1}} \quad \text{as} \quad x \to \infty.$$

2-R Watson's lemma (see related notes for the conditions on the functions, etc.). Let $f(t)$ have asymptotic series $f(t) \sim \sum_{n=0}^{\infty} c_n t^{\alpha+\beta n}$ with $\alpha > -1$, $\beta > 0$, then

$$I(x) = \int_0^T f(t)e^{-xt}\, dt \sim \sum_{n=0}^{\infty} \frac{c_n \Gamma(\alpha + \beta n + 1)}{x^{\alpha+\beta n+1}} \quad \text{as} \quad x \to \infty.$$

3-R Extension of [1-R]. (Laplace's integral) For the case when $\phi(t)$ has inverse, and $\phi'(t)$ is nonzero, and monotonic on $[a, b]$,

$$I(x) = \int_a^b f(t)e^{x\phi(t)}\, dt \sim \frac{f(b)e^{x\phi(b)}}{x\phi'(b)} \quad \text{as} \quad x \to \infty.$$

4-R More general Laplace's integral. For the case when $\phi(t)$ is non-zero, and $\phi'(t)$ has a single maximum at c, in the interval $[a, b]$ (i.e. $\phi'(c) = 0$, and $\phi''(c) < 0$; $c \in [a, b]$). Then:

i) if $c \in (a, b)$

$$I(x) = \int_a^b f(t)e^{x\phi(t)}\, dt \sim f(c)e^{x\phi(c)} \left(\frac{2\pi}{x|\phi''(c)|}\right)^{1/2} \quad \text{as} \quad x \to \infty.$$

ii) if either $c = a$, or $c = b$

$$I(x) = \int_a^b f(t)e^{x\phi(t)}\, dt \sim \frac{1}{2} f(c)e^{x\phi(c)} \left(\frac{2\pi}{x|\phi''(c)|}\right)^{1/2} \quad \text{as} \quad x \to \infty.$$

II-Results for Fourier type integrals.

(In this section the letter C in the itemisation stands for pure complex case, and $\omega \in \mathbb{R}$)

1-C Let $f(t)$ be a smooth real function bounded in the interval $[a, b]$ and such that the integral

$$I(\omega) = \int_a^b f(t)e^{i\omega t}\, dt$$

exists. Then the asymptotic behavior of this integral as $|\omega| \to \infty$ can be obtained using integraton by parts and is given by

$$I(\omega) \sim \sum_{n=0}^{\infty} \frac{(-1)^n}{(i\omega)^{n+1}} \left[f^{(n)}(b)e^{i\omega b} - f^{(n)}(a)e^{i\omega a}\right] \quad \text{as} \quad |\omega| \to \infty.$$

2-C Watson's lemma can be extended to complex variables but the region of validity has to be carefully considered.

$$I = \int_0^T f(t)e^{-zt}\, dt \sim \sum_{n=0}^{\infty} c_n \frac{\Gamma(\alpha + \beta n + 1)}{z^{\alpha+\beta n+1}}$$

IMPORTANT: Saying that this is valid for $|z| \to \infty$ is not sufficient, the theorem is valid only when $|arg(z)| < (\pi/2) - \delta$ with $0 < \delta < (\pi/2)$.

3-C The case of Laplace's integral of Fourier type. If $\phi(t)$ has inverse, and $\phi'(t)$ is nonzero, and monotonic on $[a, b]$, then it is possible to use a change of variables as in [3-R] to obtain:

$$I(\omega) = \int_a^b f(t)e^{i\omega\phi(t)}\, dt \sim \frac{1}{i\omega}\left[\frac{f(b)e^{i\omega\phi(b)}}{\phi'(b)} - \frac{f(a)e^{i\omega\phi(a)}}{\phi'(a)}\right] \quad \text{as} \quad |\omega| \to \infty.$$

4-C The equivalent result to [4-R] for Fourier type integrals is known as the stationary phase approximation (see Stationary phase file for applicability and conditions over the functions), and is given by:

$$I(\omega) = \int_a^b f(t)e^{i\omega\phi(t)}\, dt \sim f(c)e^{i\omega\phi(c)} \left(\frac{2\pi}{\omega|\phi''(c)|}\right)^{1/2} e^{i\pi/4} \quad \text{as} \quad |\omega| \to \infty.$$

II - Example of complex Watson's lemma.

$$I(z) = \int_0^1 \frac{e^{-zt}}{1+t}\, dt$$

The asymptotic series for $f(t) = 1/(1 + t)$ is

$$f(t) \sim \sum_{n=0}^{\infty} (-1)^n t^n \quad \text{as} \quad t \to 0^+$$

That is, $\alpha = 0 > -1$, and $\beta = 1 > 0$. Then, $c_n = (-1)^n$ and

$$\int_0^1 \frac{e^{-zt}}{1+t}\, dt \sim \sum_{n=0}^{\infty} (-1)^n \frac{\Gamma(n+1)}{z^{n+1}} \quad \text{as} \quad |z| \to \infty, \quad \text{and} \quad |arg(z)| < \pi/2 - \delta; 0 < \delta < \pi/2.$$

Apply Watson's lemma to

$$I(z) = \int_0^1 \frac{e^{izt}}{1+t}\, dt$$

by introducing the change of variables $z = i\bar{z}$.

The result of this asymptotic approximation should be different than the one we found in lectures using integration by parts on

$$I(\omega) = \int_0^1 \frac{e^{i\omega t}}{1+t}\, dt, \quad \omega \in \mathbb{R}$$

Why is this the case? (*Hint:* check region of validity of your calculation, and think about what are the differences between $i\omega$ with $\omega \in \mathbb{R}$, and iz with $z \in \mathbb{C}$.)