Watson's lemma for real variables

Let $0 \leq T < \infty$ be a fixed real number. Assume $f(t)$ is a bounded function in the interval $[0, T]$, such that $|f(t)| < e^{ct}$ as $t \to \infty$ for some $c \in \mathbb{R}$, and has asymptotic series representation

$$f(t) \sim \sum_{n=0}^{\infty} c_n t^{\alpha+\beta n}, \quad \text{with } \alpha > -1, \beta > 0 \text{ as } t \to 0^+.$$

Then

$$I = \int_0^T f(t)e^{-xt}\, dt \sim \sum_{n=0}^{\infty} c_n \frac{\Gamma(\alpha+\beta n+1)}{x^{\alpha+\beta n+1}} \text{ as } x \to \infty$$

Note: i) The conditions on $f(t)$ are sufficient for the integral $I(x)$ to exist. ii) The result is independent of T. iii) $\alpha = 0$, $\beta = 1$, and $x \to -x$ corresponds to the example we solved last lecture.

To prove the lemma, we will first show the preliminary result:

$$I(x) = \int_0^T f(t)e^{-xt}\, dt \sim \int_0^\infty f(t)e^{-xt}\, dt \quad \text{as} \quad x \to \infty$$

with an error exponentially small. In fact,

$$I(x) = \int_0^T f(t)e^{-xt}\, dt = \int_0^\infty f(t)e^{-xt}\, dt - \int_T^\infty f(t)e^{-xt}\, dt.$$

Then, when approximating $I(x) \sim \int_0^\infty f(t)e^{-xt}\, dt$, and considering that $|f(t)| < e^{ct}$ as $x \to \infty$, the error is given by

$$|R_T| = \left| \int_T^\infty f(t)e^{-xt}\, dt \right| \leq \left| \int_T^\infty e^{-(x-c)t}\, dt \right| = e^{cT} \frac{e^{-xT}}{x\left(1 - \frac{c}{x}\right)} = \mathcal{O}\left(\frac{e^{-xT}}{x}\right),$$

which for all $x > c$ when $x \to \infty$, is small at all orders as long as $T > 0$.

Now we are ready to apply this to our function $f(t)$,

$$I = \int_0^T f(t)e^{-xt}\, dt \sim \sum_{n=0}^{\infty} \int_0^\infty c_n t^{\alpha+\beta n} e^{-xt}\, dt + \mathcal{O}\left(\frac{e^{-xT}}{x}\right).$$

We can now use the definition of asymptotic series with the error explicitly written:

$$f(t) \sim \sum_{n=0}^{N} c_n t^{\alpha+\beta n} + R_N(t) \quad \text{as } t \to 0^+$$

where, in this case

$$|R_N(t)| = \mathcal{O}(t^{\alpha+\beta(N+1)}) \quad \text{as } t \to 0^+,$$

that is, $R_N(t) \leq Ct^{\alpha+\beta(N+1)}$ as $t \to 0^+$ for some real constant $C > 0$. Replacing the asymptotic expression for $f(t)$ into the integral we find

$$I = \int_0^T f(t)e^{-xt}\,dt \sim \sum_{n=0}^{\infty} \int_0^{\infty} c_n t^{\alpha+\beta n} e^{-xt}\,dt - \int_0^{\infty} R_N(t)e^{-xt}\,dt + \mathcal{O}\left(\frac{e^{-xT}}{x}\right) \quad \text{as } x \to \infty.$$

Introducing the change of variables $u = xt$ in the first integral yields:

$$\int_0^{\infty} t^{\alpha+\beta n} e^{-xt}\,dt = \frac{1}{x^{\alpha+\beta n+1}} \int_0^{\infty} u^{\alpha+\beta n} e^{-u}\,du = \frac{\Gamma(\alpha+\beta n+1)}{x^{\alpha+\beta n+1}}$$

and, as $x \to \infty$, the second integral will produce an error $|\mathcal{R}_N(x)|$ given by:

$$|\mathcal{R}_N(x)| \leq \left| C\int_0^{\infty} t^{\alpha+\beta(N+1)} e^{-xt}\,dt \right| = C\frac{\Gamma(\alpha+\beta(N+1)+1)}{x^{\alpha+\beta(N+1)+1}} = \mathcal{O}\left(x^{-(\alpha+\beta(N+1)+1)}\right);$$

showing that the series we found is indeed an asymptotic series for $I(x)$ given by:

$$I = \int_0^T f(t)e^{-xt}\,dt \sim \sum_{n=0}^{\infty} c_n \frac{\Gamma(\alpha+\beta n+1)}{x^{\alpha+\beta n+1}} \quad \text{as } x \to \infty$$

Extention of Watson's lemma to complex variables

This theorem can be extended to the complex plane by just replacing x by $z \in \mathbb{C}$, i.e, imposing the same conditions on $f(t)$

$$I = \int_0^T f(t)e^{-zt}\,dt \sim \sum_{n=0}^{\infty} c_n \frac{\Gamma(\alpha+\beta n+1)}{z^{\alpha+\beta n+1}}$$

IMPORTANT: Saying that this is valid for $|z| \to \infty$ is not sufficient, the theorem is valid only when $|arg(z)| < (\pi/2) - \delta$ with $0 < \delta < (\pi/2)$.

Finally the requirement that $\alpha > -1$ has to do with the integrability of the function $f(t)$ about $t = 0$. In fact, for any $\alpha > -1$, the integral about 0 is finite:

$$\int_0^x f(t)\,dt = \int_0^x t^{\alpha} dt = \left.\frac{t^{\alpha+1}}{\alpha+1}\right|_0^x = \frac{x^{\alpha+1}}{\alpha+1}$$

The first power that would violate this integrability condition is $\alpha = -1$ for which the integral would yield $\ln t$; this is what is meant above by "The conditions on $f(t)$ are sufficient for the integral $I(x)$ to exist." Since $e^{-xt}|_{t=0} = 1$, any singularity that could exist would have its order determined solely by the power law in the integral of $f(t)$.