Asymptotic solutions of differential equations: WKBJ

I - Solutions for equations with a leading small parameter

As was discussed before, equations that have a small parameter, and are such that lead to singular perturbations, can be treated with matched asymptotics. However, matched asymptotics fails when the solutions have a fast oscillatory nature. In these cases the appropriate tool to find asymptotic solutions is the Liouville Green method (also known as WKBJ approximation):

Consider

$$\epsilon^2 \frac{d^2y}{dx^2} = q(x)y, \quad q(x) \neq 0.$$

The basic idea of the method is to propose a solution of the form

$$y \sim e^{s(x)/\delta(\epsilon)}, \quad \delta(\epsilon) \to 0.$$

where $s(x)$ is a slowly varying function of x. The solution $y(x)$ may nonetheless have a fast variation because is the exponential of $s(x)$.

After substituting the proposed solution into the original equation it is found that:

$$\epsilon^2 \left(\frac{s''}{\delta} + \frac{s'^2}{\delta^2} \right) - q(x) = 0,$$

where the primes mean derivative respect to x.

The dependence of δ on ϵ is found by balancing different terms of the equation. It is in general assumed that the dependence is a single power law (i.e. $\delta = \epsilon^\alpha$), so that a perturbation scheme is feasible. In this case we balance the term with the square of the first derivative with the term $q(x)$. Other balances are, of course, possible but are not very useful (**Note:** *balancing the second derivative with the square of the first derivative leads to* $\delta = \epsilon^0 = 1$ *leaves the equation unchanged, and leads to the original problem of a singular perturbation, and balancing the second derivative with* $q(x)$ *would leave as dominant term* s'^2 *by itself, which is of not use at all).*

This means that δ is chosen so that the coefficients of the term with the square of the first derivative and then one in front of $q(x)$ are of the same order, that is:

$$\frac{\epsilon^2}{\delta^2} = 1,$$

which yields $\delta = \epsilon$. Leading to:

$$s'^2 - q(x) = -\epsilon s'', \ (*)$$

and indicating a clear path towards a controlled perturbation scheme.

Writing:

$$s(x) = \sum_{n=0}^{\infty} \epsilon^n s_n(x) \quad \text{as} \quad \epsilon \to 0,$$

substituting this expression into (*):

$$\left(s_0' + \epsilon s_1' + \cdots\right)^2 - q(x) = \epsilon \left(s_0'' + \epsilon s_1'' + \cdots\right),$$

and collecting terms of the same order in ϵ, the following set of equations are found:

$$\begin{aligned}
(s_0')^2 &= q(x) \\
2s_0' s_1' &= -s_0'' \\
&\cdots \\
2s_0' s_n' + \sum_{k=1}^{n-1} s_k' s_{n-k}' &= -s_{n-1}'' \quad \text{for} \quad n \geq 2.
\end{aligned}$$

This system of equations can be solved order by order. The first two terms of $s(x)$ are:

$$\begin{aligned}
s_0 &= \pm \int q(x)^{1/2} \, dx \,, \\
\text{and} \quad s_1' = -\frac{s_0''}{2s_0'} \quad \Longrightarrow \quad s_1 &= -\frac{1}{4} \log |q(x)|.
\end{aligned}$$

Finally, replacing these results into $y(x)$ yields the asymptotic solution:

$$y(x) \sim |q(x)|^{-\frac{1}{4}} \left(A \exp\left[\frac{1}{\epsilon} \int q(x)^{1/2} \, dx\right] + B \exp\left[-\frac{1}{\epsilon} \int q(x)^{1/2} \, dx\right]\right).$$

It is important to note that depending on the sign of $q(x)$ the solutions will be either exponential or oscillatory.
