Some useful techniques to calculate Gaussian integrals

I - Gaussian regularization

Let $f(ix)$ be a smooth oscillatory function (example: $\sin(ix)$)

$$I = \int_0^\infty f(ix)\ dx.$$

To calculate these type of integrals it is very useful to regularize it by multiplying the integrand by a gaussian and then take the limit; that is, define

$$I_\alpha = \int_0^\infty f(ix) e^{-\alpha x^2}\ dx.$$

and then

$$I = \lim_{\alpha \to 0} I_\alpha = \lim_{\alpha \to 0} \int_0^\infty f(ix) e^{-\alpha x^2}\ dx.$$

In the case of $f(ix) = e^{ix^2}$, I_α^2 is absolutely convergent and using Fubini's theorem and the change of variables seen in the lectures it is possible to obtain the value for I by taking the limit as $\alpha \to 0$ at the end. There are many other ways to calculate this integral including using an analytic extension, which is equivalent to the method shown above.

(Fubini's theorem: If a function $f(x,y)$ is integrable in a given domain $D = X \times Y$, i.e.

$$\int\int_D f(x,y)\ dx\ dy \quad < \infty$$

Then

$$\int\int_D f(x,y)\ dx\ dy = \int_X dx \left(\int_Y f(x,y)\ dy \right) = \int_Y dy \left(\int_X f(x,y)\ dx \right)$$

which means that the product of the integrals not only gives the same answer but also the the order of integration commutes.)

II - Moments of a Gaussian distribution

The moments of a gaussian distribution are defined as:

$$M_n = \int_0^\infty x^n e^{-x^2}\ dx.$$

Calculating moments for odd n only involves the change of variables $t = x^2$ and integration by parts. When n is even there is no suitable substitution that can solve the problem. However, there is a technique rather similar to the one above that makes it possible to find the value of M_{2k} with $k \in \mathbb{N}$. Let

$$M_{2k}^{\alpha} = \int_0^{\infty} x^{2k} e^{-\alpha x^2}\ dx.$$

then

$$M_{2k} = \lim_{\alpha \to 1} M_{2k}^{\alpha}$$

Noting that

$$M_{2k}^{\alpha} = \int_0^{\infty} x^{2k} e^{-\alpha x^2}\ dx = (-1)^k \int_0^{\infty} \frac{\partial^k (e^{-\alpha x^2})}{\partial \alpha^k}\ dx$$

It is now possible to find M_{2k} by taking the limit:

$$M_{2k} = \lim_{\alpha \to 1} (-1)^k \frac{\partial^k}{\partial \alpha^k} \left[\int_0^{\infty} e^{-\alpha x^2} \right]$$