Stationary Phase

I - Asymptotics

Let $f(t)$ and $\phi(t)$ be smooth functions, such that $\phi'(c) = 0$ at some point $c \in (a, b)$, and that $\phi'(t) \neq 0$ everywhere else in the closed interval. Moreover, assume that $\phi''(c) \neq 0$ and $f(c) \neq 0$. Thus, consider the behavior of the integral

$$I(\omega) = \int_a^b f(t) e^{i\omega\phi(t)}\ dt$$

for $|\omega| >> 1$ when $\phi''(c) > 0$ (the case $\phi''(c) < 0$ is analogous). We can rewrite

$$I(\omega) = e^{i\omega\phi(c)} \int_a^b f(t) e^{i\omega[\phi(t)-\phi(c)]}\ dt.$$

Note that the term $e^{i\omega[\phi(t)-\phi(c)]}$ is highly oscillatory for $t \neq c$ and $\omega >> 1$. The fast oscillations give rise to cancellations which in turn causes the integral to decay rapidly except in a small neighborhood of c. Thus, approximating $\phi(t) \simeq \phi(c) + \frac{1}{2}\phi''(c)(t-c)^2$ about the stationary point c, yields

$$\begin{aligned}
I(\omega) &\sim e^{i\omega\phi(c)} \int_{c-\epsilon}^{c+\epsilon} f(t) e^{i\omega[\phi(t)-\phi(c)]}\ dt \\
&\sim e^{i\omega\phi(c)} f(c) \int_{c-\epsilon}^{c+\epsilon} e^{i\frac{\omega}{2}\phi''(c)(t-c)^2}\ dt \\
&\sim e^{i\omega\phi(c)} f(c) \int_{-\infty}^{+\infty} e^{i\frac{\omega}{2}\phi''(c)(t-c)^2}\ dt.
\end{aligned}$$

With the change of variables $s = \left[\frac{\omega}{2}\phi''(c)\right]^{1/2}(t-c)$ we find

$$\begin{aligned}
I(\omega) &\sim e^{i\omega\phi(c)} f(c) \left(\frac{2}{\omega\phi''(c)}\right)^{1/2} \int_{-\infty}^{+\infty} e^{is^2}\ ds \\
&\sim e^{i\omega\phi(c)} f(c) \left(\frac{2\pi}{\omega\phi''(c)}\right)^{1/2} e^{i\pi/4} \quad \text{as} \quad |\omega| \to \infty.
\end{aligned}$$

II - Example

For a fixed integer n, the Bessel function of the first kind has the integral representation

$$\begin{aligned}
J_n(x) &= \int_0^1 \cos\left[n\pi t - x\sin(\pi t)\right]\ dt \\
&= Re\left[\int_0^1 e^{in\pi t} e^{-ix\sin(\pi t)}\ dt\right].
\end{aligned}$$

In this case we can set $f(t) = e^{in\pi t}$ and then, the phase is $\phi(t) = -\sin(\pi t)$. In the interval $[0, 1]$ the phase is stationary only at the interior point $c = 1/2$, for which $\phi(c) = -1$, and $\phi''(c) = \pi^2$. Thus,

$$
\begin{aligned}
J_n(x) &\sim Re\left[e^{in\pi/2}e^{-ix}\left(\frac{2}{x\pi}\right)^{1/2}e^{i\pi/4}\right] \\
&\sim \left(\frac{2}{x\pi}\right)^{1/2} Re\left[e^{-i(x-n\pi/2-\pi/4)}\right] \\
&\sim \left(\frac{2}{x\pi}\right)^{1/2} \cos\left(x - \frac{n\pi}{2} - \frac{\pi}{4}\right) \quad \text{as} \quad x \to \infty.
\end{aligned}
$$