Asymptotic versus Convergent series. Optimal truncation

I - Different asymptotic sequences (also known as gauges).

The asymptotic expansion of the function $f(x) = \tan x$ as $x \to 0$ with respect to the asymptotic sequence (or gauge) $\{\phi_n(x)\}_{n=0}^{\infty} = \{x^n\}_{n=0}^{\infty}$ is:

$$\tan x \sim \sum_{n=0}^{\infty} \frac{2^{2n}(2^{2n}-1)}{(2n)!}|B_{2n}|x^{2n-1} \quad \text{as} \quad x \to 0, \tag{1}$$

where the B_{2n} are the Bernoulli numbers, and was obtained by using Taylor expansion about $x = 0$. We can also write an asymptotic series for the same $f(x)$, as $x \to 0$ with gauge functions $\{\phi_n(x)\}_{n=0}^{\infty} = \{(\sin x)^n\}_{n=0}^{\infty}$.

$$\tan x \sim \sum_{n=0}^{\infty} \frac{(2n)!}{(n!)^2}(\sin x)^{2n+1} \quad \text{as} \quad x \to 0, \tag{2}$$

(check that $\{(\sin x)^n\}_{n=0}^{\infty}$ as $x \to 0$ is an asymptotic sequence). While it is true that both series (1) and (2) are asymptotic representations of $f(x)$, the approximations they provide to $f(x)$ for x close to 0 are different. The difference decreases as x approaches 0, but for small finite values of x it might be numerically relevant. For example for $x = 0.4$ the first two terms of (1) give 0.4213333 while those of (2) give 0.418945, this does not seem much, but the error these values have with respect to the exact result are 3%, and 9%, respectively.

II - More properties of asymptotic series.

Let $f(x)$ be an integrable function with asymptotic representation

$$f(x) \sim \sum_{n=0}^{\infty} a_n (x - x_0)^n \quad \text{as} \quad x \to x_0,$$

then,

$$\int_{x_0}^{x} f(\xi)\, d\xi \sim \sum_{n=0}^{\infty} \frac{a_n}{n+1}(x - x_0)^{n+1} \quad \text{as} \quad x \to x_0,$$

See Prof. Manton's notes page 4.

III - Asymptotic integrals (Stieltjes). Series using integration by parts.

See lecture 2 notes and Prof. Manton's notes pages 4 and 5. There you will also find an example of how to generate an asymptotic series as $x \to \infty$ other than the one below.

IV - Asymptotic versus Convergent series.

The convergence properties of a series are intrinsic to the sequence formed by its coefficients. In fact, given a sequence of constants corresponding to a series representing a function, it is possible to show that the series converges even when the function that the series represents is unknown. In contrast, to show that a series is asymptotic it is necessary to know both, the coefficients of the series and the function to which the series is asymptotic. It many cases a convergent series is not asymptotic and an asymptotic series does not converge.

To understand the difference, consider the asymptotic series given by

$$\sum_{n=0}^{\infty} a_n \phi_n(x),$$

where $\{\phi_n(x)\}_{n=0}^{\infty}$ is an asymptotic sequence as $x \to x_0$. Because we do not know if this asymptotic series is convergent we say that this is a formal series. However, the partial sum of the series is well defined for any finite N:

$$F_N(x) = \sum_{n=0}^{N} a_n \phi_n(x).$$

For each value of x, a convergent series has a unique limiting sum $F_\infty(x)$ but it does not provide any information on how well $F_N(x)$ approximates $F_\infty(x)$ for fixed N or on how fast it converges. As we noticed last lecture, an asymptotic series does not define a unique function $f(x)$, but rather a whole class of asymptotically equivalent functions. Hence, it does not provide an arbitrarily accurate approximation to $f(x)$ when $x \neq x_0$, it only provides a good approximation for values of the function when x is sufficiently close to x_0. The quality of the approximation depends on i) how close is x to x_0, and when the series is not convergent ii) on how many terms are taken into account. However, it is possible to find the optimal number of terms to be retained in the partial sum, and the resulting truncation is known as the **optimal truncation**.

II - Optimal truncation.

To illustrate these matters, consider the error function

$$\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt$$

For the n-th time ($n \to \infty$) we are surprised (NOT!) of being able to use Taylor series for $t \to 0$ (Note: for the function e^{-t^2} the series is absolutely convergent). Inserting the series into the integral, and then integrating term by term, yields

$$\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}} \left[x - \frac{1}{3}x^3 + \frac{1}{2 \cdot 5}x^5 + \cdots \right] = \frac{2}{\sqrt{\pi}} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)n!} x^{2n+1} \ .$$

That the series converges for all x can be seen from the fact that the ratio

$$\left|\frac{u_{n+1}}{u_n}\right| = \frac{(2n+1)}{(2n+3)}\frac{x^2}{(n+1)}$$

is less than 1 for any real value of x and sufficiently large n. However, as the value of x increases the convergence is very slow (to obtain an accuracy of 10^{-5} for $x = 3$ requires the summation of more than thirty terms), and a better result can be obtained with an asymptotic series.

To show that this is the case we shall find an asymptotic series for $\mathrm{erf}(x)$

$$\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}dt = 1 - \frac{2}{\sqrt{\pi}}\int_x^\infty e^{-t^2}dt$$

and with the change of variables $s^2 = t$

$$\mathrm{erf}(x) = 1 - \frac{2}{\sqrt{\pi}}\int_x^\infty e^{-s}s^{-1/2}ds$$

(Do you notice something familiar? You should, the first part of the course is about Taylor series, Gamma functions and the change of variables $s = t^2$). Now we integrate by parts (ALERT: use of the "new" technique... Integration by parts...)

$$\mathrm{erf}(x) = 1 - \frac{1}{\sqrt{\pi}}\left[\frac{e^{-x^2}}{x} - \frac{1}{2}\int_{x^2}^\infty s^{-3/2}e^{-s}ds\right].$$

The procedure can be iterated to any order N to obtain:

$$\begin{aligned}
E_n(x) &= \int_{x^2}^\infty s^{-n-1/2}e^{-s}ds \\
&= -\left[s^{-n-1/2}e^{-s}\Big|_{x^2}^\infty + \left(n+\frac{1}{2}\right)\int_{x^2}^\infty s^{-n-1/2-1}e^{-s}ds\right] \\
&= \frac{e^{-x^2}}{x^{2n+1}} - \left(n+\frac{1}{2}\right)E_{n+1}(x).
\end{aligned}$$

Which leads to:

$$\begin{aligned}
\mathrm{erf}(x) =& 1 - \frac{1}{\sqrt{\pi}}E_0(x) \\
=& 1 - \frac{1}{\sqrt{\pi}}\left[\frac{e^{-x^2}}{x} - \frac{1}{2}E_1(x)\right] \\
=& \cdots \\
=& 1 - \frac{e^{-x^2}}{\sqrt{\pi}}\sum_{n=0}^{N-1}(-1)^n\frac{(2n-1)!!}{2^n x^{2n+1}} + R_N(x).
\end{aligned}$$

where

$$R_N(x) = (-1)^N\frac{1}{\sqrt{\pi}}\frac{(2N-1)!!}{2^N}E_N(x).$$

This series diverges for any value of x because

$$\left|\frac{u_{n+1}}{u_n}\right| = \frac{(2n+1)!!}{2^{n+1}x^{2n+3}}\frac{2^n x^{2n+1}}{(2n-1)!!} = \frac{2n+1}{2x^2}$$

is larger than 1 when $2n+1 > 2x^2$. However,

$$\begin{aligned}|R_N(x)| &= C_N\left|\int_{x^2}^{\infty} s^{-N-1/2-1}e^{-s}ds\right| \\ &\le \frac{C_N}{x^{2N+1}}\int_{x^2}^{\infty} e^{-s}ds \\ &\le \frac{C_N e^{-x^2}}{x^{2N+1}}\end{aligned}$$

where $C_N = (2N-1)!!/(2^N\sqrt{\pi})$, so that

$$\text{erf}(x) - \left[1 - \frac{e^{-x^2}}{\sqrt{\pi}}\sum_{n=0}^{N-1}(-1)^n\frac{(2n-1)!!}{2^n x^{2n+1}}\right] = \mathcal{O}\left(\frac{e^{-x^2}}{x^{2N+1}}\right) \quad \text{as} \quad x \to \infty.$$

which means that the series

$$\text{erf}(x) \sim 1 - \frac{e^{-x^2}}{\sqrt{\pi}}\sum_{n=0}^{\infty}(-1)^n\frac{(2n-1)!!}{2^n x^{2n+1}} \qquad (*)$$

is an asymptotic expansion for $\text{erf}(x)$ as $x \to \infty$. Moreover, one could proceed to calculate $\text{erf}(3)$ for comparison and would find that to get the same accuracy it is sufficient just adding the first two terms in $(*)$. Unfortunately it is not possible to improve the approximation at will by adding more terms because the ratio between two successive terms of the series become larger than 1 as $n \ge x^2$. Hence, the asymptotic series $(*)$ has an optimal truncation for N equal to the largest integer smaller than x^2.

In this example we have encountered an asymptotic expansion of the form

$$f(x) \sim g(x)\sum_{n=0}^{\infty} a_n\phi_n(x) \quad \text{as} \quad x \to \infty,$$

where the function $g(x) \propto e^{-x^2}$ is bounded as $x \to \infty$. This reflects the non uniqueness of gauge functions: since $g(x)$ is bounded as $x \to \infty$ the functions $\tilde{\phi}_n(x) = g(x)\phi_n(x)$ form an asymptotic sequence as $x \to \infty$, thus we could equally well write $f(x) \sim \sum_{n=0}^{\infty} a_n\tilde{\phi}_n(x)$ as $x \to \infty$. This provides additional flexibility, but should be used with care because, as we saw in the previous example, differences between the results obtained with different asymptotic sequences might be numerically relevant for x not too close to point of interest.

Important result.

In general to find the **optimal truncation**, i.e., the one that gives the best approximation, compute the expression for the ratio between successive terms of the asymptotic series and, for your choice of x, find the maximum value of $n = N$ that satisfies $|u_{n+1}/u_n| < 1$. $N \equiv N(x)$ is the number of terms to be summed to obtain the best approximation for the chosen x. Once $N(x)$ is known, the error $R_N(x)$ can be determined.

V - Application of Optimal truncation.

We now go back to the Stieltjes Integral, for which we have found an asymptotic expansion. Now we wish to know how to find the optimal truncation when $\rho(t)$ is given. In our example $\rho(t) = e^{-t}$ (the simplest possible non-trivial choice). With this choice the coefficients of the series are the trusty old:

$$c_n = \int_0^\infty \rho(t) t^n \, dt = \int_0^\infty e^{-t} t^n \, dt,$$

that is $c_n = n!$, and the ratio between two successive terms, $|u_n/u_{n-1}| = nx$, becomes larger than 1 as $n > 1/x$ which means that the series does not converge for any $x > 0$. However, for any N we have shown that $R_N(x) = o(x^N)$ as $x \to 0$ (Translation: it is asymptotic but not convergent), and we can write

$$\int_0^\infty \frac{\rho(t)}{1 + xt} \, dt \sim \sum_{n=0}^\infty (-1)^n n! x^n \quad \text{as} \quad x \to 0^+$$

There is a good reason we put $x \to 0^+$ in this case: the lack of convergence of the series can be traced back to the fact that the original integral is not defined for $x < 0$ because the integrand has a non-integrable singularity at $t = 1/x$, and we know that to be convergent at a point $x = x_0$, a power series must converge in a disk of finite radius centered at said point, in this case $x = 0$.

Since the asymptotic power series does not converge, its partial sum does not provide an arbitrarily accurate approximation to the integral for any fixed $x > 0$. However, for any fixed N the error is a polynomial in x.

QUESTION: What is the optimal truncation of the series, that is, the truncation which yields the best approximation?

ANSWER: Because $|u_n/u_{n-1}| = nx$ becomes larger than 1 as $n > 1/x$, then the best approximation occurs at $N \sim 1/x$, for which $R_N(x) \sim (1/x)! x^{1/x}$. It is also possible to calculate the approximate error using the Stirling formula for small x (i.e. large N): $N! \sim \sqrt{2\pi N} N^N e^{-N}$. Then, the optimal truncation has an approximated error

$$R_N(x) \sim \sqrt{\frac{2\pi}{x}} e^{-1/x} \quad \text{as} \quad x \to 0.$$

Note the remarkable result that the optimal truncated sum is exponentially accurate.