Stokes phenomenon. Application to Airy function.

I - Stokes phenomenon

Suppose that $g(z)$ is the asymptotic representation of $f(z)$ as $z \to z_0$

$$f(z) \sim g(z) \quad \text{as} \quad z \to z_0.$$

From this expression it is unclear which path in the complex plane we are specifying as $z \to z_0$. When the function $g(z)$ is entire this is not an issue because any path would produce the same asymptotic behaviour. However, in many cases the asymptotic approximations are not entire functions, even when the true function is, and this produces an ambiguity that needs to be addressed. A good example of such a case are the solutions to the Airy equation.

For example, the asymptotic expressions for the solutions of the Airy equation are:

$$Ai(z) \sim \frac{1}{2\sqrt{\pi}} z^{-1/4} e^{-\frac{2}{3}z^{3/2}}, (*) \quad \text{and} \quad Bi(z) \sim \frac{1}{2\sqrt{\pi}} z^{-1/4} e^{\frac{2}{3}z^{3/2}}. (**)$$

which are multivalued functions with branch points. However, we know that $Ai(z)$ is an enitre function and its Taylor series converges for all finite values of $|z|$, and (*) only holds in some region of the plane. In the case of the $Bi(z)$, the fact that it grows exponentially along the real axis indicates that a possible approach is to restrict z such that

$$|\arg(z^{2/3})| < \frac{\pi}{2}, \implies \quad |\arg(z)| < \frac{\pi}{3}$$

Thus the sector of validity for $Bi(z)$ to have the behaviour in (*) is $|\arg(z)| < \pi/3$.

In general if $f(z) \sim g(z)$ as $z \to z_0$, then $f(z) - g(z) = o(g(z))$ as $z \to z_0$, and thus $f(z) = g(z) + (f(z) - g(z))$. When this condition is satisfied, it is said that when z lies in a certain sector where $g(z)$ is dominant and $f(z) - g(z)$ is small, subdominant or recessive. As the values of z approach the boundaries of the sector, $f(z) - g(z)$ is not small any more. In fact, outside the sector $f - g$ becomes larger than g. This effect is called the Stokes Phenomenon after Stokes(1857) who first observed it. The edges of the sector, that is the place where the change of behaviour occur, are called Stokes Lines.

As an application we now consider some of the second order ODEs we studied earlier, where we found solutions of the form

$$y \sim e^{S_1(z)} \quad \text{and} \quad y \sim e^{S_2(z)} \quad \text{as} \quad z \to z_0.$$

To find the Stokes lines it is necessary to locate the curve where the two solutions have comparable size. For our exponential solutions, these lines are determined by the relationship:

$$Re(S_1(z) - S_2(z)) = 0.$$

The curves/lines where the solutions differ the most are called anti-Stokes lines,and are determined (in most cases) by the relationship

$$Im(S_1(z) - S_2(z)) = 0.$$

EXAMPLE: consider the Airy functions. The Stokes lines are given by

$$Re(z^{2/3}) = 0$$

Which corresponds to

$$\arg(z) = \pm\frac{\pi}{3}, \pi \quad \text{as} \quad |z| \to \infty.$$

The function $Bi(z)$ has the behaviour (**) valid only in the sector $|\arg(z)| < \pi/3$. However for $Ai(z)$ it can be shown from the integral representation that the behaviour (*) as $z \to \infty$ holds for the much larger sector $|\arg(z)| < \pi$.

II - Airy functions: Linear relations. Regions of validity of their asymptotics.

Symmetry of the Airy equation: The change of variables $t = \omega z$, where $\omega = e^{-2i\pi 3}$ is a cubic root of unity transforms the original Airy equation into:

$$\frac{d^2y}{dt^2} - ty = 0$$

hence, $y = Ai(\omega z)$ is also a solution of Airy's equation. Similarly $Ai(z)$, $Ai(\omega z)$, $Ai(\omega^2 z)$, and $Bi(z)$ are all solutions of Airy's equation. The Airy equation is only second order, hence some of these solutions can be written as linear cominations of each other:

$$Ai(z) = CAi(\omega z) + BAi(\omega^2 z)$$

Using the Taylor series for $Ai(z)$, and comparing the coefficents of the terms z^0 and z^1 it is possble to find B and C:

$$\begin{aligned} C + B =& 1, \quad \text{and} \quad C\omega + B\omega^2 = 1. \\ C =& -\omega \quad \text{and} \quad B = -\omega^2. \end{aligned}$$

Then, the expression for $Ai(z)$ becomes

$$Ai(z) = -\omega Ai(\omega z) - \omega^2 Ai(\omega^2 z),$$

and similarly

$$Bi(z) = i\omega Ai(\omega z) - i\omega^2 Ai(\omega^2 z).$$

These relationships, together with

$$Ai(z) \sim \frac{1}{2\sqrt{\pi}} z^{-1/4} e^{-\frac{2}{3}z^{3/2}} \sum_{n=0}^{\infty} (-1)^n c_n z^{-3n/2} \quad \text{for} \quad |\arg(z)| < \pi$$

where

$$c_n = \frac{1}{2\pi}\left(\frac{3}{4}\right)^n \frac{\Gamma(n+\frac{5}{6})\Gamma(n+\frac{1}{6})}{n!},$$

can now be used to obtain asymptotic expansions for $Ai(z)$, $Bi(z)$ valid in other sectors of the complex plane.

To be able to replace the asymptotic behaviour into the linear combination for $Ai(z)$ it is necessary to make them compatible by requiring that:

$$-\pi < \arg(\omega z) < \pi \quad \text{and} \quad -\pi < \arg(\omega^2 z) < \pi$$

Thus, for $\pi/3 < \arg(z) < 5\pi/3$

$$\begin{aligned} Ai(z) \sim & -\omega \frac{1}{2\sqrt{\pi}}(\omega z)^{-1/4} e^{-\frac{2}{3}(\omega z)^{3/2}} \sum_{n=0}^{\infty}(-1)^n c_n (\omega z)^{-3n/2} \\ & -\omega^2 \frac{1}{2\sqrt{\pi}}(\omega^2 z)^{-1/4} e^{-\frac{2}{3}(\omega^2 z)^{3/2}} \sum_{n=0}^{\infty}(-1)^n c_n (\omega^2 z)^{-3n/2}. \end{aligned}$$

After replacing the value for ω we obtain, for the sector $\pi/3 < \arg(z) < 5\pi/3$,

$$Ai(z) \sim \frac{1}{2\sqrt{\pi}} z^{-1/4} e^{-\frac{2}{3}z^{3/2}} \sum_{n=0}^{\infty}(-1)^n c_n z^{-3n/2} + \frac{i}{2\sqrt{\pi}} z^{-1/4} e^{\frac{2}{3}z^{3/2}} \sum_{n=0}^{\infty}(-1)^n c_n z^{-3n/2}$$

Using the reflection symmetry of the solutions: $Ai(z) = [Ai(z^\dagger)]^\dagger$ we find the other possible asymptotic solution for the sector $-5\pi/3 < \arg(z) < -\pi/3$:

$$Ai(z) \sim \frac{1}{2\sqrt{\pi}} z^{-1/4} e^{-\frac{2}{3}z^{3/2}} \sum_{n=0}^{\infty}(-1)^n c_n z^{-3n/2} - \frac{i}{2\sqrt{\pi}} z^{-1/4} e^{\frac{2}{3}z^{3/2}} \sum_{n=0}^{\infty}(-1)^n c_n z^{-3n/2},$$

which is the solution that corresponds to the other cubic root of unity, that is $\omega = e^{2i\pi 3}$.
