VIII. *On the numerical Calculation of a Class of Definite Integrals and Infinite Series.* By G. G. STOKES, M.A., *Fellow of Pembroke College, and Lucasian Professor of Mathematics in the University of Cambridge.*

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IN a paper "On the Intensity of Light in the neighbourhood of a Caustic*," Mr. Airy the Astronomer Royal has shown that the undulatory theory leads to an expression for the illumination involving the square of the definite integral $\int_0^\infty \cos\frac{\pi}{2}(w^3 - mw)\,dw$, where m is proportional to the perpendicular distance of the point considered from the caustic, and is reckoned positive towards the illuminated side. Mr. Airy has also given a table of the numerical values of the above integral extending from $m = -4$ to $m = +4$, at intervals of 0.2, which was calculated by the method of quadratures. In a Supplement to the same paper† the table has been re-calculated by means of a series according to ascending powers of m, and extended to $m = \pm 5.6$. The series is convergent for all values of m, however great, but when m is at all large the calculation becomes exceedingly laborious. Thus, for the latter part of the table Mr. Airy was obliged to employ 10-figure logarithms, and even these were not sufficient for carrying the table further. Yet this table gives only the first two roots of the equation $W = 0$, W denoting the definite integral, which answer to the theoretical places of the first two dark bands in a system of spurious rainbows, whereas Professor Miller was able to observe 30 of these bands. To attempt the computation of 30 roots of the equation $W = 0$ by means of the ascending series would be quite out of the question, on account of the enormous length to which the numerical calculation would run.

After many trials I at last succeeded in putting Mr. Airy's integral under a form from which its numerical value can be calculated with extreme facility when m is large, whether positive or negative, or even moderately large. Moreover the form of the expression points out, without any numerical calculation, the law of the progress of the function when m is large. It is very easy to deduce from this expression a formula which gives the i^{th} root of the equation $W = 0$ with hardly any numerical calculation, except what arises from merely passing from $\left(\frac{m}{3}\right)^{\frac{3}{2}}$, the quantity given immediately, to m itself.

The ascending series in which W may be developed belongs to a class of series which are of constant occurrence in physical questions. These series, like the expansions of ϵ^{-x}, $\sin x$, $\cos x$, are convergent for all values of the variable x, however great, and are easily calculated numerically when x is small, but are extremely inconvenient for calculation when x is large,

* *Camb. Phil. Trans.* Vol. VI. p. 379.

† Vol. VIII. p. 595.

give no indication of the law of progress of the function, and do not even make known what the function becomes when $x = \infty$. These series present themselves, sometimes as developements of definite integrals to which we are led in the first instance in the solution of physical problems, sometimes as the integrals of linear differential equations which do not admit of integration in finite terms. Now the method which I have employed in the case of the integral W appears to be of very general application to series of this class. I shall attempt here to give some sort of idea of it, but it does not well admit of being described in general terms, and it will be best understood from examples.

Suppose then that we have got a series of this class, and let the series be denoted by y or $f(x)$, the variable according to ascending powers of which it proceeds being denoted by x. It will generally be easy to eliminate the transcendental function $f(x)$ between the equation $y = f(x)$ and its derivatives, and so form a linear differential equation in y, the coefficients in which involve powers of x. This step is of course unnecessary if the differential equation is what presented itself in the first instance, the series being only an integral of it. Now by taking the terms of this differential equation in pairs, much as in Lagrange's method of expanding implicit functions which is given by Lacroix*, we shall easily find what terms are of most importance when x is large: but this step will be best understood from examples. In this way we shall be led to assume for the integral a circular or exponential function multiplied by a series according to descending powers of x, in which the coefficients and indices are both arbitrary. The differential equation will determine the indices, and likewise the coefficients in terms of the first, which remains arbitrary. We shall thus have the complete integral of the differential equation, expressed in a form which admits of ready computation when x is large, but containing a certain number of arbitrary constants, according to the order of the equation, which have yet to be determined.

For this purpose it appears to be generally requisite to put the infinite series under the form of a definite integral, if the series be not itself the developement of such an integral which presented itself in the first instance. We must now endeavour to determine by means of this integral the leading term in $f(x)$ for indefinitely large values of x, a process which will be rendered more easy by our previous knowledge of the *form* of the term in question, which is given by the integral of the differential equation. The arbitrary constants will then be determined by comparing the integral just mentioned with the leading term in $f(x)$.

There are two steps of the process in which the mode of proceeding must depend on the particular example to which the method is applied. These are, first, the expression of the ascending series by means of a definite integral, and secondly, the determination thereby of the leading term in $f(x)$ for indefinitely large values of x. Should either of these steps be found impracticable, the method does not on that account fall to the ground. The arbitrary constants may still be determined, though with more trouble and far less elegance, by calculating the numerical value of $f(x)$ for one or more values of x, according to the number of arbitrary constants to be determined, from the ascending and descending series separately, and equating the results.

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* *Traité du Calcul*, &c. Tom. I. p. 104.

In this paper I have given three examples of the method just described. The first relates to the integral W, the second to an infinite series which occurs in a great many physical investigations, the third to the integral which occurs in the case of diffraction with a circular aperture in front of a lens. The first example is a good deal the most difficult. Should the reader wish to see an application of the method without involving himself in the difficulties of the first example, he is requested to turn to the second and third examples.

FIRST EXAMPLE.

1. Let it be required to calculate the integral

$$W = \int_0^\infty \cos \frac{\pi}{2} (w^3 - mw)\, dw \quad . \quad . \quad . \quad (1)$$

for different values of m, especially for large values, whether positive or negative, and in particular to calculate the roots of the equation $W = 0$.

2. Consider the integral

$$u = \int_0^\infty \epsilon^{-(\cos 3\theta + \sqrt{-1}\sin 3\theta)(x^3 - nx)}\, dx, \quad . \quad . \quad (2)$$

where θ is supposed to lie between $-\frac{\pi}{6}$ and $+\frac{\pi}{6}$, in order that the integral may be convergent.

Putting

$$x = (\cos\theta - \sqrt{-1}\sin\theta)\, z,$$

we get $dx = (\cos\theta - \sqrt{-1}\sin\theta)\, dz$, and the limits of z are 0 and ∞; whence, writing for shortness

$$p = (\cos 2\theta + \sqrt{-1}\sin 2\theta)\, n, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

we get

$$u = (\cos\theta - \sqrt{-1}\sin\theta) \int_0^\infty \epsilon^{-(z^3 - pz)}\, dz^*. \quad . \quad . \quad . \quad . \quad (4)$$

3. Let now θ, which hitherto has been supposed less than $\frac{\pi}{6}$, become equal to $\frac{\pi}{6}$. The integral obtained from (2) by putting $\theta = \frac{\pi}{6}$ under the integral sign may readily be proved to

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* The legitimacy of this transformation rests on the theorem that if $f(x)$ be a continuous function of x, which does not become infinite for any real or imaginary, but finite, value of x, we shall obtain the same result for the integral of $f(x)\,dx$ between two given real or imaginary limits through whatever series of real or imaginary values we make x pass from the inferior to the superior limit. It is unnecessary here to enunciate the theorem which applies to the case in which $f(x)$ becomes infinite for one or more real or imaginary values of x. In the present case the limits of x are 0 and real infinity, and accordingly we may first integrate with respect to z from 0 to a large real quantity z_1, θ' (which is supposed to be written for θ in the expression for x) being constant, then leave z equal to z_1, make θ' vary, and integrate from θ to 0, and lastly make z_1 infinite. But it may be proved without difficulty, (and the proof may be put in a formal shape as in Art. 8,) that the second integral vanishes when z_1 becomes infinite, and consequently we have only to integrate with respect to z from 0 to real infinity.

be convergent. But this is not sufficient in order that we may be at liberty to assert the equality of the results obtained from (2), (4) by putting $\theta = \frac{\pi}{6}$ before integration. It is moreover necessary that the convergency of the integral (2) should not become infinitely slow when θ approaches indefinitely to $\frac{\pi}{6}$, in other words, that if X be the superior limit to which we must integrate in order to render the remainder, or rather its modulus, less than a given quantity which may be as small as we please, X should not become infinite when θ becomes equal to $\frac{\pi}{6}$*. This may be readily proved in the present case, since the integral (2) is even more convergent than the integral

$$\int_0^\infty \epsilon^{-\sqrt{-1}\sin 3\theta\,(x^3 - nx)}\,dx,$$

which may be readily proved to be convergent.

Putting then $\theta = \frac{\pi}{6}$ in (2) and (4), we get

$$u = \int_0^\infty \cos(x^3 - nx)\,dx - \sqrt{-1}\int_0^\infty \sin(x^3 - nx)\,dx, \quad . \quad . \quad (5)$$

$$u = \left(\cos\frac{\pi}{6} - \sqrt{-1}\sin\frac{\pi}{6}\right)\int_0^\infty \epsilon^{-(z^3 - pz)}\,dz, \quad . \quad . \quad . \quad (6)$$

where

$$p = \left(\cos\frac{\pi}{3} + \sqrt{-1}\sin\frac{\pi}{3}\right)n. \quad . \quad . \quad . \quad . \quad . \quad (7)$$

Let

$$u = U - \sqrt{-1}\,U',$$

and in the expression for U got from (5) put

$$x = \left(\frac{\pi}{2}\right)^{\frac{1}{3}} w, \qquad n = \left(\frac{\pi}{2}\right)^{\frac{2}{3}} m; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

then we get

$$W = \left(\frac{\pi}{2}\right)^{-\frac{1}{3}} U. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

4. By the transformation of u from the form (5) to the form (6), we are enabled to differentiate it as often as we please with respect to n by merely differentiating under the integral sign. By expanding the exponential ϵ^{pz} in (6) we should obtain u, and therefore U, in a series according to ascending powers of n. This series is already given in Mr. Airy's Supplement. It is always convergent, but is not convenient for numerical calculation when n is large.

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* See Section III. of a paper, "On the Critical Values of the sums of Periodic Series." *Camb. Phil. Trans.* Vol. VIII. p. 561.

We get from (6)

$$\frac{d^2u}{dp^2} - \frac{pu}{3} = \left(\cos\frac{\pi}{6} - \sqrt{-1}\sin\frac{\pi}{6}\right)\int_0^\infty \epsilon^{-(z^3-pz)}\left(z^2-\frac{p}{3}\right)dz = \frac{1}{3}\left(\cos\frac{\pi}{6} - \sqrt{-1}\sin\frac{\pi}{6}\right),$$

which becomes by (7)

$$\frac{d^2u}{dn^2} + \frac{n}{3}u = \tfrac{1}{3}\sqrt{-1}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

Equating to zero the real part of the first member of this equation, we get

$$\frac{d^2U}{dn^2} + \frac{n}{3}U = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

5. We might integrate this equation by series according to ascending powers of n, and we should thus get, after determining the arbitrary constants, the series which have been already mentioned. What is required at present is, to obtain for U an expression which shall be convenient when n is large.

The form of the differential equation (11) already indicates the general form of U for large values of n. For, suppose n large and positive, and let it receive a small increment δn. Then the proportionate increment of the coefficient $\frac{n}{3}$ will be very small; and if we regard this coefficient as constant, and δn as variable, we shall get for the integral of (11)

$$U = N\cos\left\{\sqrt{\left(\frac{n}{3}\right)}.\delta n\right\} + N'\sin\left\{\sqrt{\left(\frac{n}{3}\right)}.\delta n\right\}, \quad . \quad . \quad (12)$$

where N, N' are regarded as constants, δn being small, which does not prevent them from being in the true integral of (11) slowly varying functions of n. The approximate integral (12) points out the existence of circular functions such as $\cos f(n)$, $\sin f(n)$ in the true integral; and since $\sqrt{\left(\frac{n}{3}\right)}.\delta n$ must be the small increment of $f(n)$, we get $f(n) = \frac{2}{3}\sqrt{\frac{n^3}{3}}$, omitting the constant, which it is unnecessary to add. When n is negative, and equal to $-n'$, the same reasoning would point to the existence of exponentials with $\pm\frac{2}{3}\sqrt{\frac{n'^3}{3}}$ in the index. Of course the exponential with a positive index will not appear in the particular integral of (11) with which we are concerned, but both exponentials would occur in the complete integral. Whether n be positive or negative, we may, if we please, employ exponentials, which will be real or imaginary as the case may be.

6. Assume then to satisfy (11)

$$U = \epsilon^{\frac{2}{3}\sqrt{-\frac{n^3}{3}}}\{An^\alpha + Bn^\beta + Cn^\gamma + \ldots\}^*, \quad . \quad . \quad . \quad (13)$$

* The idea of multiplying the circular functions by a series according to descending powers of n was suggested to me by

where $A, B, C \ldots \alpha, \beta, \gamma \ldots$ are constants which have to be determined. Differentiating, and substituting in (11), we get

$$\alpha(\alpha-1)An^{\alpha-2}+\beta(\beta-1)Bn^{\beta-2}+\ldots$$
$$+\frac{\sqrt{-1}}{2\sqrt{3}}\left\{(4\alpha+1)An^{\alpha-\frac{1}{2}}+(4\beta+1)Bn^{\beta-\frac{1}{2}}+\ldots\right\}=0.$$

As we want a series according to descending powers of n, we must put

$$4\alpha+1=0,\quad \beta=\alpha-\tfrac{3}{2},\quad \gamma=\beta-\tfrac{3}{2}\ldots$$

$$B=2\sqrt{-3}\;\frac{\alpha(\alpha-1)}{4\beta+1}A,\quad C=2\sqrt{-3}\;\frac{\beta(\beta-1)}{4\gamma+1}B\ldots\ldots$$

whence

$$U=An^{-\frac{1}{4}}\epsilon^{\frac{2}{3}\sqrt{-\frac{n^3}{3}}}\left\{1-\frac{1.5}{1}\,\frac{\sqrt{-1}}{16\sqrt{(3n^3)}}+\frac{1.5.7.11}{1.2}\left(\frac{\sqrt{-1}}{16\sqrt{(3n^3)}}\right)^2\right.$$
$$\left.-\frac{1.5.7.11.13.17}{1.2.3}\left(\frac{\sqrt{-1}}{16\sqrt{(3n^3)}}\right)^3+\ldots\right\}. \quad . \quad . \quad (14)$$

By changing the sign of $\sqrt{-1}$ both in the index of ϵ and in the series, writing B for A, and adding together the results, we shall obtain the complete integral of (11) with its two arbitrary constants. The integral will have different forms according as n is positive or negative.

First, suppose n positive. Putting the function of n of which A is the coefficient at the second side of (14) under the form $P+\sqrt{-1}\,Q$, and observing that an expression of the form

$$A(P+\sqrt{-1}\,Q)+B(P-\sqrt{-1}\,Q),$$

where A and B are imaginary arbitrary constants, and which is supposed to be real, is equivalent to $AP+BQ$, where A and B are real arbitrary constants, we get

$$U=An^{-\frac{1}{4}}\left(R\cos\frac{2}{3}\sqrt{\frac{n^3}{3}}+S\sin\frac{2}{3}\sqrt{\frac{n^3}{3}}\right)+Bn^{-\frac{1}{4}}\left(R\sin\frac{2}{3}\sqrt{\frac{n^3}{3}}-S\cos\frac{2}{3}\sqrt{\frac{n^3}{3}}\right), \quad . \quad (15)$$

where

$$\left.\begin{aligned}R&=1-\frac{1.5.7.11}{1.2.16^2.3n^3}+\frac{1.5.7.11.13.17.19.23}{1.2.3.4.16^4.3^2n^6}\;\ldots\ldots\\ S&=\frac{1.5}{1.16(3n^3)^{\frac{1}{2}}}-\frac{1.5.7.11.13.17}{1.2.3.16^3(3n^3)^{\frac{3}{2}}}+\ldots\ldots\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (16)$$

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seeing in Moigno's *Repertoire d'optique moderne*, p. 189, the following formulæ which M. Cauchy has given for the calculation of Fresnel's integrals for large, or moderately large, values of the superior limit:

$$\int_0^m \cos\frac{\pi}{2}z^2\,dz=\frac{1}{2}-N\cos\frac{\pi}{2}m^2+M\sin\frac{\pi}{2}m^2;$$
$$\int_0^m \sin\frac{\pi}{2}z^2\,dz=\frac{1}{2}-M\cos\frac{\pi}{2}m^2-N\sin\frac{\pi}{2}m^2;$$

where

$$M=\frac{1}{m\pi}-\frac{1.3}{m^5\pi^3}+\frac{1.3.5.7}{m^9\pi^5}-\ldots;\quad N=\frac{1}{m^3\pi^2}-\frac{1.3.5}{m^7\pi^4}+\ldots.$$

The demonstration of these formulæ will be found in the 15th Volume of the *Comptes Rendus*, pp. 554 and 573. They may be readily obtained by putting $\pi z^2=2x$, and integrating by parts between the limits $\frac{1}{2}\pi m^2$ and ∞ of x.

Secondly, suppose n negative, and equal to $-n'$. Then, writing $-n'$ for n in (14), and changing the arbitrary constant, and the sign of the radical, we get

$$U = Cn'^{-\frac{1}{4}} \epsilon^{-\frac{2}{3}\sqrt{\frac{n'^3}{3}}} \left\{1 - \frac{1 \,.\, 5}{1 \,.\, 16\,(3n^3)^{\frac{1}{2}}} + \frac{1 \,.\, 5 \,.\, 7 \,.\, 11}{1 \,.\, 2 \,.\, 16 \,.\, 3n^3} - \ldots\ldots\right\}. \quad . \quad . \quad . \qquad (17)$$

It is needless to write down the part of the complete integral of (11) which involves an exponential with a positive index, because, as has been already remarked, it does not appear in the particular integral with which we are concerned.

7. When n or n' is at all large, the series (16) or (17) are at first rapidly convergent, but they are ultimately in all cases hypergeometrically divergent. Notwithstanding this divergence, we may employ the series in numerical calculation, provided we do not take in the divergent terms. The employment of the series may be justified by the following considerations.

Suppose that we stop after taking a finite number of terms of the series (16) or (17), the terms about where we stop being so small that we may regard them as insensible; and let U_1 be the result so obtained. From the mode in which the constants $A, B, C, \ldots\ \alpha, \beta, \gamma \ldots$ in (13) were determined, it is evident that if we form the expression

$$\frac{d^2 U_1}{dn^2} + \frac{n}{3} U_1, \text{ or } \frac{d^2 U_1}{dn'^2} - \frac{n'}{3} U_1,$$

according as n is positive or negative, the terms will destroy each other, except one or two at the end, which remain undestroyed. These terms will be of the same order of magnitude as the terms at the part of the series (16) or (17) where we stopped, and therefore will be insensible for the value of n or n' for which we are calculating the series numerically, and, much more, for all superior values. Suppose the arbitrary constants A, B in (16) determined by means of the ultimate form of U for $n = \infty$, and C in (17) by means of the ultimate form of U for $n' = \infty$. Then U_1 satisfies exactly a differential equation which differs from (11) by having the zero at the second side replaced by a quantity which is insensible for the value of n or n' with which we are at work, and which is still smaller for values comprised between that and the particular value, (namely ∞,) by means of which the arbitrary constants were determined so as to make U_1 and U agree. Hence U_1 will be a near approximation to U. But if we went too far in the series (16) or (17), so as, after having gone through the insensible terms, to take in some terms which were not insensible, the differential equation which U_1 would satisfy exactly would differ sensibly from (11), and the value of U_1 obtained would be faulty.

8. It remains to determine the arbitrary constants A, B, C. For this purpose consider the integral

$$Q = \int_0^\infty \epsilon^{-x^3 + 3q^2 x}\, dx, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (18)$$

where q is any imaginary quantity whose amplitude does not lie beyond the limits $-\frac{\pi}{6}$ and

$+\frac{\pi}{6}$. Since the quantity under the integral sign is finite and continuous for all finite values of x, we may, without affecting the result, make x pass from its initial value 0 to its final value ∞ through a series of imaginary values. Let then $x = q + y$, and we get

$$Q = \epsilon^{2q^3} \int_{-q}^{\infty} \epsilon^{-y^3 - 3qy^2} dy,$$

where the values through which y passes in the integration are not restricted to be such as to render x real. Putting $y = (3q)^{-\frac{1}{2}} t$, where that value of the radical is supposed to be taken which has the smallest amplitude, we get

$$Q = (3q)^{-\frac{1}{2}} \epsilon^{2q^3} \int \epsilon^{-(3q)^{-\frac{3}{2}} t^3 - t^2} dt. \quad . \quad . \quad . \quad . \quad (19)$$

The limits of t are $-3^{\frac{1}{2}} q^{\frac{3}{2}}$ and an imaginary quantity with an infinite modulus and an amplitude equal to $\frac{1}{2}\alpha$, where α denotes the amplitude of q. But we may if we please integrate up to a real quantity ρ, and then, putting $t = \rho \epsilon^{\theta\sqrt{-1}}$, and leaving ρ constant, integrate with respect to θ from 0 to $\frac{1}{2}\alpha$, and lastly put $\rho = \infty$. The first part of the integral will be evidently convergent at the limit ∞, since the amplitude of the coefficient of t^3 in the index does not lie beyond the limits $-\frac{\pi}{2}$ and $+\frac{\pi}{2}$; and calling the two parts of the integral with respect to t in (19) T, T_4, we get

$$T = \int_{-3^{\frac{1}{2}} q^{\frac{3}{2}}}^{\infty} \epsilon^{-(3q)^{-\frac{3}{2}} t^3 - t^2} dt, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (20)$$

$$T_4 = \text{limit}_{(\rho=\infty)}\, \rho\sqrt{-1} \int_0^{\frac{1}{2}\alpha} \epsilon^{-(3q)^{-\frac{3}{2}} \rho^3 \epsilon^{3\theta\sqrt{-1}} - \rho^2 \epsilon^{2\theta\sqrt{-1}} + \theta\sqrt{-1}} d\theta. \quad . \quad (21)$$

We shall evidently obtain a superior limit to either the real or the imaginary part of T_4 by reducing the expression under the integral sign to its modulus. The modulus is $\epsilon^{-\Theta}$ where

$$\Theta = (3c)^{-\frac{3}{2}} \rho^3 \cos(3\theta - \tfrac{3}{2}\alpha) + \rho^2 \cos 2\theta,$$

c being the modulus of q. The first term in this expression is never negative, being only reduced to zero in the particular case in which $\theta = 0$ and $\alpha = \pm\frac{\pi}{6}$. The second term is never less than $\rho^2 \cos\frac{\pi}{3}$ or $\frac{1}{2}\rho^2$, and is in general greater. Hence both the real and the imaginary parts of the expression of which T_4 is the limit are numerically less that $\frac{1}{2}\alpha\rho\epsilon^{-\frac{1}{2}\rho^2}$, which vanishes when $\rho = \infty$, and therefore $T_4 = 0$. Hence we have rigorously

$$Q = (3q)^{-\frac{1}{2}} \epsilon^{2q^3} T \quad . \quad . \quad . \quad . \quad . \quad . \quad (22)$$

Let us now seek the limit to which T tends when c becomes infinite. For this purpose divide the integral T into three parts T_1, T_2, T_3, where T_1 is the integral taken from $-3^{\frac{1}{2}} q^{\frac{3}{2}}$ to a real negative quantity $-a$, T_2 from $-a$ to a real positive quantity $+b$, and T_3 from b to ∞; and suppose c first to become infinite, a and b remaining constant, and lastly make a and b infinite.

Changing the sign of t in T_1, and the order of the limits, we get

$$T_1 = \int_a^{3^{\frac{1}{2}} c^{\frac{3}{2}}} \epsilon^{(3q)^{-\frac{1}{2}} t^3 - t^2} dt.$$

Put $t = \rho\epsilon^{\theta\sqrt{-1}}$. Then we may integrate first from $\rho = a$ to $\rho = 3^{\frac{1}{2}} c^{\frac{3}{2}}$ while θ remains equal to 0, and afterwards from $\theta = 0$ to $\theta = \frac{3}{2}\alpha$ while ρ remains equal to $3^{\frac{1}{2}} c^{\frac{3}{2}}$. Let the two parts of the integral be denoted by T', T''. We shall evidently obtain a superior limit to T' by making the following changes in the integral: first, replacing the quantity under the integral sign by its modulus; secondly, replacing t^3 in the index by the product of t^2 and the greatest value (namely $3^{\frac{1}{2}} c^{\frac{3}{2}}$) which t receives in the integration; thirdly, replacing a by the smallest quantity (namely 0) to which it can be equal, and, fourthly, extending the superior limit to ∞. Hence the real and imaginary parts of T' are both numerically less than $\int_a^{\infty} \epsilon^{-\frac{2}{3}t^2} dt$, a quantity which vanishes in the limit, when a becomes infinite.

We shall obtain a superior limit to the real or imaginary part of T'' by reducing the quantity under the integral sign to its modulus, and omitting $\sqrt{-1}$ in the coefficient. Hence L will be such a limit if

$$L = 3^{\frac{1}{2}} c^{\frac{3}{2}} \int_0^{\frac{3}{2}\alpha} \epsilon^{-c^3 f(\theta)} d\theta, \text{ where } f(\theta) = 3 \cos 2\theta - \cos(3\theta - \tfrac{3}{2}\alpha).$$

We may evidently suppose α to be positive, if not equal to zero, since the case in which it is negative may be reduced to the case in which it is positive by changing the signs of α and θ. When $\theta = \dfrac{\pi}{6}$, the first term in $f(\theta)$ is equal to $\frac{3}{2}$, which, being greater than 1, determines the sign of the whole, and therefore $f(\theta)$ is positive; and $f(\theta)$ is evidently positive from $\theta = 0$ to $\theta = \dfrac{\pi}{6}$, since for such values $\cos 2\theta > \frac{1}{2}$. Also in general $f'(\theta) = -6 \sin 2\theta + 3 \sin(3\theta - \frac{3}{2}\alpha)$, which is evidently positive from $\theta = \dfrac{\pi}{6}$ to $\theta = \dfrac{\pi}{4}$, and the latter is the largest value we need consider, being the extreme value of θ when α has its extreme value $\dfrac{\pi}{6}$. When θ has its extreme value $\frac{3}{2}\alpha$, $f(\theta) = 2 \cos 3\alpha$, which is positive when $\alpha < \dfrac{\pi}{6}$, and vanishes when $\alpha = \dfrac{\pi}{6}$. Hence $f(\theta)$ is positive when $\theta < \frac{3}{2}\alpha$; for it has been shewn to be positive when $\theta < \dfrac{\pi}{6}$, which meets the case in which $\alpha < \dfrac{\pi}{9}$ or $= \dfrac{\pi}{9}$, and to be constantly decreasing from $\theta = \dfrac{\pi}{6}$ to $\theta = \frac{3}{2}\alpha$, which meets the case in which $\theta > \dfrac{\pi}{9}$. Hence when $\alpha < \dfrac{\pi}{6}$ the limit of L for $c = \infty$ is zero, inasmuch as the coefficient of c^3 in the index of ϵ is negative and finite; and when $\alpha = \dfrac{\pi}{6}$ the same is true, for the same reason, if it be not for a range of integration lying as near as we please to the superior limit. In this case put for shortness $f(\theta) = \delta$, regard $\frac{3}{2}a - \theta$ as a function of δ, $F(\delta)$, and integrate from $\delta = 0$ to $\delta = \beta$, where β is a constant which may be as small as we please. By what precedes, $F'(\delta)$ will be finite in the integration, and may be made as nearly as we please equal to the constant $F'(0)$ by diminishing β. Hence the integral ultimately becomes $3^{\frac{1}{2}}F'(0)c^{\frac{3}{2}}\int_0^\beta \epsilon^{-c^3\delta}d\delta$, which vanishes when c becomes infinite. Hence the limit of T_1 is zero.

We have evidently

$$T_3 < \int_b^\infty \epsilon^{-t^2}dt,$$

which vanishes when b becomes infinite. Hence the limit of T is equal to that of T_2. Now making c first infinite and afterwards a and b, we get

$$\text{limit of } T_2 = \text{limit of } \int_{-a}^{b} \epsilon^{-t^2}dt = \int_{-\infty}^{+\infty} \epsilon^{-t^2}dt = \sqrt{\pi},$$

and therefore we have ultimately, for very large values of c,

$$Q = \left(\frac{\pi}{3q}\right)^{\frac{1}{2}} \epsilon^{2q^3}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (22)$$

In order to apply this expression to the integral u given by (6) we must put

$$3q^2 = n\epsilon^{\frac{\pi}{3}\sqrt{-1}}, \text{ whence } q = \left(\frac{n}{3}\right)^{\frac{1}{2}} \epsilon^{\frac{\pi}{6}\sqrt{-1}},\ \epsilon^{-\frac{\pi}{6}\sqrt{-1}}\left(\frac{\pi}{3q}\right)^{\frac{1}{2}} = \frac{\pi^{\frac{1}{2}}}{(3n)^{\frac{1}{4}}}\epsilon^{-\frac{\pi}{4}\sqrt{-1}},\ 2q^3 = 2\left(\frac{n}{3}\right)^{\frac{3}{2}}\sqrt{-1},$$

whence we get ultimately

$$u = \frac{\pi^{\frac{1}{2}}}{(3n)^{\frac{1}{4}}}\epsilon^{\left\{2\left(\frac{n}{3}\right)^{\frac{3}{2}} - \frac{\pi}{4}\right\}\sqrt{-1}}, \qquad U = \frac{\pi^{\frac{1}{2}}}{(3n)^{\frac{1}{4}}}\cos\left\{2\left(\frac{n}{3}\right)^{\frac{3}{2}} - \frac{\pi}{4}\right\}^{*} \quad . \quad (23)$$

Comparing with (15) we get

$$A = B = \frac{\pi^{\frac{1}{2}}}{2^{\frac{1}{2}}3^{\frac{1}{4}}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (24)$$

9. We cannot make n pass from positive to negative through a series of real values, so long as we employ the series according to descending powers, because these series become illusory when n is small. When n is imaginary we cannot speak of the integrals which appear at the right hand side of (5), because the exponential with a positive index which would appear under the integral signs would render each of these integrals divergent. If however we take equation (6) as the definition of u, and suppose U always derived from u by changing the sign of $\sqrt{-1}$ in the coefficient of the integral and in the value of p, but not in the expression for n, and taking half the sum of the results, we may regard u and U as certain functions of n whether n be real or imaginary. According to this definition, the series involving ascending

* This result might also have been obtained from the integral U in its original shape, namely, $\int_0^\infty \cos(x^3 - nx)\,dx$, by a method similar to that employed in Art. 21. If x_1 be the positive value of x which renders $x^3 - nx$ a minimum, we have $x_1 = 3^{-\frac{1}{2}}n^{\frac{1}{2}}$. Let the integral U be divided into three parts, by integrating separately from $x = 0$ to $x = x_1 - a$, from $x = x_1 - a$ to $x = x_1 + b$, and from $x = x_1 + b$ to $x = \infty$; then make n infinite while a and b remain finite, and lastly, let a and b vanish. In this manner the second of equations (23) will be obtained, by the assistance of the known formulæ

$$\int_{-\infty}^\infty \cos x^2\,dx = \int_{-\infty}^\infty \sin x^2\,dx = 2^{-\frac{1}{2}}\pi^{\frac{1}{2}}.$$

integral powers of n, which is convergent for all values of n, real or imaginary, however great be the modulus, will continue to represent u when n is imaginary. The differential equation (11), and consequently the descending series derived from it, will also hold good when n is imaginary; but since this series contains radicals, while U is itself a rational function of n, we might expect beforehand that in passing from one imaginary value of n to another it should sometimes be necessary to change the sign of a radical, or make some equivalent change in the coefficients A, B. Let $n = n_1 \epsilon^{\nu\sqrt{-1}}$, where n_1 is positive. Since both values of $2\left(\frac{n}{3}\right)^{\frac{3}{2}}$ are employed in the series, with different arbitrary constants, we may without loss of generality suppose that value of $n^{\frac{3}{2}}$ which has $\frac{3}{2}\nu$ for its amplitude to be employed in the circular functions or exponentials, as well as in the expression for S. In the multiplier we may always take $-\frac{\nu}{4}$ for the amplitude of $n^{-\frac{1}{4}}$ by including in the constant coefficients the factor by which one fourth root of n differs from another; but then we must expect to find the arbitrary constants discontinuous. In fact, if we observe the forms of R and S, and suppose the circular functions in (15) expanded in ascending series, it is evident that the expression for U will be of the form

$$A n^{-\frac{1}{4}} N + B n^{\frac{1}{4}} N', \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (25)$$

where N and N' are rational functions of n. At least, this will be the case if we regard as a rational function a series involving descending integral powers of n, and which is at first rapidly convergent, though ultimately divergent, or rather, if we regard as such the function to which the convergent part of the series is a very close approximation when the modulus of n is at all large. Now, if A and B retained the same values throughout, the above expression would not recur till ν was increased by 8π, whereas U recurs when ν is increased by 2π. If we write $\nu + 2\pi$ for ν, and observe that N and N' recur, the expression (25) will become

$$-\sqrt{-1}\, A n^{-\frac{1}{4}} N + \sqrt{-1}\, B n^{\frac{1}{4}} N';$$

and since U recurs it appears that A, B become $\sqrt{-1}\, A$, $-\sqrt{-1}\, B$, respectively, when ν is increased by 2π. Also the imaginary part of the expression (25) changes sign with ν, as it ought; so that, in order to know what A and B are generally, it would be sufficient to know what they are from $\nu = 0$ to $\nu = \pi$.

If we put $n_1 \epsilon^{\pi\sqrt{-1}}$ for n in the second member of equation (15), and write β for $2 \,.\, 3^{-\frac{3}{2}} n_1^{\frac{3}{2}}$, and R_1, S_1 for what R, S become when n_1 is put for n in the second members of equations (16) and all the terms are taken positively, we shall get as our result

$$\tfrac{1}{2}\epsilon^{-\frac{\pi}{4}\sqrt{-1}} n_1^{-\frac{1}{4}} \{(A - \sqrt{-1}\, B)(R_1 + S_1)\,\epsilon^{\beta} + (A + \sqrt{-1}\, B)(R_1 - S_1)\,\epsilon^{-\beta}\}.$$

Now the part of this expression which contains $(R_1 + S_1)\epsilon^{\beta}$ ought to disappear, as appears from (17). If we omit the first part of the expression, and in the second part put for A and B their values given by (24), we shall obtain an expression which will be identical with the second member of (17) provided

$$C = \frac{\pi^{\frac{1}{2}}}{2 \,.\, 3^{\frac{1}{4}}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (26)$$

This mode of determining the constant C is anything but satisfactory. I have endeavoured in vain to deduce the leading term in U for n negative from the integral itself, whether in the original form in which it appears in (5), or in the altered form in which it is obtained from (6). The correctness of the above value of C will however be verified further on.

10. Expressing n, U in terms of m, W by means of (8) and (9), putting for shortness

$$\phi = 2\left(\frac{n}{3}\right)^{\frac{3}{2}} = \pi\left(\frac{m}{3}\right)^{\frac{3}{2}}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (27)$$

where the numerical values of m and n are supposed to be taken when these quantities are negative, observing that $16\sqrt{(3n^3)} = 72\phi$, and reducing, we get when m is positive

$$W = 2^{\frac{1}{2}}(3m)^{-\frac{1}{4}}\left\{R\cos\left(\phi - \frac{\pi}{4}\right) + S\sin\left(\phi - \frac{\pi}{4}\right)\right\}, \quad . \quad . \quad . \quad (28)$$

where

$$\left.\begin{aligned} R &= 1 - \frac{1.5.7.11}{1.2\,(72\phi)^2} + \frac{1.5.7.11.13.17.19.23}{1.2.3.4\,(72\phi)^4} - \ldots, \\ S &= \frac{1.5}{1.72\phi} - \frac{1.5.7.11.13.17}{1.2.3\,(72\phi)^3} + \ldots. \end{aligned}\right\} \quad . \quad . \quad (29)$$

When m is negative, so that W is the integral expressed by writing $-m$ for m in (1), we get

$$W = 2^{-\frac{1}{2}}(3m)^{-\frac{1}{4}}\epsilon^{-\phi}\left\{1 - \frac{1.5}{1.72\phi} + \frac{1.5.7.11}{1.2\,(72\phi)^2} - \ldots\right\}. \quad . \quad . \quad . \quad (30)$$

11. Reducing the coefficients of ϕ^{-1}, $\phi^{-2}\ldots$ in the series (29) for numerical calculation, we have, not regarding the signs,

order	i	ii	iii	iv	v	vi
logarithm	$\bar{2}.841638$;	$\bar{2}.569766$;	$\bar{2}.579704$;	$\bar{2}.760793$;	$\bar{1}.064829$;	$\bar{1}.464775$
coefficient	.0694444;	.0371335;	.0379930;	.0576490;	.116099;	.291592.

Thus, for $m = 3$, in which case $\phi = \pi$, we get for the successive terms after the first, which is 1,

$$.022105, \quad .003762, \quad .001225, \quad .000592, \quad .000379, \quad .000303.$$

We thus get for the value of the series in (30), by taking half the last term but one and a quarter of its first difference, .980816; whence for $m = 3$, $W = 6^{-\frac{1}{2}} \times .980816\,\epsilon^{-\pi} = .0173038$, of which the last figure cannot be trusted. Now the number given by Mr. Airy to 5 decimal places, and calculated from the ascending series and by quadratures separately, is .01730, so that the correctness of the value of C given by (26) is verified.

For $m = +3$ we have from (28)

$$W = -3^{-\frac{1}{2}}(R - S) = -3^{-\frac{1}{2}}(.9965 - .0213) = -.5632,$$

which agrees with Mr. Airy's result $-.56322$ or $-.56323$. As m increases, the convergency of the series (29) or (30) increases rapidly.

12. The expression (28) will be rendered more easy of numerical calculation by assuming $R = M\cos\psi$, $S = M\sin\psi$, and expanding M and $\tan\psi$ in series to a few terms. These series will evidently proceed, the first according to even, and the second according to odd inverse powers of ϕ. Putting the several terms, taken positively, under the form 1, $a\phi^{-1}$, $ab\phi^{-2}$, $abc\phi^{-3}$, $abcd\phi^{-4}$ &c., and proceeding to three terms in each series, we get

$$M = 1 - a\left(b - \frac{a}{2}\right)\phi^{-2} + a\left\{bc(d-a) + \frac{a^2}{2}\left(b - \frac{a}{4}\right)\right\}\phi^{-4}, \quad . \quad . \quad (31)$$

$$\tan\psi = a\phi^{-1} - ab\,(c-a)\,\phi^{-3} + ab\{cd\,(e-a) - ab\,(c-a)\}\phi^{-5}. \quad . \quad . \quad (32)$$

The roots of the equation $W = 0$ are required for the physical problem to which the integral W relates. Now equations (28), (29) shew that when m is at all large the roots of this equation are given very nearly by the formula $\phi = (i - \frac{1}{4})\,\pi$, where i is an integer. From the definition of ψ it follows that the root satisfies exactly the equation

$$\phi = (i - \tfrac{1}{4})\,\pi + \psi. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (33)$$

By means of this equation we may expand ϕ in a series according to descending powers of Φ, where $\Phi = (i - \frac{1}{4})\,\pi$. For this purpose it will be convenient first to expand ψ in a series according to descending powers of ϕ, by means of the expansion of $\tan^{-1}x$ and the equation (32), and having substituted the result in (33) to expand by Lagrange's theorem. The result of the expansion carried as far as to Φ^{-5} is

$$\phi = \Phi + a\Phi^{-1} - \{ab\,(c-a) + \tfrac{1}{3}a^3 + a^2\}\,\Phi^{-3}$$
$$+ \{ab[cd\,(e-a) - ab\,(c-a)] + a^2b\,(c-a) + \tfrac{1}{5}a^5 + 4a\,[ab\,(c-a) + \tfrac{1}{3}a^3] + 2a^3\}\,\Phi^{-5}. \ldots (34)$$

13. To facilitate the numerical calculation of the coefficients let

$$a = \frac{a'}{1\,.\,D}; \quad b = \frac{b'}{2\,.\,D}; \quad c = \frac{c'}{3\,.\,D}; \text{ \&c.,}$$

and let the coefficients of ϕ^{-2}, ϕ^{-4} in (31) be put under the forms $-\dfrac{A_2}{1\,.\,2\,D^2}$, $\dfrac{A_4}{1\,.\,2\,.\,3\,.\,4\,D^4}$, and similarly with respect to (32), (34). Then to calculate W for a given value of m, we have

$$W = 2^{\frac{1}{2}}\,(3m)^{-\frac{1}{4}}\,M\cos\left(\phi - \frac{\pi}{4} - \psi\right), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (35)$$

where

$$M = 1 - \frac{A_2}{1\,.\,2\,D^2}\phi^{-2} + \frac{A_4}{1\,.\,2\,.\,3\,.\,4\,D^4}\phi^{-4}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (36)$$

$$\tan\psi = \frac{C_1}{1\,.\,D}\phi^{-1} - \frac{C_3}{1\,.\,2\,.\,3\,D^3}\phi^{-3} + \frac{C_5}{1\,.\,2\,.\,3\,.\,4\,.\,5\,D^5}\phi^{-5}, \quad . \quad . \quad (37)$$

and for calculating the roots of the equation $W = 0$, we have

$$\phi = \Phi + \frac{E_1}{1\,.\,D}\Phi^{-1} - \frac{E_3}{1\,.\,2\,.\,3\,D^3}\Phi^{-3} + \frac{E_5}{1\,.\,2\,.\,3\,.\,4\,.\,5\,D^5}\Phi^{-5}. \quad . \quad . \quad . \quad (38)$$

The coefficients in these formulæ are given by the equations

$$\left.\begin{aligned} A_2 = a'(b'-a')\,;\quad A_4 = a'\left\{b'c'(d'-4a') + 3a'^2(2b'-a')\right\};\\ C_1 = a'\,;\quad C_3 = a'b'(c'-3a')\,;\quad C_5 = a'b'\left\{c'd'(c'-5a') - 10C_3\right\};\\ E_1 = a'\,;\quad E_3 = C_3 + 2a'^2(3D + a')\,;\\ E_5 = C_5 + 20a'(4D + a')\,C_3 + 24a'^5 + 80a'^3D(3D + 2a').\end{aligned}\right\} \quad . \quad (39)$$

14. Putting in these formulæ

$$a' = 1\,.\,5\,;\quad b' = 7\,.\,11\,;\quad c' = 13\,.\,17\,;\quad d' = 19\,.\,23\,;\quad e' = 25\,.\,29\,;\quad D = 72\,;$$

we get

$$A_2 = 5\,.\,72\,;\quad A_4 = 3\,.\,5\,.\,72^2.\,457\,;\quad C_1 = 5\,;\quad C_3 = 2\,.\,5\,.\,7\,.\,11\,.\,103\,;$$

$$C_5 = 4^2.\,5^3.\,7^2.\,11\,.\,23861\,;\quad E_1 = 5\,;\quad E_3 = 72\,.\,1255\,;\quad E_5 = 4\,.\,5^3.\,72^2.\,10883\,;$$

whence we obtain, on substituting in (36), (37), (38),

$$M = 1 - \frac{5}{144}\phi^{-2} + \frac{2285}{41472}\phi^{-4},$$

$$\tan\psi = \frac{5}{72}\phi^{-1} - \frac{39655}{1119744}\phi^{-3} + \frac{321526975}{2902376448}\phi^{-5},$$

$$\phi = \Phi + \frac{5}{72}\Phi^{-1} - \frac{1255}{31104}\Phi^{-3} + \frac{272075}{2239488}\Phi^{-5}.$$

Reducing to decimals, having previously divided the last equation by π, and put for Φ its value $(i - \frac{1}{4})\pi$, we get

$$M = 1 - .034722\,\phi^{-2} + .055097\,\phi^{-4}, \quad . \quad . \quad . \quad . \quad . \quad (40)$$

$$\tan\psi = .069444\,\phi^{-1} - .035414\,\phi^{-3} + .110781\,\phi^{-5}, \quad . \quad . \quad (41)$$

$$\frac{\phi}{\pi} = i - .25 + \frac{.028145}{4i-1} - \frac{.026510}{(4i-1)^3} + \frac{.129402}{(4i-1)^5} \quad . \quad . \quad . \quad (42)$$

15. Supposing $i = 1$ in (42), we get

$$\frac{\phi}{\pi} = .75 + .0094 - .0010 + 0005 = .7589\,;$$

whence $m = 3\left(\frac{\phi}{\pi}\right)^{\frac{2}{3}} = 2.496$. The descending series obtained in this paper fail for small values of m; but it appears from Mr. Airy's table that for such values the function W is positive, the first change of sign occurring between $m = 2.4$ and $m = 2.6$. Hence the integer i in (42) is that which marks the order of the root. A more exact value of the first root, obtained by interpolation from Mr. Airy's table, is 2.4955. For $i = 1$ the series (42) is not convergent enough to give the root to more than three places of decimals, but the succeeding roots are given by this series with great accuracy. Thus, even in the case of the second root the value of the last term in (42) is only .000007698. It appears then that this term might have been left out altogether.

16. To determine when W is a maximum or minimum we must put $\frac{dW}{dm} = 0$. We might get $\frac{dW}{dm}$ by direct differentiation, but the law of the series will be more easily obtained from the differential equation. Resuming equation (11), and putting V for $\frac{dU}{dn}$, we get by dividing by n and then differentiating

$$\frac{d^2V}{dn^2} - \frac{1}{n}\frac{dV}{dn} + \frac{n}{3}V = 0.$$

This equation may be integrated by descending series just as before, and the arbitrary constants will be determined at once by comparing the result with the derivative of the second member of (15), in which A, B are given by (24). As the process cannot fail to be understood from what precedes, it will be sufficient to give the result, which is

$$V = 3^{-\frac{2}{3}}\pi^{\frac{1}{2}}n^{\frac{1}{2}}\left\{R'\cos\left(\phi + \frac{\pi}{4}\right) + S'\sin\left(\phi + \frac{\pi}{4}\right)\right\}, \quad . \quad . \quad . \quad . \quad (43)$$

where

$$\left.\begin{aligned} R' &= 1 - \frac{-1.7.5.13}{1.2\,(72\phi)^2} + \frac{-1.7.5.13.11.19.17.25}{1.2.3.4\,(72\phi)^4} - \dots, \\ S' &= \frac{-1.7}{1.72\phi} - \frac{-1.7.5.13.11.19}{1.2.3\,(72\phi)^3} + \dots. \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (44)$$

17. The expression within brackets in (43) may be reduced to the form $M\cos\left(\phi + \frac{\pi}{4} - \psi\right)$ just as before, and the formulæ of Art. 13 will apply to this case if we put

$$a' = -1.7: \ b' = 5.13; \ c' = 11.19; \ \&\text{c.}, \ D = 72.$$

The roots of the equation $\frac{dW}{dm} = 0$ are evidently the same as those of $V = 0$. They are given approximately by the formula $\phi = (i - \frac{3}{4})\pi$, and satisfy exactly the equation $\phi = (i - \frac{3}{4})\pi + \psi$. The root corresponding to any integer i may be expanded in a series according to the inverse odd powers of $4i - 3$ by the formulæ of Art. 13. Putting $(i - \frac{3}{4})\pi$ for Φ, and taking the series to three terms only, we get

$$E_1 = -7; \quad E_3 = -84168;$$

whence

$$\phi = \Phi - \frac{7}{72}\Phi^{-1} + \frac{1169}{31104}\Phi^{-3};$$

or, reducing as before,

$$\frac{\phi}{\pi} = i - .75 - \frac{.039403}{4i - 3} + \frac{.024693}{(4i - 3)^3}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (45)$$

This series will give only a rough approximation to the first root, but will answer very well for the others.

For $i = 1$ the series gives $\pi^{-1}\phi = .25 - .039 + .025$, which becomes on taking half the second term and a quarter of its first difference $.25 - .019 - .004 = .227$, whence $m = 1.12$.

The value of the first root got by interpolation from Mr. Airy's table is 1.0845. For the 2nd and 3rd roots we get from (45)

$$\text{for } i = 2,\ \pi^{-1}\phi = 1.25 - .00788 + .00020 = 1.24232\ ;$$
$$\text{for } i = 3,\ \pi^{-1}\phi = 2.25 - .00438 + .00003 = 2.24565.$$

For higher values of i the last term in (45) may be left out altogether.

18. The following table contains the first 50 roots of the equation $W = 0$, and the first 10 roots of the derived equation. The first root in each case was obtained by interpolation from Mr. Airy's table; the series (42) and (45) were sufficiently convergent for the other roots. In calculating the 2nd root of the derived equation, a rough value of the first term left out in (45) was calculated, and its half taken since the next term would be of opposite sign. The result was only $-$ .000025, so that the series (45) may be used even when i is as small as 2. By far

i	m	diff.	i	m	diff.
1	2.4955	1.8676	26	26.1602	·6730
2	4.3631	1.5291	27	26.8332	.6647
3	5.8922	1.3514	28	27.4979	.6567
4	7.2436	1.2352	29	28.1546	.6491
5	8.4788	1.1512	30	28.8037	.6419
6	9.6300	1.0861	31	29.4456	.6349
7	10.7161	1.0335	32	30.0805	.6284
8	11.7496	.9899	33	30.7089	.6219
9	12.7395	.9529	34	31.3308	.6159
10	13.6924	.9208	35	31.9467	.6100
11	14.6132	.8927	36	32.5567	.6043
12	15.5059	.8676	37	33.1610	.5989
13	16.3735	.8452	38	33.7599	.5936
14	17.2187	.8250	39	34.3535	.5885
15	18.0437	.8065	40	34.9420	.5836
16	18.8502	.7897	41	35.5256	.5788
17	19.6399	.7740	42	36.1044	.5742
18	20.4139	.7597	43	36.6786	.5698
19	21.1736	.7463	44	37.2484	.5655
20	21.9199	.7337	45	37.8139	.5612
21	22.6536	.7221	46	38.3751	.5572
22	23.3757	.7111	47	38.9323	.5532
23	24.0868	.7008	48	39.4855	.5494
24	24.7876	.6909	49	40.0349	.5456
25	25.4785	.6817	50	40.5805	
1	1.0845	2.3824	6	9.0599	1.1175
2	3.4669	1.6777	7	10.1774	1.0590
3	5.1446	1.4336	8	11.2364	1.0111
4	6.5782	1.2903	9	12.2475	.9710
5	7.8685	1.1914	10	13.2185	

the greater part of the calculation consisted in passing from the values of $\pi^{-1}\phi$ to the corresponding values of m. In this part of the calculation 7-figure logarithms were used in obtaining the value of $\frac{1}{3}m$, and the result was then multiplied by 3.

A table of differences is added, for the sake of exhibiting the decrease indicated by theory in the interval between the consecutive dark bands seen in artificial rainbows. This decrease will be readily perceived in the tables which contain the results of Professor Miller's observations *. The table of the roots of the derived equation, which gives the maxima of W^2, is calculated for the sake of meeting any observations which may be made on the supernumerary bows accompanying a natural rainbow, since in that case the maximum of the red appears to be what best admits of observation.

SECOND EXAMPLE.

19. Let us take the integral

$$u = \frac{2}{\pi}\int_0^{\frac{\pi}{2}} \cos(x\cos\theta)\, d\theta = 1 - \frac{x^2}{2^2} + \frac{x^4}{2^2 4^2} - \frac{x^6}{2^2 4^2 6^2} + \dots \dagger \quad . \quad (46)$$

which occurs in a great many physical investigations. If we perform the operation $x\dfrac{d}{dx}$ twice in succession on the series we get the original series multiplied by $-x^2$, whence

$$\frac{d^2u}{dx^2} + \frac{1}{x}\frac{du}{dx} + u = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (47)$$

20. The form of this equation shews that when x is very large, and receives an increment δx, which, though not necessarily a very small fraction itself, is very small compared with x, u is expressed by $A\cos\delta x + B\sin\delta x$, where under the restrictions specified A and B are sensibly constant ‡. Assume then, according to the plan of Art. 5,

$$u = \epsilon^{x\sqrt{-1}}\left\{Ax^{\alpha} + Bx^{\beta} + Cx^{\gamma} + \dots\right\}. \quad . \quad . \quad . \quad . \quad (48)$$

On substituting in (47) we get

$$\sqrt{-1}\left\{(2\alpha+1)Ax^{\alpha-1} + (2\beta+1)Bx^{\beta-1} + \dots\right\}$$
$$+ \alpha^2 Ax^{\alpha-2} + \beta^2 Bx^{\beta-2} + \dots = 0.$$

Since we want a descending series, we must put

$$2\alpha + 1 = 0; \ \beta = \alpha - 1; \ \gamma = \beta - 1 \dots;$$
$$(2\beta+1)B = \sqrt{-1}\,\alpha^2 A; \ (2\gamma+1)C = \sqrt{-1}\,\beta^2 B \dots;$$

whence

$$\alpha = -\tfrac{1}{2}; \ \beta = -\tfrac{3}{2}; \ \gamma = -\tfrac{5}{2} \dots;$$

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* *Cambridge Philosophical Transactions*, Vol. VII. p. 277.

† This integral has been tabulated by Mr. Airy from $x=0$ to $x=10$, at intervals of 0.2. The table will be found in the 18th Volume of the *Philosophical Magazine*, page 1.

‡ That the 1st and 3rd terms in (47) are ultimately the important terms, may readily be seen by trying the terms two and two in the way mentioned in the introduction. Thus, if we suppose the first two to be the important terms, we get ultimately $U=A$ or $U=B\log x$, either of which would render the last term more important than the 1st or 2nd, and if we suppose the 2nd and 3rd to be the important terms, we get ultimately $u=A\epsilon^{-\frac{x^2}{2}}$, which would render the 1st term more important than either of the others.

$$B = -\frac{1^2}{1.8}\sqrt{-1}A\,; \quad C = +\frac{1^2.3^2}{1.2.8^2}(\sqrt{-1})^2 A\,; \quad D = -\frac{1^2.3^2.5^2}{1.2.3.8^3}(\sqrt{-1})^3 A \ldots.$$

Substituting in (48), reducing the result to the form $A(P+\sqrt{-1}\,Q)$, adding another solution of the form $B(P-\sqrt{-1}\,Q)$, and changing the arbitrary constants, we get

$$u = Ax^{-\frac{1}{2}}(R\cos x + S\sin x) + Bx^{-\frac{1}{2}}(R\sin x - S\cos x), \quad . \quad (49)$$

where

$$\left.\begin{aligned} R &= 1 - \frac{1^2.3^2}{1.2\,(8x)^2} + \frac{1^2.3^2.5^2.7^2}{1.2.3.4\,(8x)^4}\ldots, \\ S &= \frac{1^2}{1.8x} - \frac{1^2.3^2.5^2}{1.2.3\,(8x)^3} + \ldots. \end{aligned}\right\} \quad . \quad . \quad . \quad (50)$$

21. It remains to determine the arbitrary constants A, B. In equation (46) let $\cos\theta = 1-\mu$, whence

$$d\theta = \frac{d\mu}{\sin\theta} = \frac{d\mu}{(2\mu-\mu^2)^{\frac{1}{2}}} = \frac{d\mu}{(2\mu)^{\frac{1}{2}}} + M\,d\mu,$$

where

$$M = (2\mu-\mu^2)^{-\frac{1}{2}} - (2\mu)^{-\frac{1}{2}},$$

a quantity which does not become infinite between the limits of μ. Substituting in (46) we get

$$u = \frac{\sqrt{2}}{\pi}\int_0^1 \cos\{(1-\mu)\,x\}\,\mu^{-\frac{1}{2}}d\mu + \frac{2}{\pi}\int_0^1 \cos\{(1-\mu)\,x\}\,M\,d\mu. \quad . \quad (51)$$

By considering the series whose n^{th} term is the part of the latter integral for which the limits of μ are $n\pi x^{-1}$ and $(n+1)\pi x^{-1}$ respectively, it would be very easy to prove that the integral has a superior limit of the form Hx^{-1}, where H is a finite constant, and therefore this integral does not furnish any part of the leading terms in u. Putting $\mu x = \nu$ in the first integral in (51), so that

$$\mu^{-\frac{1}{2}}d\mu = x^{-\frac{1}{2}}\nu^{-\frac{1}{2}}d\nu,$$

observing that the limits of ν are 0 and x, of which the latter ultimately becomes ∞, and that

$$\int_0^\infty \cos\nu\,.\,\nu^{-\frac{1}{2}}d\mu = 2\int_0^\infty \cos\lambda^2 d\lambda = \sqrt{\frac{\pi}{2}} = 2\int_0^\infty \sin\lambda^2 d\lambda = \int_0^\infty \sin\nu\,.\,\nu^{-\frac{1}{2}}d\nu,$$

we get ultimately for very large values of x

$$u = (\pi x)^{-\frac{1}{2}}(\cos x + \sin x).$$

Comparing with (49) we get

$$A = B = \pi^{-\frac{1}{2}},$$

whence

$$u = \left(\frac{2}{\pi x}\right)^{\frac{1}{2}} R\cos\left(x-\frac{\pi}{4}\right) + \left(\frac{2}{\pi x}\right)^{\frac{1}{2}} S\sin\left(x-\frac{\pi}{4}\right).^{*} \quad . \quad . \quad (52)$$

* This expression for u, or rather an expression differing from it in nothing but notation and arrangement, has been already obtained in a different manner by Sir William R. Hamilton, in a memoir *On Fluctuating Functions*. See *Transactions of the Royal Irish Academy*, Vol. XIX. p. 313.

For example, when $x = 10$ we have, retaining 5 decimal places in the series,

$$R = 1 - .00070 + .00001 = .99931\,;\quad S = .01250 - .00010 = .01240$$

$$\text{Angle } x - \frac{\pi}{4} = 527^0.95780 = 3 \times 180^0 - 12^0 2' 32''\,;\ \text{ whence } u = -.24594,$$

which agrees with the number (– .2460) obtained by Mr. Airy by a far more laborious process, namely, by calculating from the original series.

22. The second member of equation (52) may be reduced to the same form as that of (28), and a series obtained for calculating the roots of the equation $u = 0$ just as before. The formulæ of Art. 13 may be used for this purpose on putting

$$a' = 1^2\,;\ b' = 3^2\,;\ c' = 5^2\,;\ \&\text{c.}\,;\ D = 8,$$

and writing x, X for ϕ, Φ, where $X = (i - \frac{1}{4})\pi$. We obtain

$$A_2 = 8\,;\ A_4 = 3\,.\,8^2.\,53\,;\ C_1 = 1\,;\ C_3 = 2\,.\,3^2.\,11\,;\ C_5 = 3^2.\,4^2.\,5\,.\,1139\,;$$

$$E_1 = 1\,;\ E_3 = 8\,.\,31\,;\ E_5 = 4^4.\,3779\,;$$

whence we get for calculating u for a given value of x

$$M = 1 - \frac{1}{16}x^{-2} + \frac{53}{512}x^{-4},$$

$$\tan\psi = \frac{1}{8}x^{-1} - \frac{33}{512}x^{-3} + \frac{3417}{16384}x^{-5},$$

$$u = \left(\frac{2}{\pi x}\right)^{\frac{1}{2}} M \cos\left(x - \frac{\pi}{4} - \psi\right). \quad . \quad . \quad . \quad . \quad . \quad (53)$$

For calculating the roots of the equation $u = 0$ we have

$$x = X + \frac{1}{8}X^{-1} - \frac{31}{384}X^{-3} + \frac{3779}{15360}X^{-5}.$$

Reducing to decimals as before, we get

$$M = 1 - .0625x^{-2} + .103516x^{-4}, \quad . \quad . \quad . \quad . \quad . \quad (54)$$

$$\tan\psi = .125x^{-1} - .064453x^{-3} + .208557x^{-5}, \quad . \quad . \quad (55)$$

$$\frac{x}{\pi} = i - {\cdot}25 + \frac{{\cdot}050661}{4i-1} - \frac{{\cdot}053041}{(4i-1)^3} + \frac{{\cdot}262051}{(4i-1)^5}. \quad . \quad (56)$$

As before, the series (56) is not sufficiently convergent when $i = 1$ to give a very accurate result. In this case we get

$$\pi^{-1}x = .75 + .017 - .002 + .001 = {\cdot}766,$$

whence $x = 2.41$. Mr. Airy's table gives $u = +{\cdot}0025$ for $x = 2{\cdot}4$, and $u = -{\cdot}0968$ for $x = 2{\cdot}6$, whence the value of the root is 2.4050 nearly.

The value of the last term in (56) is .0000156 for $i = 2$, and .00000163 for $i = 3$, so that all the roots after the first may be calculated very accurately from this series.

THIRD EXAMPLE.

23. Consider the integral

$$v = \frac{2}{\pi}\int_0^x \int_0^{\frac{\pi}{2}} \cos(x\cos\theta)\, x\, dx\, d\theta = \int_0^x ux\, dx = \frac{x^2}{2} - \frac{x^4}{2^2 . 4} + \frac{x^6}{2^2 . 4^2 . 6} - \ldots * \quad (57)$$

which occurs in investigating the diffraction of an object-glass with a circular aperture.

By performing on the series the operation denoted by $x\dfrac{d}{dx}x^{-1}\dfrac{d}{dx}$, we get the original series with the sign changed, whence

$$\frac{d^2v}{dx^2} - \frac{1}{x}\frac{dv}{dx} + v = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (58)$$

We may obtain the integral of this equation in a form similar to (49). As the process is exactly the same as before, it will be sufficient to write down the result, which is

$$v = A'x^{\frac{1}{2}}(R\cos x + S\sin x) + B'x^{\frac{1}{2}}(R\sin x - S\cos x), \quad . \quad (59)$$

where

$$\left.\begin{aligned} R &= 1 - \frac{-1.3.1.5}{1.2\,(8x)^2} + \frac{-1.3.1.5.3.7.5.9}{1.2.3.4\,(8x)^4} - \ldots, \\ S &= \frac{-1.3}{1.8x} - \frac{-1.3.1.5.3.7}{1.2.3\,(8x)^3} + \ldots, \end{aligned}\right\} \quad . \quad . \quad (60)$$

the last two factors in the numerator of any term being formed by adding 2 to the last two factors respectively in the numerator of the term of the preceding order.

The arbitrary constants may be easily determined by means of the equation

$$\frac{dv}{dx} = ux. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (61)$$

Writing down the leading terms only in this equation, we have

$$x^{\frac{1}{2}}(-A'\sin x + B'\cos x) = \pi^{-\frac{1}{2}}x^{\frac{1}{2}}(\cos x + \sin x),$$

whence

$$-A' = B' = \pi^{-\frac{1}{2}},$$

$$v = \left(\frac{2x}{\pi}\right)^{\frac{1}{2}}\left\{R\cos\left(x - \frac{3\pi}{4}\right) + S\sin\left(x - \frac{3\pi}{4}\right)\right\}. \quad . \quad . \quad (62)$$

24. Putting in the formulæ of Art. 13,

$$a' = -1.3; \quad b' = 1.5; \quad c' = 3.7; \quad d' = 5.9; \quad e' = 7.11; \quad D = 8;$$

we get

$$A_2 = -3.8; \quad A_4 = -3^3.8^2.11; \quad C_1 = -3; \quad C_3 = -2.3^2.5^2; \quad C_5 = -3^3.4^2.5^2.127;$$

$$E_1 = -3; \quad E_3 = -3^2.8; \quad E_5 = -3^3.4.8^2.131;$$

* The series $1 - \dfrac{x^2}{2.4} + \dfrac{x^4}{2.4^2.6} \ldots$ or $\dfrac{2v}{x^2}$ has been tabulated by Mr. Airy from $x = 0$ to $x = 12$ at intervals of 0.2. See *Camb. Phil. Trans.* Vol. v. p. 291. The same function has also been calculated in a different manner and tabulated by M. Schwerd in his work on diffraction. The argument in the latter table is the angle $\dfrac{180^0}{\pi}x$, and the table extends from 0^0 to 1125^0 at intervals of 15^0, that is, from $x = 0$ to $x = 19.63$ at intervals of 0.262 nearly.

whence we get for the formulæ answering to those of Art. 22,

$$M = 1 + \frac{3}{16}x^{-2} - \frac{99}{512}x^{-4},$$

$$\tan\psi = -\frac{3}{8}x^{-1} + \frac{75}{512}x^{-3} - \frac{5715}{16384}x^{-5},$$

$$x = X - \frac{3}{8}X^{-1} + \frac{3}{128}X^{-3} + \frac{1179}{5120}X^{-5},$$

X being in this case equal to $(i+\frac{1}{4})\pi$.

Reducing to decimals as before, we get for the calculation of v for a given value of x,

$$M = 1 + .1875x^{-2} + .193359x^{-4}, \quad . \quad . \quad . \quad (63)$$

$$\tan\psi = -.375x^{-1} + .146484x^{-3} - .348817x^{-5}, \quad . \quad . \quad (64)$$

$$v = \left(\frac{2x}{\pi}\right)^{\frac{1}{2}} M\cos\left(x - \frac{3\pi}{4} - \psi\right); \quad . \quad . \quad . \quad . \quad (65)$$

and for calculating the roots of the equation $v = 0$,

$$\frac{x}{\pi} = i + .25 - \frac{.151982}{4i+1} + \frac{.015399}{(4i+1)^3} - \frac{.245835}{(4i+1)^5}. \quad . \quad . \quad (66)$$

25. The following table contains the first 12 roots of each of the equations $u = 0$, and $x^{-2}v = 0$. The first root of the former was got by interpolation from Mr. Airy's table, the others were calculated from the series (56). The roots of the latter equation were all calculated from the series (66), which is convergent enough even in the case of the first root. The columns which contain the roots are followed by columns which contain the differences between consecutive roots, which are added for the purpose of shewing how nearly equal these differences are to 1, which is what they ultimately become when the order of the root is indefinitely increased.

i	$\frac{x}{\pi}$ for $u=0$	diff.	$\frac{x}{\pi}$ for $v=0$	diff.
1	.7655	.9916	1.2197	1.0133
2	1.7571	.9975	2.2330	1.0053
3	2.7546	.9988	3.2383	1.0028
4	3.7534	.9993	4.2411	1.0017
5	4.7527	.9995	5.2428	1.0011
6	5.7522	.9997	6.2439	1.0009
7	6.7519	.9997	7.2448	1.0006
8	7.7516	.9998	8.2454	1.0005
9	8.7514	.9999	9.2459	1.0004
10	9.7513	.9999	10.2463	1.0003
11	10.7512	.9999	11.2466	1.0003
12	11.7511		12.2469	

26. The preceding examples will be sufficient to illustrate the general method. I will remark in conclusion that the process of integration applied to the equations (11), (47), and (58) leads very readily to the complete integral in finite terms of the equation

$$\frac{d^2y}{dx^2} - \left\{q^2 + \frac{i(i+1)}{x^2}\right\} y = 0, \quad . \quad . \quad . \quad . \quad . \quad (67)$$

where i is an integer, which without loss of generality may be supposed positive. The form under which the integral immediately comes out is

$$y = A\epsilon^{qx}\left\{1 - \frac{i(i+1)}{1.2qx} + \frac{(i-1)\,i\,(i+1)(i+2)}{1.2(2qx)^2} - \ldots\right\},$$

$$+ B\epsilon^{-qx}\left\{1 + \frac{i(i+1)}{1.2qx} + \frac{(i-1)\,i\,(i+1)(i+2)}{1.2(2qx)^2} + \ldots\right\},$$

where each series will evidently contain $i+1$ terms. It is well known that (67) is a general integrable form which includes as a particular case the equation which occurs in the theory of the figure of the earth, for q in (67) is any quantity real or imaginary, and therefore the equation formed from (67) by writing $+q^2y$ for $-q^2y$ may be supposed included in the form (67).

It may be remarked that the differential equations discussed in this paper can all be reduced to particular cases of the equation obtained by replacing $i(i+1)$ in (67) by a general constant. By taking $gn^{\frac{1}{2}}$, where g is any constant, for the independent variable in place of n in the differential equations which U, V in the first example satisfy, these equations are reduced to the form

$$\frac{d^2y}{dx^2} + \frac{2a}{x}\frac{dy}{dx} + \left(\frac{b}{x^2} + c\right) y = 0,$$

and (47), (58) are in this form already. Putting now $y = x^{-a}z$, we shall reduce the last equation to the form required.

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