Some more functions and asymptotic series. Properties of Asymptotic series in general

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I - Gamma-function and Airy function

1 $\underline{\Gamma(z)}$:

As we saw in last lecture:

$$\Gamma(n+1) = n! = \int_0^\infty e^{-t}\ t^n\ dt.$$

This definition can be extended from $n \in \mathbb{N}$ to all $z \in \mathbb{C}$:

$$\Gamma(z) = \int_0^\infty e^{-t}\ t^{z-1}\ dt.$$

In particular, if $z = 1/2$

$$\Gamma(1/2) = \int_0^\infty e^{-t}\ t^{-1/2}\ dt.$$

To prove $\Gamma(1/2) = \sqrt{\pi}$ try the change of variables $s^2 = t$. To find the answer we need the value of one of the most famous integrals in mathematics:

$$I = \int_{-\infty}^\infty e^{-x^2}\ dx.$$

Here, since x is a dummy variable

$$I = \int_{-\infty}^\infty e^{-x^2}\ dx = \int_{-\infty}^\infty e^{-y^2}\ dy.$$

Thus,

$$I^2 = \int_{-\infty}^\infty \int_{-\infty}^\infty e^{-(x^2+y^2)}\ dx\ dy.$$

Now try the change of variables from cartesian to polar coordinates $(x, y) \to (r, \theta)$.

2 Airy function $\underline{Ai(z)}$:

The Airy function (named after George Bidell Airy who first discovered it) is the solution to the differential equation

$$\frac{d^2y(z)}{dz^2} - zy(z) = 0.$$

As usual and for simplicity we first consider the real case, that is $z \to x$. To find the solution we first Fourier transform the equation:

$$\mathcal{F}\left[\frac{d^2y(x)}{dx^2} - zy(x)\right] = -k^2\hat{y}(k) - \int_{-\infty}^\infty xe^{-ikx}y(x)\ dx = 0, \tag{1}$$

where

$$\hat{y}(k) = \int_{-\infty}^{\infty} e^{-ikx} y(x)\ dx.$$

Notice that eqn (1) can be rewritten as

$$-k^2\hat{y}(k) - (-\frac{1}{i})\frac{d\hat{y}(k)}{dk} = 0,$$

which yields $\hat{y}(k) = ik^3/3$ and after antitransforming we find the famous expression:

$$Ai(x) \equiv y(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i(kx+\frac{k^3}{3})}\ dk.$$

Note that one more time we have generated an integral of the type introduced in lecture 1, i. e., $F(\lambda) = \int_0^\infty e^{\lambda R(z)} g(z) dz$.

II - Some properties of asymptotic series

1 A Taylor series about $x = x_0$ is an Asymptotic series as $x \to x_0$:

This is an **important** result:

Let $f(x)$ be a smooth function that admits a Taylor expansion about $x = 0$ (in general it could be about $x = x_0$, because a simple change of variables $\xi = x - x_0$ renders both calculations identical), then:

$$f(x) = \sum_{n=0}^{\infty} f_n x^n \quad \text{where} \quad f_n = \frac{f^{(n)}(0)}{n!},$$

where $f^{(n)}(x)$ denotes the n-th derivative of the function. To show that this is an asymptotic series as $x \to 0$ we need to apply our definition, that is, we need to show that

$$f(x) - \sum_{n=0}^{N} a_n \phi_n(z) = o(\phi_N(x)) \quad \text{as} \quad x \to 0,$$

where in this case $\{\phi_n(z)\}_{n=0}^{\infty} = \{x^n\}_{n=0}^{\infty}$, with $n \in \mathbb{N}$. We proceed by first calculating the remainder:

$$f(x) - \sum_{n=0}^{N} f_n x^n = R_N(x) = \sum_{n=N+1}^{\infty} f_n x^n,$$

and then applying the limiting process. First we are going to see that it is $\mathcal{O}(x^{N+1})$

$$\lim_{x\to 0}\ \sup\left|\frac{R_N(x)}{x^{N+1}}\right| \to K$$

where K is some constant. From the expression of the remainder it is easy to see that the limit has the value $K = f^{(n)}(0)/n!$, that is

$$f(x) - \sum_{n=0}^{N} f_n x^n = \mathcal{O}(x^{N+1}) \quad \text{as} \quad x \to 0.$$

Now we note that $\mathcal{O}(x^{N+1})$ for the sequence $\{x^n\}_{n=0}^{\infty}$ is equivalent to $o(x^N)$ since $\lim_{x\to 0} \sup\left|\frac{Kx^{N+1}}{x^N}\right| \to 0$.

Important points:

i) To show that a series is asymptotic, calculate the remainder and apply the limit.

ii) For asymptotic series where the asymptotic sequences $\{(x-x_0)^n\}_{n=0}^{\infty}$ as $x \to x_0$ or $\{(x-x_0)^{-n}\}_{n=0}^{\infty}$ as $x \to \infty$ are used, $\mathcal{O}(x^{N+1})$ and $o(x^N)$ are equivalent. In this course, unless stated explicitly otherwise we will always be using these asymptotic sequences.(One should be careful with Big-$\mathcal{O}$ and little-o when using other asymptiotic sequences as basis).

2 Lemma on uniqueness:

Given a function $f(z)$, it is not necessarily the case that there will be an asymptotic series to represent is. At the same time if $f(z)$ admits an asymptotic series representation, there could be many different ones. In fact, for each possible choice of asymptotic sequence that produces a representation of $f(z)$ there will be a different sequence of coefficients for the series. **However**, for a given asymptotic sequence the asymptotic series for a given function, if it exists, is unique.

Lemma: For a given function $f(x)$ which admits an asymptotic series representation with a given asymptotic sequence $\{\phi_n(z)\}_{n=0}^{\infty}$ as $x \to x_0$ on a given domain, the asymptotic series for $f(z)$ is unique.

To show this, we assume that there are more than one representations of $f(z)$ with the same asymptotic sequence:

$$f(z) \sim \sum_{n=0}^{\infty} a_n \phi_n(z) \quad \text{and} \quad f(z) \sim \sum_{n=0}^{\infty} b_n \phi_n(z)$$

substracting both expressions we find

$$\sum_{n=0}^{\infty} (a_n - b_n)\phi_n(z) = 0.$$

Because the $\phi_n(z)$ are a complete set of independent functions the only possible way to satisfy the identity is for all coefficients to be zero, i.e., $a_n = b_n$, which contradicts our hypothesis, then the series must be unique.

3 Sums and products of asymptotic series:

Let $f(z)$ and $g(z)$ be two functions with asymptotic series with respect to the same asymptotic sequences. Then, the sum $h(z) = f(z) + g(z)$ is also an asymptotic series.

$$f(z) \sim \sum_{n=0}^{\infty} a_n \phi_n(z) \quad \text{and} \quad g(z) \sim \sum_{n=0}^{\infty} b_n \phi_n(z).$$

Then,

$$h(z) = f(z) + g(z) \sim \sum_{n=0}^{\infty} a_n \phi_n(z) + \sum_{n=0}^{\infty} b_n \phi_n(z) \tag{2}$$

That is, we want to show that:

$$h(z) - \sum_{n=0}^{N} c_n \phi_n(z) = o(\phi_N(z)), \tag{3}$$

where

$$h(z) - \sum_{n=0}^{N} c_n \phi_n(z) = f(z) - \sum_{n=0}^{N} a_n \phi_n(z) + g(z) - \sum_{n=0}^{N} b_n \phi_n(z). \tag{4}$$

We know that $f(z) - \sum_{n=0}^{N} a_n \phi_n(z)$ and $g(z) - \sum_{n=0}^{N} b_n \phi_n(z)$ are both $o(\phi_N(z))$. Aplying the limiting procedure it is easy to verify that the

$$\lim_{x \to x_0} \sup \left| \frac{f(z) - \sum_{n=0}^{N} a_n \phi_n(z) + g(z) - \sum_{n=0}^{N} b_n \phi_n(z)}{\phi_N(z)} \right| \quad \text{as} \quad x \to 0. \tag{5}$$

then $h(z) - \sum_{n=0}^{N} c_n \phi_n(z) = o(\phi_N(z))$ and is also an asymptotic series (verify the result. Always: TRUST, BUT VERIFY!)

4 Small at all orders:

Because we are going to use only the few asymptotic sequences listed above, there are certain functions that will not have an asymptotic representation with respect to those sequences. In particular we are interested on the exponential function.

Suppose that we wish to represent the exponential as $x \to \infty$ with a series of the form

$$e^{-x} \sim \sum_{n=0}^{N} a_n x^{-n} \quad \text{as} \quad x \to \infty. \tag{6}$$

We have been told many times that as $x \to \infty$ the exponential decays faster than any power law, this should make it clear that there is no possible sequence of constant coefficients that could reflect this fact for the series above. To show this we can make a change of variable $x = 1/u$ s.t. $x \to \infty$ corresponds to $u \to 0$, then

$$e^{-1/u} \sim \sum_{n=0}^{N} a_n u^n \quad \text{as} \quad u \to 0. \tag{7}$$

This change of variables makes it possible to use all the results we found for the Taylor expansion. This is a part of a question in your sample sheets, and when you answer that question you should find that all the coefficients are zero. Both cases, e^{-x} as $x \to \infty$, and $e^{-1/x}$ as $x \to 0$ are equivalent. We say that e^{-x} is "small at all orders" with respect to the sequence $\{x^{-n}\}_{n=0}^{\infty}$ as $x \to \infty$.

This has some interesting consequences, if a given function $f(x)$ has an asymptotic series representation

$$f(z) \sim \sum_{n=0}^{N} a_n x^{-n} \quad \text{as} \quad x \to \infty, \tag{8}$$

the asymptotic series representation of the functions $Ae^{-x}+f(z)$, for any constant A, is the same.

$$Ae^{-x} + f(z) \sim \sum_{n=0}^{N} a_n x^{-n} \quad \text{as} \quad x \to \infty. \tag{9}$$

That means that while given a function the asymtotic series is unique, given an asymptotic series the functions that are represented are **not** unique.

II - Stieltjes Integral

1 <u>The integral itself:</u>

One of the integrals that we will be studying is the Stieltjes Integral, given by

$$S(x) = \int_0^{\infty} \frac{\rho(t)}{1+xt}\, dt$$

for $x \to 0$. The objective is to find a suitable asymoptotic approximation to this integral (you should look up Mr Steiltjes' biography, it is very interesting)

2 <u>Asymptotic series for the Stieltjes Integral</u>

Now that we know that a Taylor series is an asymptotic series as $x \to 0$ we can try to find an asymptotic series for our integral by inserting into it the Taylor series of the term multiplying $\rho(t)$

$$\frac{1}{1+xt} = \sum_{n=0}^{\infty} (-1)^n x^n t^n$$

and then checking that the remainder is indeed $o(x^N)$ or equivalently $\mathcal{O}(x^{N+1})$. Thus,

$$S(x) = \int_0^{\infty} \frac{\rho(t)}{1+xt}\, dt \sim \sum_{n=0}^{\infty} (-1)^n x^n \int_0^{\infty} \rho(t) t^n\, dt.$$

To find the remainder we note that $\sum_{n=0}^{N}(-1)^n x^n t^n + \sum_{n=N+1}^{\infty}(-1)^n x^n t^n$ can be written as

$$\sum_{n=0}^{N}(-1)^n x^n t^n + (-1)^{N+1} x^{N+1} t^{N+1} \sum_{n=0}^{\infty}(-1)^n x^n t^n,$$

that is $\sum_{n=0}^{\infty}(-1)^n x^n t^n = \sum_{n=0}^{N}(-1)^n x^n t^n + R(x)$, with

$$R(x) = \frac{(-1)^{N+1} x^{N+1} t^{N+1}}{1+xt}.$$

Now we can write:

$$S(x) - \sum_{n=0}^{N}(-1)^n x^n \int_0^{\infty} \rho(t) t^n\, dt = (-1)^{N+1} x^{N+1} \int_0^{\infty} \frac{\rho(t) t^{N+1}}{1+xt}\, dt$$

and from this we can see that because the reminder is indeed $\mathcal{O}(x^{N+1})$, then the desired limit is $o(x^N)$, and the series we found is asymptotic as $x \to 0$.