Asymptotic solutions of differential equations

I - Types of Singularities of differential equations

Differential equations may have two main types of singularities, fixed and movable ones:

i) **fixed singularities:**

A fixed singularity is one that is intrinsic to the equation, and the location of any existing singularities is determined by the character of the coefficients of the equation. For example:

$$\frac{d^2y}{dx^2} + \frac{1}{x}\frac{dy}{dx} + y = 0$$

has a fixed singularity at $x = 0$, while

$$\frac{d^2y}{dx^2} - xy = 0$$

has a singularity at $x \to \infty$.

The singularities of a differential equation can be "integrable" or "non-integrable". What is meant by "integrable" is that the singularities in the coefficients will manifest themselves not in the solution itself, but in its derivatives.

In particular, equations of the type:

$$\frac{d^2y}{dx^2} + p(x)\frac{dy}{dx} + q(x)y = 0,$$

for which the poles in $p(x)$ are at most of order one, and/or the poles in $q(x)$ at most of order two are "integrable".

(Note: *The equation above is a particular case of the Sturm-Liouville operator:*

$$\frac{d}{dx}\left(p(x)\frac{dy}{dx}\right) + \left[\lambda w(x) - q(x)\right] y = 0. \;)$$

ii) **Movable singularities:**

A movable singularity is one that depends on the initial or boundary conditions. For example:

$$\frac{dy}{dx} = y^2 \quad \text{with} \quad y(0) = \frac{1}{a}, \quad a \quad \text{constant}$$

has a general solution

$$y^{-1} = C - x$$

when the boundary conditions are applied:

$$y(x) = -\frac{1}{x-a}.$$

That is, the place where $y(x)$ has a singularity depends on the value of the initial condition, and not on the features of the differential equation.

Before embarking on the calculation of asymptotic approximations for the solution of any differential equation it is important to determine if there are any singualrities, and to what type they belong.

II - Classification of type of perturbations

It is always best to recast the diferential equation in terms of dimensionless variables because this makes it possible to determine if there is a small parameter, say ϵ, suitable for the application of perturbative methods. When such a parameter exists, there are two possible situations, the perturbative scheme can be either regular or singular. If the perturbation is regular, then the limit $\epsilon \to 0$ exists and is regular. If this is not the case it is said that the perturbation is singular.

i) **Example of regular perturbation:** We seek solutions for the equation:

$$\frac{d^2y}{dx^2} + 2\epsilon\frac{dy}{dx} + y = 0$$

In this case we already know how to obtain the exact solution by proposing $y = e^{\alpha x}$, which yields the characteristic polynomial:

$$\alpha^2 + 2\epsilon\alpha + 1 = 0$$

with roots

$$\alpha_{\pm} = -\epsilon \pm \sqrt{\epsilon^2 - 1}.$$

In the limit $\epsilon \to 0$ both roots are perfectly well behaved, and this indicates that an approximate solution found perturbatively will be regular.

ii) **Example of singular perturbation:** We seek solutions for the equation:

$$\epsilon\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y = 0$$

Again, we know how to obtain the exact solution by proposing $y = e^{\alpha x}$, which yields the characteristic polynomial:

$$\epsilon\alpha^2 + 2\alpha + 1 = 0$$

with roots

$$\alpha_{\pm} = \frac{1}{\epsilon}\left(-1 \pm \sqrt{1-\epsilon}\right).$$

In the limit $\epsilon \to 0$ the plus root is perfectly well behaved, but the minus root diverges. This is a clear indication that an approximate solution found perturbatively will be singular.

In general, neglecting the highest derivative in the first order of a perturbative scheme leads to singular behaviour.

III - Perturbative methods

One of the most widespread methods to deal with singular perturbations is known as **matched asymptotics**. Given a differential equation in an interval $[a, b]$ with a singularity located about a, the method consists of finding the "solution" to the original equation neglecting the highest derivative; this solution is called the "outer solution" and is valid only when $x \to b$. Then the equation is re-scaled so that the dominant terms are different than before and a new solution, called the "inner solution", valid when $x \to a$ is found. Using these two solutions and van Dyke's matching rule, it is possible to find a uniform solution valid in the entire interval. Matched asymptotics works when both solutions, inner and outer, are slowly varying functions.

i) **Incomplete example of matched asymptotics (non examinable):** We seek solutions for the equation:

$$\epsilon \frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y = 0 \quad \text{with} \quad y(0) = 0 \quad \text{and} \quad y(1) = 1$$

The first approximation, the "outer solution", is obtained from

$$2\frac{dy}{dx} + y = 0 \quad \text{with} \quad y(1) = 1$$

and is given by:

$$y(x) = e^{\frac{(1-x)}{2}}$$

Notice that we do not have any other constant to adjust so that the other boundary condition is satisfied. This is in itself another symptom of a singular perturbation.

To find the other solution we rescale the variable x as $X = \epsilon^{\beta} x$, then

$$\epsilon^{1+2\beta} \frac{d^2y}{dX^2} + 2\epsilon^{\beta} \frac{dy}{dX} + y = 0$$

Choosing $\beta = -1$ yields

$$\frac{d^2y}{dX^2} + 2\frac{dy}{dX} + \epsilon y = 0$$

with approximate solution $y(X) = Ae^{2X} + B$ with $y(0) = 0$, or

$$y(X) = A\left(1 - e^{-2X}\right).$$

This is the inner solution, which in the original variable x is

$$y(x) = A\left(1 - e^{-(2x/\epsilon)}\right).$$

The matching method consists in determining the constant A using van Dyke's matching rule. We are not going to go into the details of the method; however, after the matching process the approximate solution is:

$$y(x) = e^{1/2}\left(e^{x/2} - e^{-(2x/\epsilon)}\right),$$

which satisfies both boundary conditions and has as its exponents the correct approximations to the exact characteristic roots.

This method is of fundamental importance in fluid mechanics and dynamical systems. In particular, in fluid mechanics is the method of choice for the treatment of boundary layers.

ii) **When matched asymptotics does not work (WKBJ):**

Suppose that our equation is:

$$\epsilon \frac{d^2 y}{dx^2} + by = 0 \quad \text{with} \quad y(0) = 0 \quad \text{and} \quad y(L) = 0$$

The exact solution of this equation is:

$$y(x) = A \sin\left(\sqrt{\frac{b}{\epsilon}} x\right) + B \cos\left(\sqrt{\frac{b}{\epsilon}} x\right)$$

Notice that once again the limit $\epsilon \to 0$ is singular. In this case we may try the matched asymptotic methods listed above. In which case the equation for the outer solution is:

$$y(x) = 0$$

which matches the outer condition. After re-scaling, the inner equation becomes

$$\frac{d^2 y}{dX^2} = 0 \quad \text{with} \quad y(0) = 0$$

which corresponds to $y = BX + C$: An extremely bad approximation to the true solution!

We should have expected this. The solution as $\epsilon \to 0$ is a very fast oscillation, that is: it is not a slowly varying function. Hence, matched asymptotics does not work at all well in this case.

Problems such as the one just described, namely, singular perturbations for solutions with fast oscillations can be treated successfully with the WKBJ method. The WKBJ method is of great importance in quantum mechanics because the wave equation produces solutions with regions of fast oscillations and in the semi-classical limit there is an obvious small parameter that multiplies the derivative of highest order.